Spring, 2022

ME 597 – Solid Mechanics II

Lecture 11
Viscoelastic solids

PURDUE UNIVERSITY | Mechanical Engineering

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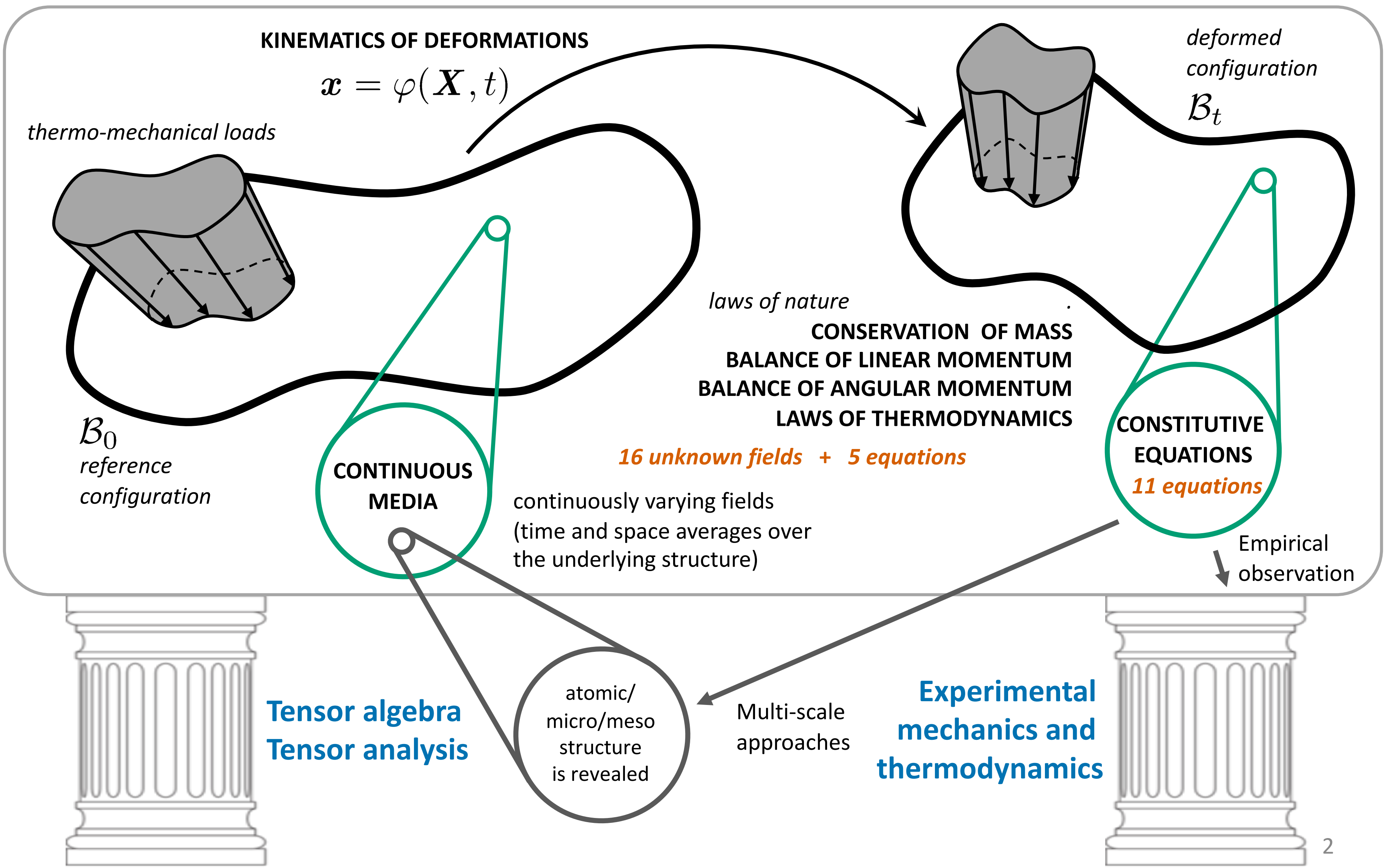
Lecture 11 – Constitutive relations



Constitutive relations

Constraints on constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g., $\boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ T$ (11 constitutive equations)

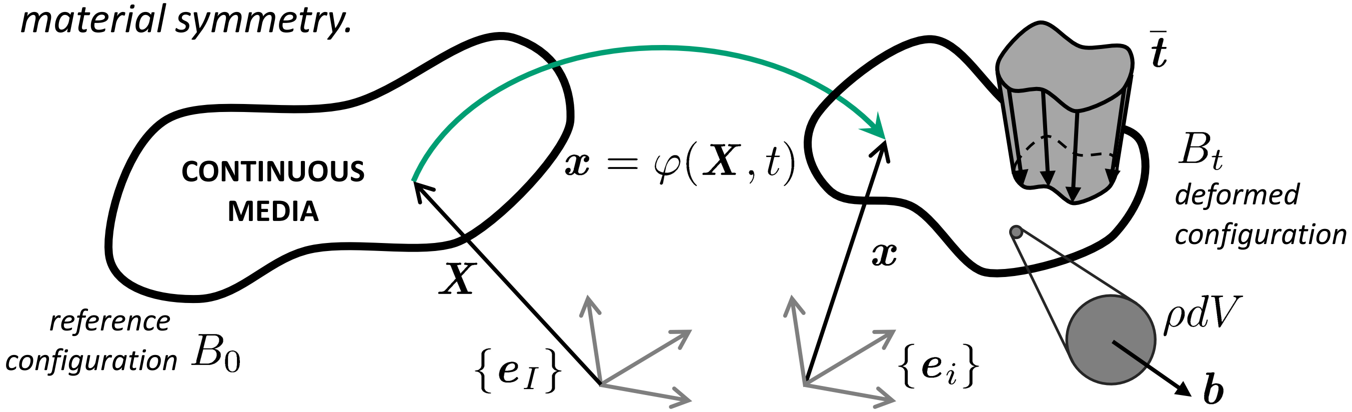
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)},\ \ \boldsymbol{q} = \boldsymbol{0}, \rho_0 u(\boldsymbol{F}, s),\ \ \dot{T} \neq 0\ \ (\text{with } \dot{s} = 0) \quad \text{isothermal processes}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)},\ \ \boldsymbol{q} = \boldsymbol{0}, W(\boldsymbol{F}, T),\ \ \dot{s} \neq 0\ \ (\text{with } \dot{T} = 0) \quad \text{isentropic processes}$$

- Can these constitutive relations be selected arbitrarily? NO!

  They must follow the following fundamental principles:

  Principle of determinism, principle of local action, second law of thermodynamics restrictions (Clausius-Duhem inequality), principle of material frame indifference, material symmetry.



Constitutive relations

Constraints on constitutive relations

- Principle of determinism & Principle of local action

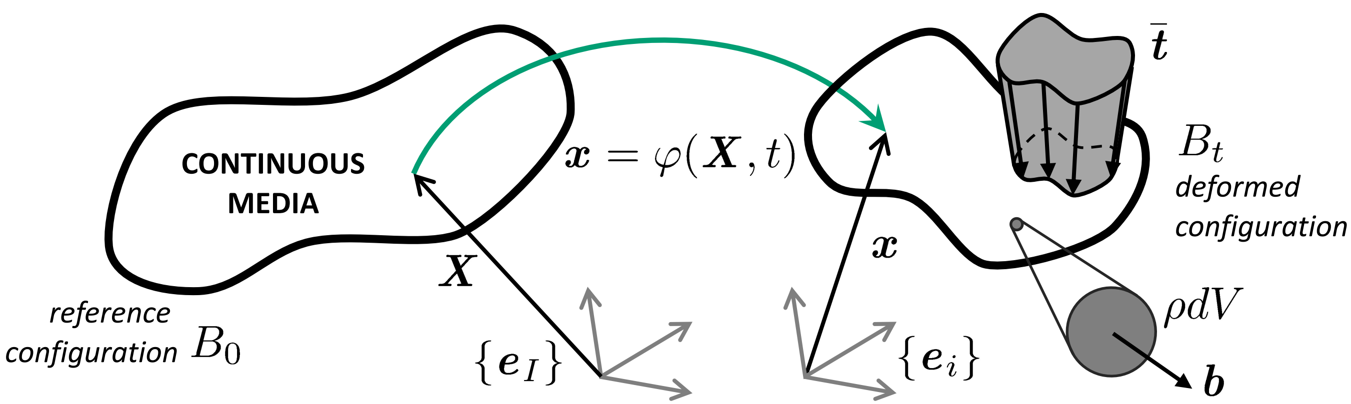
$$\boldsymbol{\sigma}(\boldsymbol{X},t) = \boldsymbol{f}(\varphi^t, \boldsymbol{F}^t, ..., T^t, \nabla_0 T^t, ..., \boldsymbol{X}, t)$$

time series $\square^t$ materials with memory

$\boldsymbol{f}(..., t)$ materials with aging

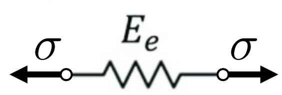
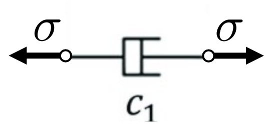
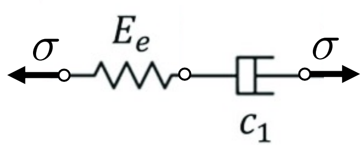
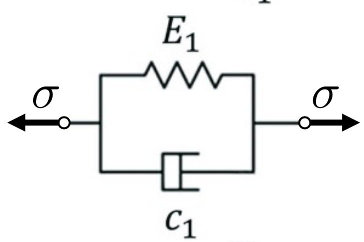
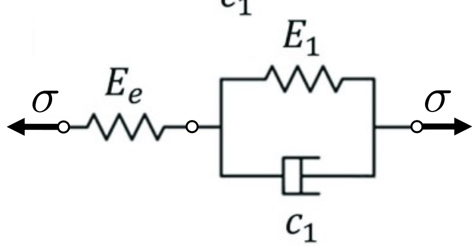
Simple elastic material $W = \overline{W}(\boldsymbol{C}, T)$ $\boldsymbol{S}^{(e)} = 2\frac{\partial \overline{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}$ $\boldsymbol{\sigma}^{(v)}(\boldsymbol{C}, T, \boldsymbol{d})$ — simple fluids —

Material with memory (infinitesimal strain) $\sigma(t) = \int_{-\infty}^{t} G(t-t')\dot{\epsilon}(t')dt'$



Materials with memory

Creep, relaxation and hysteresis

		Creep	Relaxation	Model Parameters
Spring		✗	✗	1
Dashpot		✗	✗	1
Maxwell model		✗	✓	2
Kelvin/Voigt model		✓	✗	2
Standard linear model		✓	✓	3

Materials with memory

Wave propagation

- 1D viscoelastic solid (balance of linear momentum) $\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$ $\quad u(x,t) = u_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)}$, $\epsilon = \frac{\partial u}{\partial x}$ — **complex wavenumber** k, **frequency** ω
- Standard linear model $\sigma + \frac{\eta_1}{E_0+E_1}\dot{\sigma} = \frac{E_0 E_1}{E_0+E_1}\left[\epsilon + \frac{\eta_1}{E_1}\dot{\epsilon}\right]$ — **stress-strain-time**
- Fourier transform $\hat{\sigma}(\omega) = M(\omega)\hat{\epsilon}(\omega)$ $\quad M(\omega)$ **complex modulus**

DIY

(cont.)

Materials with memory

Wave propagation

- 1D viscoelastic solid (balance of linear momentum) $\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$ $u(x,t) = u_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)}$ $\epsilon = \frac{\partial u}{\partial x}$ **complex wavenumber** k **frequency** ω
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- Fourier transform $\hat{\sigma}(\omega) = M(\omega)\hat{\epsilon}(\omega)$ $M(\omega)$ **complex modulus**

DIY

$M(\omega) = \ldots$

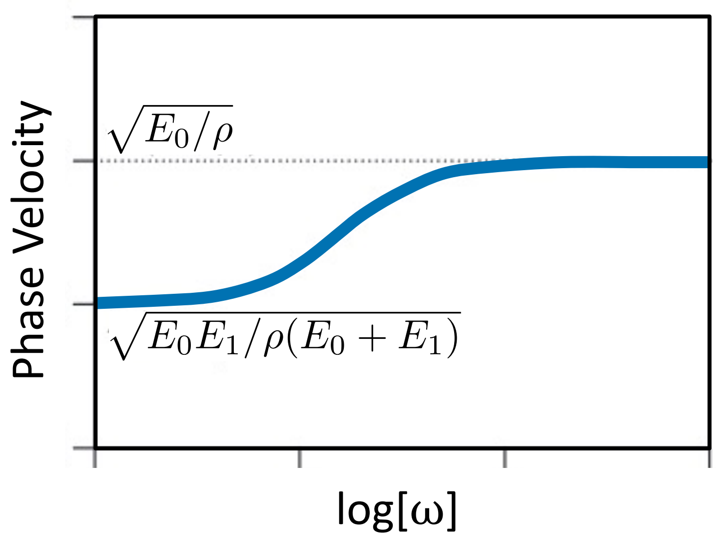
Materials with memory

Wave propagation – Phase velocity – Dispersion

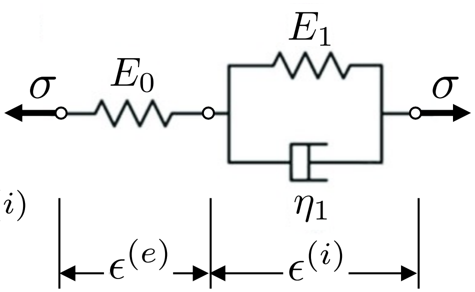
- 1D viscoelastic solid (frequency domain) $Mk^2 = \omega^2\rho$ **dispersion relation**

complex velocity $v_c(\omega) = \sqrt{\frac{M(\omega)}{\rho}} = \frac{\omega}{k}$

phase velocity $v_p(\omega) = [\mathrm{Re}(1/v_c)]^{-1}$



$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0\epsilon^{(e)}$$
$$\sigma = E_1\epsilon^{(i)} + \eta_1\dot{\epsilon}^{(i)}$$



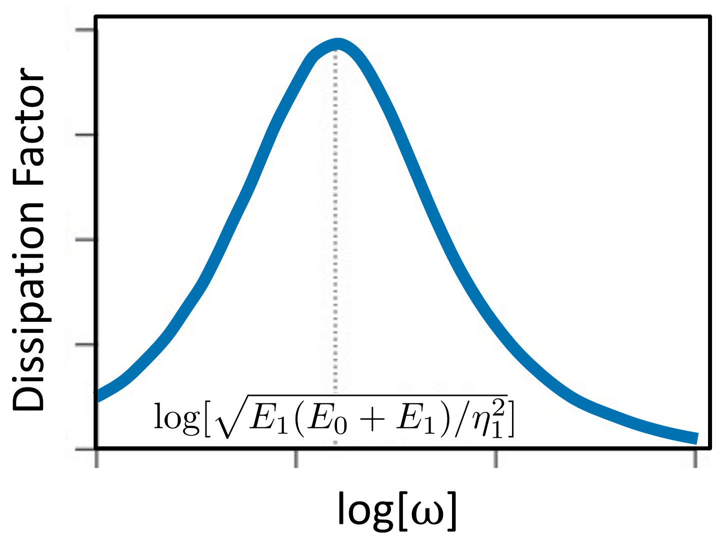
Materials with memory

Wave propagation – Dissipation

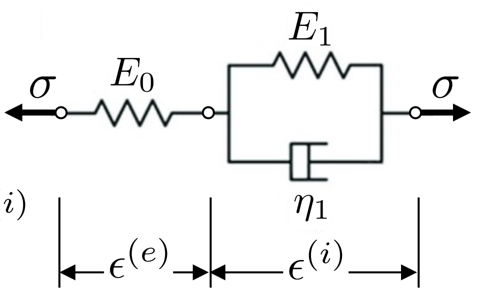
- Quality factor: twice the time-averaged strain-energy density divided by the time-averaged dissipated-energy density

$$Q = \frac{\mathrm{Re}(M)}{\mathrm{Im}(M)}$$

dissipation factor Q^{-1}



$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0\epsilon^{(e)}$$
$$\sigma = E_1\epsilon^{(i)} + \eta_1\dot{\epsilon}^{(i)}$$



Materials with memory

Internal variables and evolution equation

- The memory or path-dependency of a material can be represented through an array of **internal variables** (scalars and second-order tensors)

$$\boldsymbol{\sigma}(\boldsymbol{X},t) = \boldsymbol{f}(\boldsymbol{\epsilon},T,\boldsymbol{\epsilon}^t) \equiv \begin{cases} \boldsymbol{f}(\boldsymbol{\epsilon},T,\boldsymbol{\xi}) & \\ \dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma},T,\boldsymbol{\xi}) & \text{evolution equations} \end{cases}$$

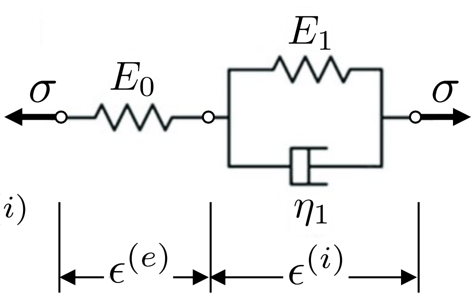
  The presence of additional variables in the constitutive laws requires additional constitutive equations, namely **evolution equations**. The hypothesis is that the rate of evolution of the internal variables is also determined from the local state.

- Example: standard solid model

$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$

$$\sigma = E_0 \epsilon^{(e)}$$

$$\sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$



  internal variable

$$\xi_1 = \epsilon^{(i)}$$

  evolution equation (kinetic equation)

$$\dot{\epsilon}^{(i)} = g_1(\sigma,\xi_1) = \frac{1}{\eta_1}\sigma - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

$$\dot{\epsilon}^{(i)} = g_1(\epsilon,\xi_1) = \frac{E_0}{\eta_1}(\epsilon - \epsilon^{(i)}) - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

(an evolution equation in terms of strain is not always possible)

Materials with memory

Internal variables and evolution equation

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$$\boldsymbol{\sigma}(\boldsymbol{X}, t) = \boldsymbol{f}(\boldsymbol{\epsilon}, T, \boldsymbol{\epsilon}^t) \equiv \begin{cases} \boldsymbol{f}(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) \\ \dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) \quad \text{evolution equations} \end{cases}$$

  The presence of additional variables in the constitutive laws requires additional constitutive equations, namely **evolution equations**. The hypothesis is that the rate of evolution of the internal variables is also determined from the local state.

Two types …

- *Physical* internal variables: describe aspects of the local physical-chemical structure which may change spontaneously (e.g., extent of chemical reaction or phase change, density of structural defects).
- *Phenomenological* internal variables: mathematical constructs (e.g., inelastic strain) with a functional dependence with stress (strain) is assumed a priori (i.e., evolution equations).

Materials with memory

Internal variables and thermodynamics

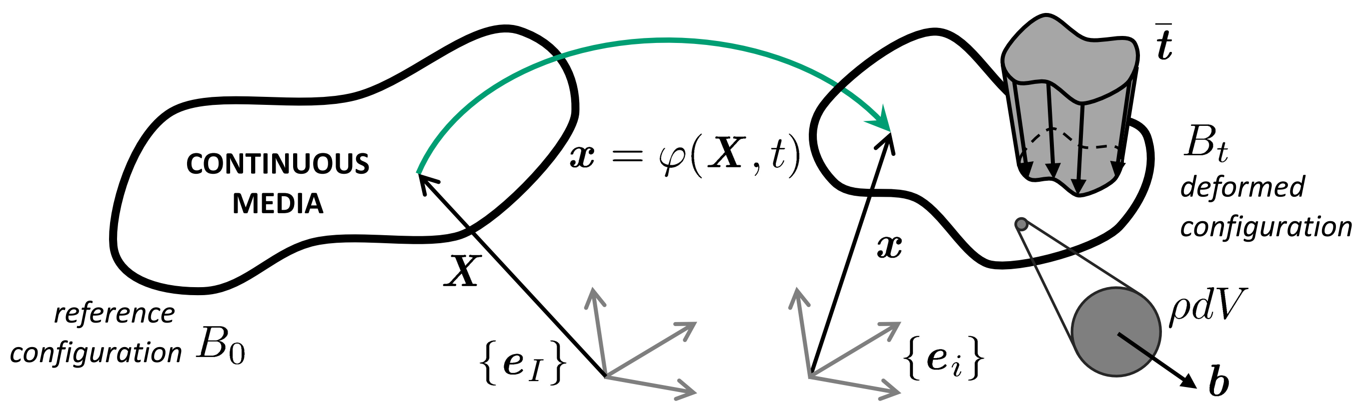
Using conservation of energy, the Clausius-Duhem inequality can be written as

$$\rho T \dot{s}^{\text{int}} = -\rho \left[\frac{1}{\rho_0} \frac{\partial W}{\partial T} + s \right] \dot{T} + \left[\boldsymbol{\sigma} - \frac{\rho}{\rho_0} \frac{\partial W}{\partial \boldsymbol{\epsilon}} \right] : \dot{\boldsymbol{\epsilon}} - \frac{1}{T} \boldsymbol{q} \cdot \nabla T \geq 0$$

Coleman and Noll made the argument that this inequality must be satisfied for <u>every</u> admissible process.

Let's now consider path-dependent behavior characterized by a set of internal variables and corresponding evolution equations, that is

$$W = \overline{W}(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) \qquad \dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma}, T, \boldsymbol{\xi})$$



Materials with memory

Internal variables and thermodynamics

- Isentropic/Adiabatic processes $\boldsymbol{q}=\mathbf{0}, W(\boldsymbol{\epsilon},T,\boldsymbol{\xi}),\ \dot{s}\neq 0\ \ (\text{with } \dot{T}=0)$

Assuming $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^{(e)}(\boldsymbol{\sigma},T)+\boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi})$, then there exists $W(\boldsymbol{\epsilon},T,\boldsymbol{\xi})$
if and only if

$$W(\boldsymbol{\epsilon},T,\boldsymbol{\xi})=W^{(e)}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}),T)+W^{(i)}(\boldsymbol{\xi},T)$$

DIY

$$\dot{W}=\frac{\partial W}{\partial \boldsymbol{\epsilon}}:\dot{\boldsymbol{\epsilon}}+\frac{\partial W}{\partial T}\dot{T}+\sum_\alpha \frac{\partial W}{\partial \xi_\alpha}\dot{\xi}_\alpha$$

(cont.)

Materials with memory

Internal variables and thermodynamics

- Isentropic/Adiabatic processes $\boldsymbol{q}=\boldsymbol{0}, W(\boldsymbol{\epsilon},T,\boldsymbol{\xi}),\ \dot{s}\neq 0\ \ (\text{with } \dot{T}=0)$

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$$W(\boldsymbol{\epsilon},T,\boldsymbol{\xi})=W^{(e)}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}),T)+W^{(i)}(\boldsymbol{\xi},T)$$

DIY

(cont.)

$$\boldsymbol{\sigma}=\frac{\rho}{\rho_0}\frac{\partial W^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}} \qquad \boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}}^{(i)}-\frac{\rho}{\rho_0}\sum_\alpha \frac{\partial W^{(i)}}{\partial \xi_\alpha}\dot{\xi}_\alpha\geq 0$$

Viscoelasticity

Internal variables and thermodynamics

- Example: standard solid model

DIY

$$W(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}), T) + W^{(i)}(\boldsymbol{\xi}, T) = \frac{1}{2}E_0(\epsilon - \epsilon^{(i)})^2 + \frac{1}{2}E_1(\epsilon^{(i)})^2$$

Internal variable $\epsilon^{(i)}(\xi) = \xi$

$\rho = \rho_0 = 1$

Evolution equation $\dot{\xi} = g(\sigma, \xi) = \frac{\sigma - E_1\xi}{\eta_1}$

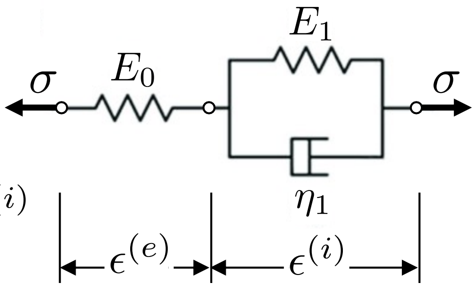
Clausius-Duhem inequality

$$\eta_1(\dot{\epsilon}^{(i)})^2 \geq 0$$

$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$

$$\sigma = E_0\epsilon^{(e)}$$

$$\sigma = E_1\epsilon^{(i)} + \eta_1\dot{\epsilon}^{(i)}$$



Viscoelasticity

Internal variables and thermodynamics

- Example: multiaxial behavior and time integration

DIY

$$W(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}), T) + W^{(i)}(\boldsymbol{\xi}, T)$$

Internal variable $\boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}) = \boldsymbol{\xi}$

$$W^{(i)}(\boldsymbol{\xi}, T) = \frac{1}{2} E_1 \boldsymbol{\epsilon}^{(i)} : \boldsymbol{\epsilon}^{(i)}$$

Evolution equation $\dot{\boldsymbol{\epsilon}}^{(i)} = \frac{\boldsymbol{\sigma} - E_1 \boldsymbol{\epsilon}^{(i)}}{\eta_1}$

Time integration $t_n \to t_{n+1} = t_n + \Delta t$

- Given $\left\{ \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_n, \boldsymbol{\epsilon}_n^{(i)} \right\}$
- Update internal variable $\boldsymbol{\epsilon}_{n+1}^{(i)} = \frac{1}{\eta_1/\Delta t} \boldsymbol{\sigma}_n + \boldsymbol{\epsilon}_n^{(i)} \left[1 - \frac{E_1}{\eta_1/\Delta t} \right]$
- Update stress .... $\boldsymbol{\sigma}_{n+1} = \frac{\rho_{n+1}}{\rho_0} \frac{\partial W_{n+1}^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}}$

Any questions?