Lecture 13
Isotropic plastic solids

PURDUE UNIVERSITY | Mechanical Engineering
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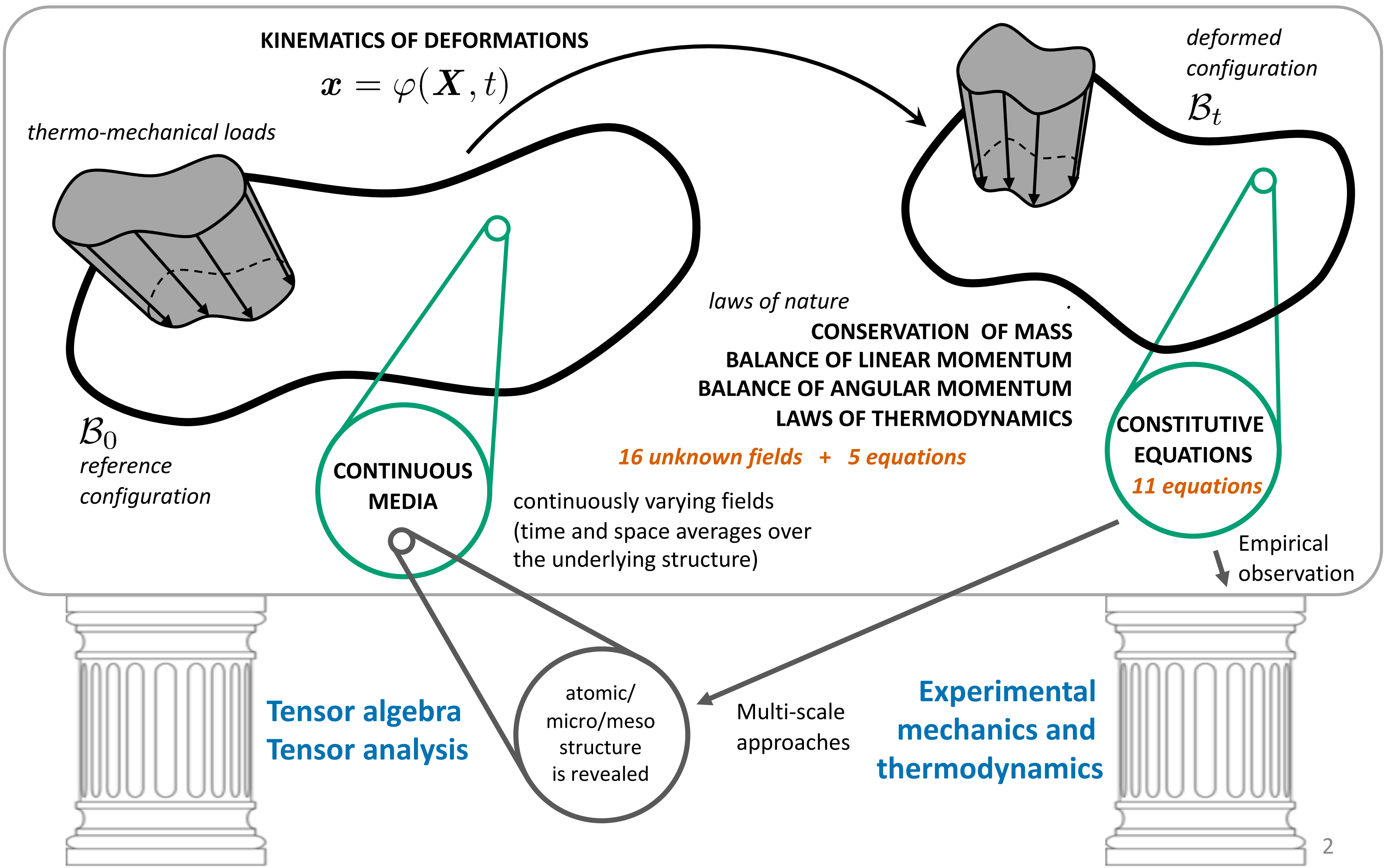
Lecture 13 – Isotropic plastic solids



Materials with memory

Recall: Internal variable, evolution equation, small strains

- The memory or path-dependency of a material can be represented through an array of **internal variables** (scalars and second-order tensors)

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}, T, \boldsymbol{\epsilon}^t) \equiv \begin{cases} \boldsymbol{\sigma}(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) \\ \dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) \quad \text{evolution equations} \end{cases}$$

  The presence of additional variables in the constitutive laws requires additional constitutive equations, namely **evolution equations**. The hypothesis is that the rate of evolution of the internal variables is also determined from the local state.

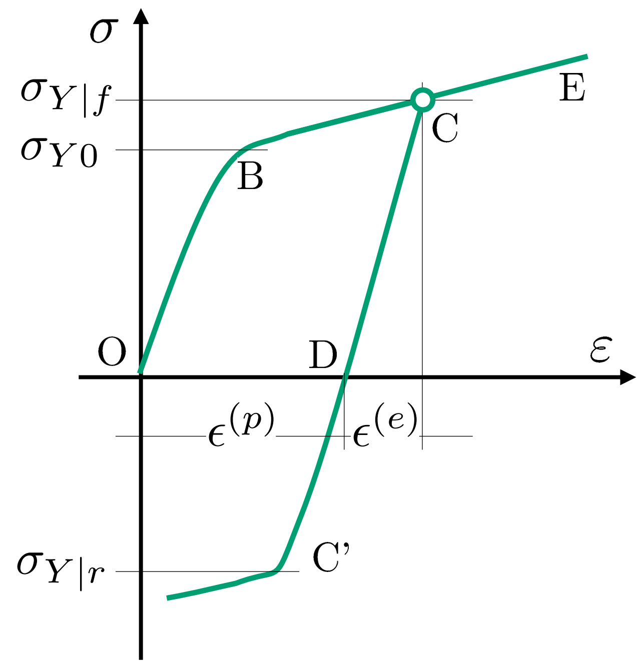
Examples:

- ***viscoelasticity****, i.e., time- and rate-dependent reversible behavior (the stress–strain relation depends on the loading rate); internal variables, e.g., the inelastic strain.*
- ***plasticity****, i.e., history-dependent irreversible behavior (the stress–strain relation depends on the loading history); internal variables, e.g., plastic strain, accumulated plastic strain.*
- ***viscoplasticity****, i.e., history- and time-dependent irreversible behavior.*
- ***damage****, i.e., irreversible degradation of the elastic stiffness with loading; internal variable is a damage parameter, e.g., a scalar measure.*

Isotropic plastic solids

Phenomenological observations

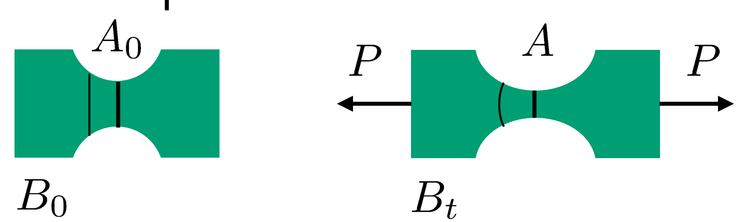
- Elasto-plastic behavior (elastic limit corresponds to the yield stress σ_Y)
- Strain-hardening behavior $\sigma_Y \geq \sigma_{Y0}$
- Beyond elastic limit:
  + elastic strain: $\epsilon^{(e)}$
  + plastic strain: $\epsilon^{(p)}$
- Metals: plastic flow is incompressible (any change in volume is elastic)
- Bauschinger effect $|\sigma_{Y|r}| \leq |\sigma_{Y|f}|$



Engineering stress
Engineering strain
$$\frac{P}{A_0} \leftrightarrow \lambda - 1$$

True stress
True strain
$$\frac{P}{A} = \frac{P}{A_0}\lambda \leftrightarrow \ln(\lambda)$$

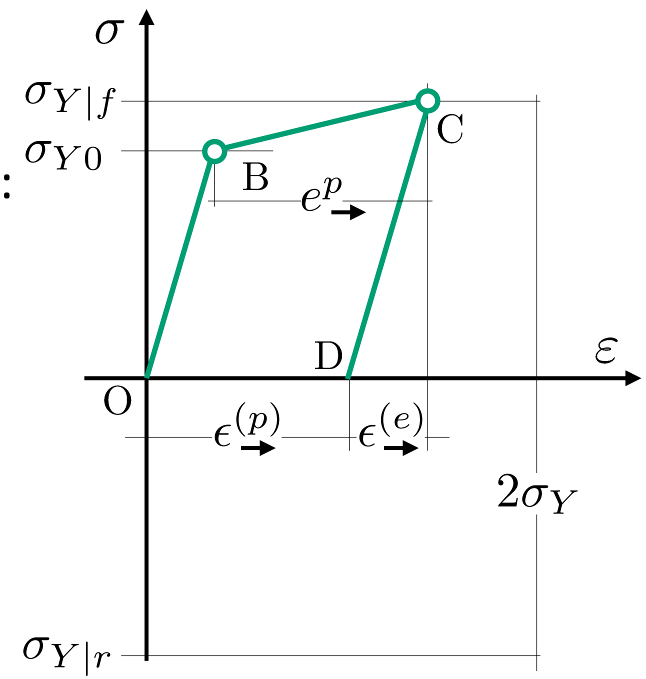
Note: $\frac{D}{Dt}\ln(\lambda) = d_{11}$



Isotropic plastic solids

Ideal plastic solids – Small strains

- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function
  $$\phi(\sigma, \sigma_Y) \leq 0$$
- Additive decomposition of (small) strain:
  + elastic part is related to the stress
  + plastic part depends on material history
  $$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$$
- Consistency condition ($\dot{\lambda} = \dot{e}^p = |\dot{\epsilon}^{(p)}|$)
  + within elastic domain
  $$\phi < 0 \quad \dot{\lambda} = 0 \implies \dot{\lambda}\phi = 0$$
  + on the yield surface
  … elastic unloading
  $$\phi = 0, \dot{\phi} < 0 \quad \dot{\lambda} = 0 \implies \dot{\lambda}\phi = 0$$
  … neutral loading
  $$\phi = 0, \dot{\phi} = 0 \quad \dot{\lambda} = 0 \implies \dot{\lambda}\phi = 0$$
  … plastic loading
  $$\dot{\phi} = 0 \quad \dot{\lambda} > 0 \implies \dot{\lambda}\phi = 0$$



$$\dot{\lambda} \geq 0 \ , \ \phi \leq 0 \ , \ \dot{\lambda}\phi = 0$$

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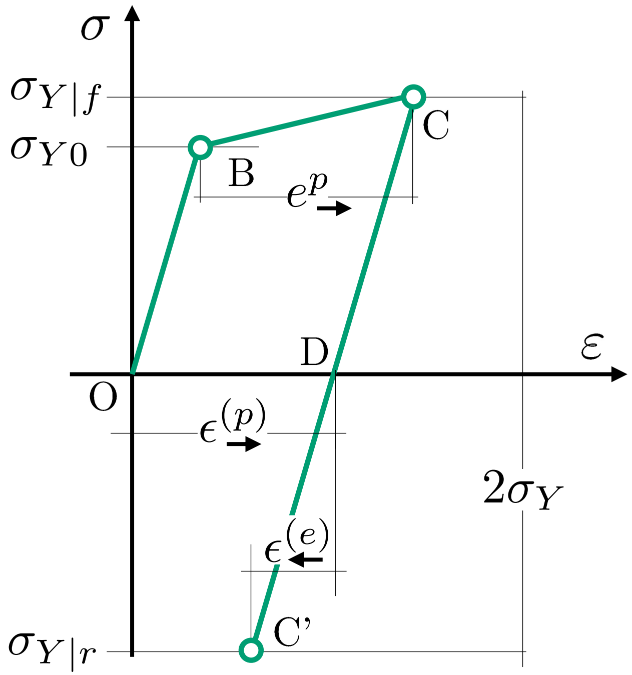

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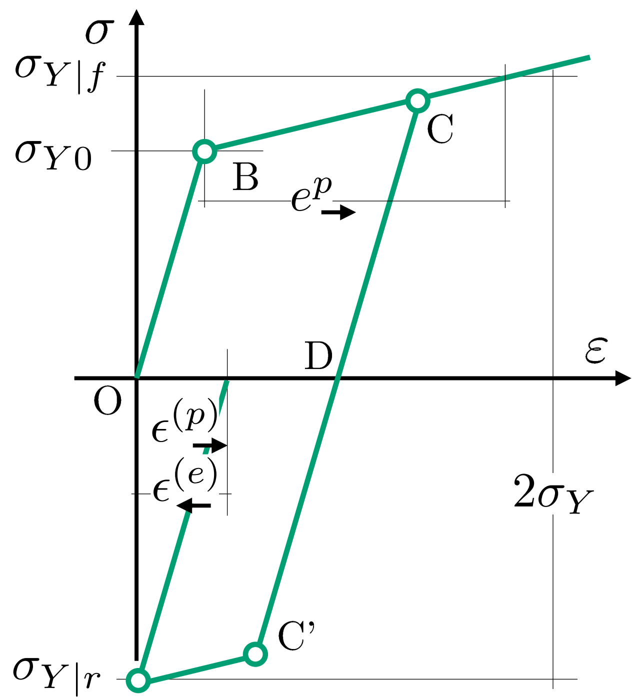

$$\dot{\lambda} \ge 0 \;,\; \phi \le 0 \;,\; \dot{\lambda}\phi = 0$$

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Isotropic plastic solids

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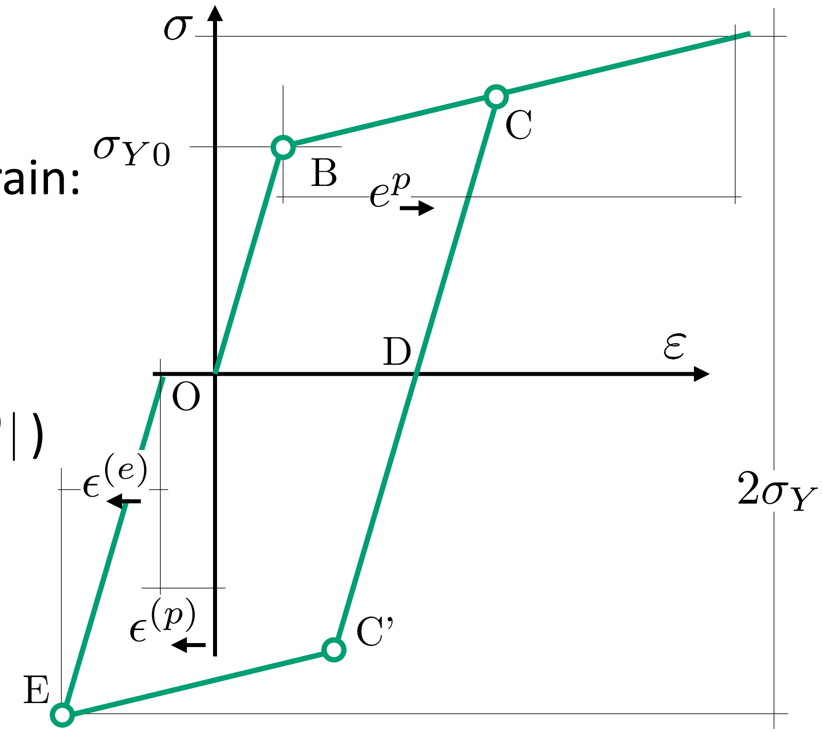

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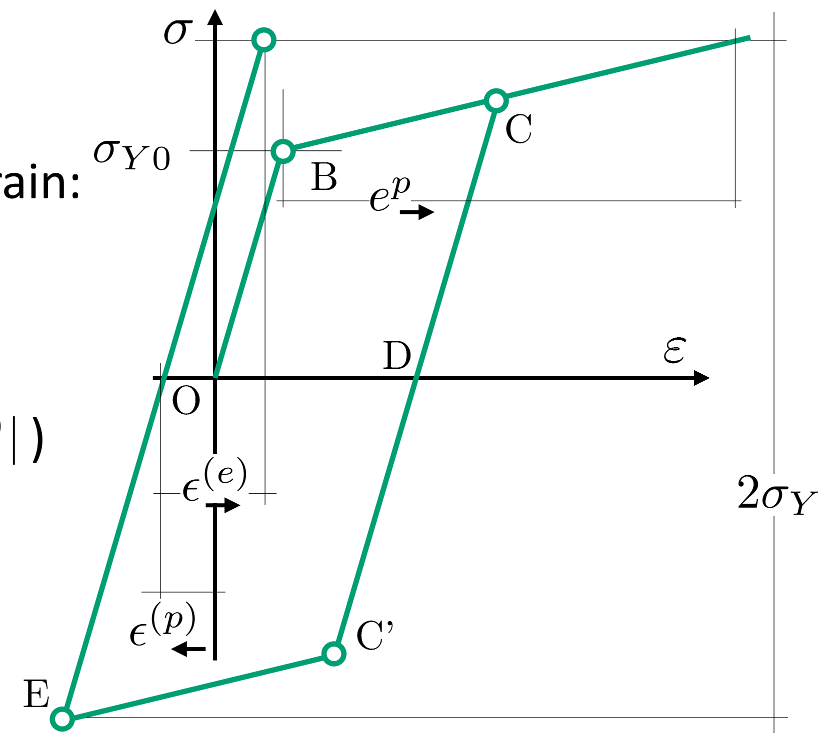

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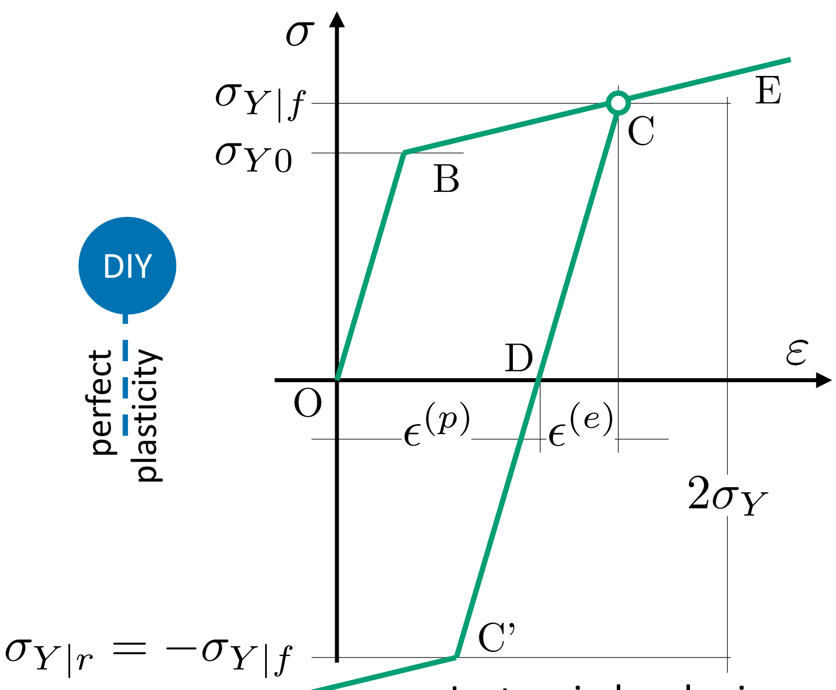
Isotropic plastic solids

Ideal plastic solids – Small strains

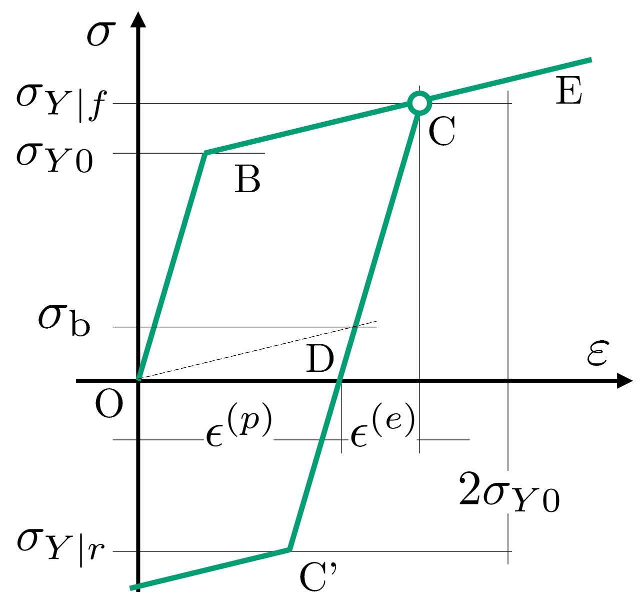
- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function and consistency condition: $\phi(\sigma, \sigma_Y) \leq 0$
- Additive decomposition of (small) strain: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$
  + elastic part is related to the stress + plastic part depends on material history

DIY

perfect plasticity



Isotropic-hardening



Kinematic-hardening

Isotropic plastic solids

Ideal plastic solids – Small strains

- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function $\phi(\boldsymbol{\sigma}, \xi) \leq 0$
  Internal variable (*for isotropic hardening*) $\xi = \int_0^t ||\dot{\boldsymbol{\epsilon}}^{(p)}||\mathrm{d}t$
- Additive decomposition of (small) strain: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$
- Consistency condition
  (*used to determine* $\dot{\lambda}$ *under plastic loading; otherwise* $\dot{\lambda} = 0$)

$$\dot{\lambda} \geq 0 \ , \ \ \phi \leq 0 \ , \ \ \dot{\lambda}\phi = 0$$

$$\text{if } \phi = 0 \text{ then } \ \ \dot{\lambda}\dot{\phi} = 0$$

- Strain energy density: $W(\boldsymbol{\epsilon}, \xi) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(p)}(\xi)) + W^{(p)}(\xi)$
  (Lecture 11)
- Evolution equations (Lecture 11):
  - Flow rule (flow potential) $\dot{\boldsymbol{\epsilon}}^{(p)} = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma}, \xi) = \dot{\lambda}\frac{\partial G}{\partial \boldsymbol{\sigma}}$
  - Hardening rule $\dot{\xi} = \dot{\lambda}h(\boldsymbol{\sigma}, \xi)$
- Cauchy stress tensor: $\boldsymbol{\sigma} = \frac{\rho}{\rho_0}\frac{\partial W^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}} := \boldsymbol{c}^e : \boldsymbol{\epsilon}^{(e)}$

Isotropic plastic solids

Ideal plastic solids – Small strains

- Consistency condition
  (*used to determine* $\dot{\lambda}$ *under plastic loading*)

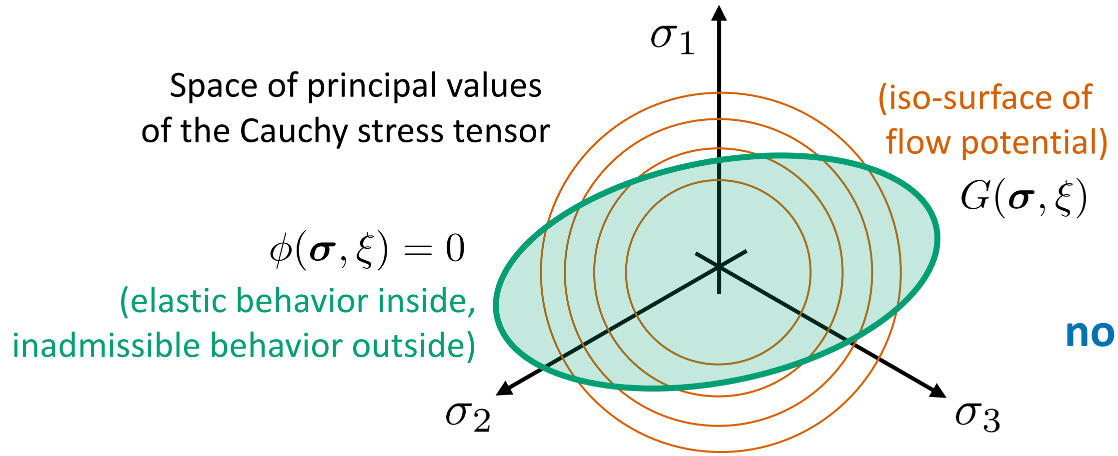
$$\dot{\lambda} \geq 0 \;,\; \phi \leq 0 \;,\; \dot{\lambda}\phi = 0$$

$$\text{if } \phi = 0 \text{ then } \;\dot{\lambda}\dot{\phi} = 0$$

DIY

$\dot{\phi} = 0 \rightarrow \text{solve for } \dot{\lambda}$

$$\dot{\phi} = 0 = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} : \boldsymbol{c}^e : \dot{\boldsymbol{\epsilon}} - \frac{\partial \phi}{\partial \boldsymbol{\sigma}} : \boldsymbol{c}^e : \dot{\lambda}\frac{\partial G}{\partial \boldsymbol{\sigma}} + \frac{\partial \phi}{\partial \xi}\dot{\lambda}h$$



$$G(\boldsymbol{\sigma}, \xi) \neq \phi(\boldsymbol{\sigma}, \xi)$$

non-associated flow rule

Isotropic plastic solids

Ideal plastic solids – Small strains

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  (*used to determine* $\dot{\lambda}$ *under plastic loading*)

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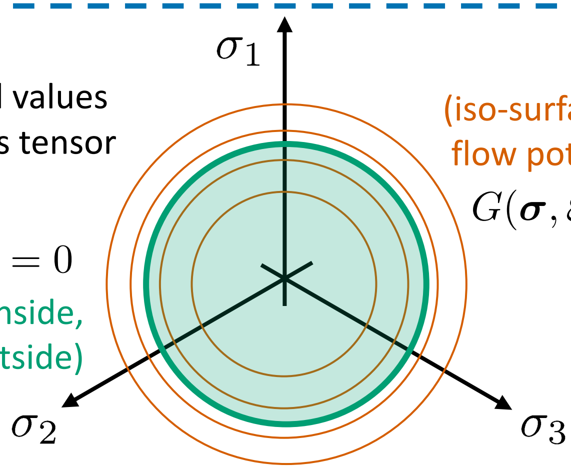
Space of principal values of the Cauchy stress tensor



$\phi(\boldsymbol{\sigma}, \xi) = 0$

(elastic behavior inside, inadmissible behavior outside)

(iso-surface of flow potential)

$G(\boldsymbol{\sigma}, \xi)$

$$G(\boldsymbol{\sigma}, \xi) = \phi(\boldsymbol{\sigma}, \xi)$$

associated flow rule

Isotropic plastic solids

Summary: Ideal plastic solids – Small strains

- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function $\phi(\boldsymbol{\sigma}, \xi) \leq 0$
  Internal variable (*for isotropic hardening*) $\xi = \int_0^t ||\dot{\boldsymbol{\epsilon}}^{(p)}|| \mathrm{d}t$

  Need …

- Additive decomposition of (small) strain: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$

- Consistency condition $\dot{\phi} = 0 \rightarrow \text{solve for } \dot{\lambda}$
  (*determine* $\dot{\lambda}$ *under plastic loading;* $\dot{\lambda} = 0$, *otherwise*)

- Strain energy density: $W(\boldsymbol{\epsilon}, \xi) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(p)}(\xi)) + W^{(p)}(\xi)$
  (Lecture 11)

- Evolution equations (Lecture 11):
  - Flow rule (flow potential) $\dot{\boldsymbol{\epsilon}}^{(p)} = \dot{\lambda} \boldsymbol{r}(\boldsymbol{\sigma}, \xi) = \dot{\lambda} \frac{\partial G}{\partial \boldsymbol{\sigma}}$
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  Need …

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Any questions?