Spring, 2022

ME 597 – Solid Mechanics II

Lecture 14

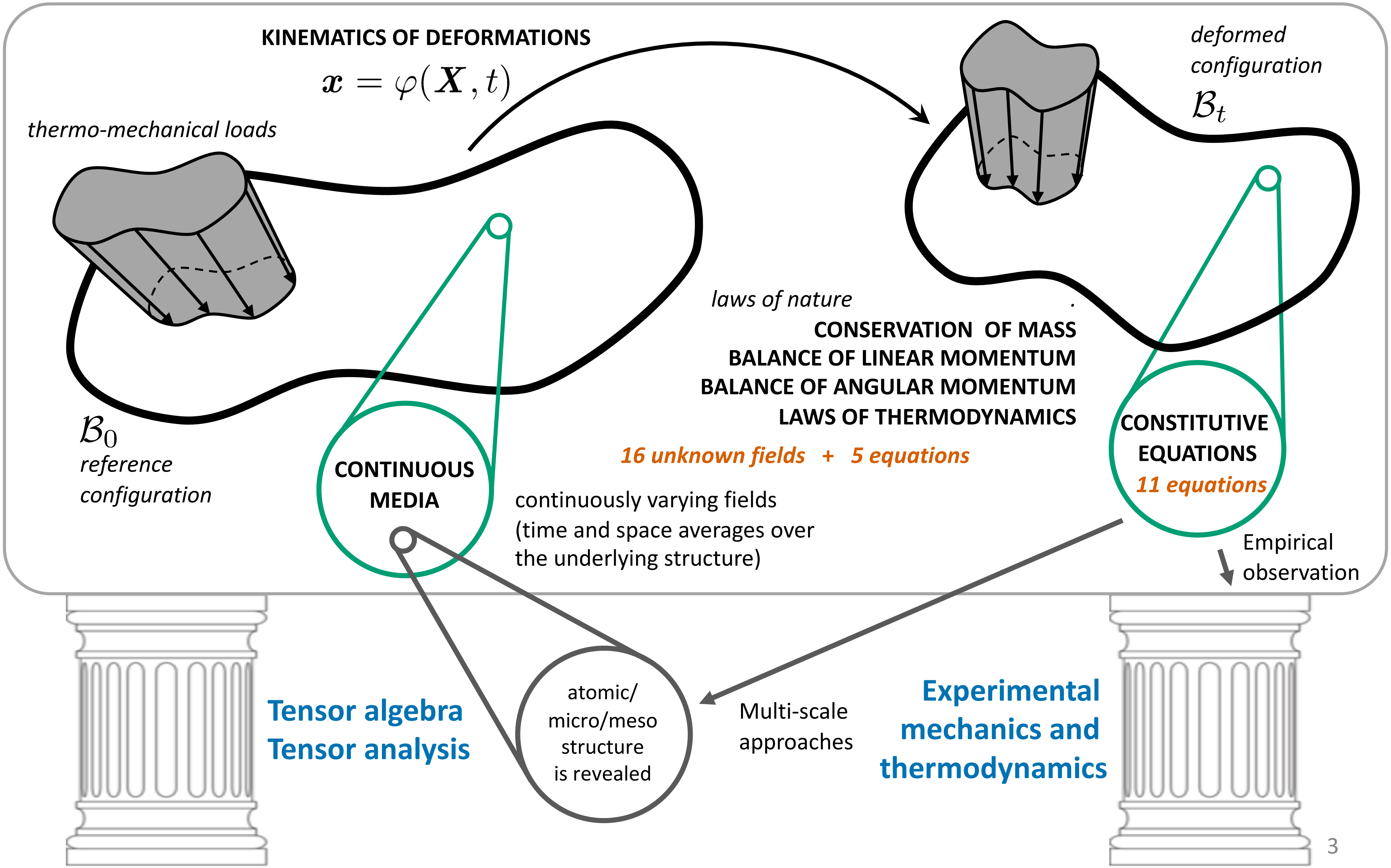
Isotropic plastic solids – J_2 or von Mises

PURDUE UNIVERSITY | Mechanical Engineering

Instructor: Prof. Marcial Gonzalez

Last modified: 3/3/22 8:36:28 AM

Lecture 14 – Isotropic plastic solids



KINEMATICS OF DEFORMATIONS

$$\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$$

thermo-mechanical loads

deformed configuration $\mathcal{B}_t$

$\mathcal{B}_0$ *reference configuration*

laws of nature

CONSERVATION OF MASS
BALANCE OF LINEAR MOMENTUM
BALANCE OF ANGULAR MOMENTUM
LAWS OF THERMODYNAMICS

16 unknown fields + 5 equations

CONTINUOUS MEDIA

continuously varying fields (time and space averages over the underlying structure)

CONSTITUTIVE EQUATIONS ***11 equations***

Empirical observation

atomic/ micro/meso structure is revealed

Multi-scale approaches

Tensor algebra
Tensor analysis

Experimental mechanics and thermodynamics

Isotropic plastic solids

Review: Phenomenological observations

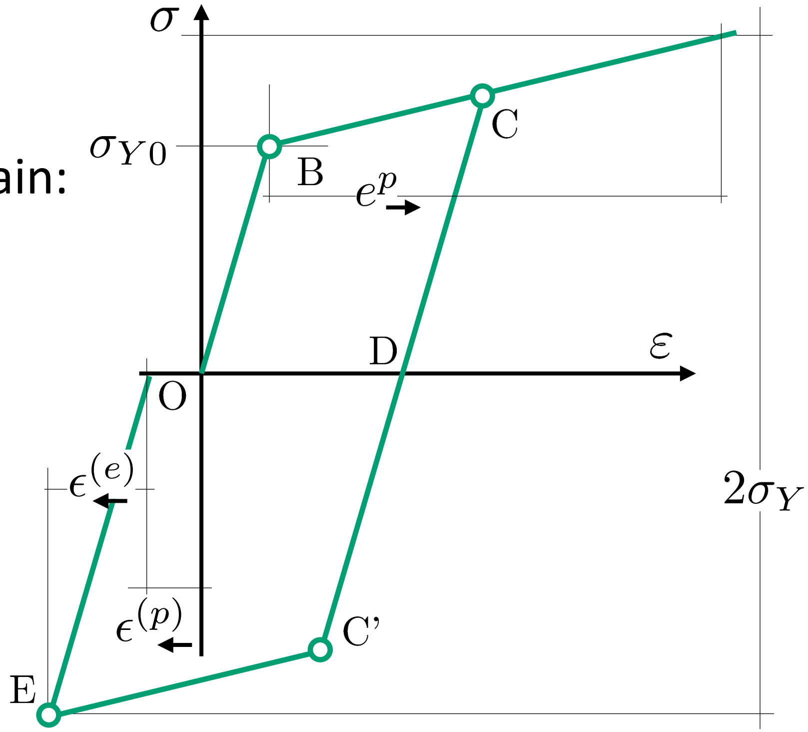
- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function

$$\phi(\sigma, \sigma_Y) \leq 0$$

- Additive decomposition of (small) strain:
  + elastic part is related to the stress
  + plastic part depends on material history

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$$

- Metals: plastic flow is incompressible
  (any change in volume is elastic)
- Bauschinger effect $|\sigma_{Y|r}| \leq |\sigma_{Y|f}|$
  Strain hardening:
  + isotropic hardening (e^p)
  + kinematic hardening
- Metals: Bridgman (c. 1940)
  Yield function is independent of hydrostatic pressure



Isotropic plastic solids

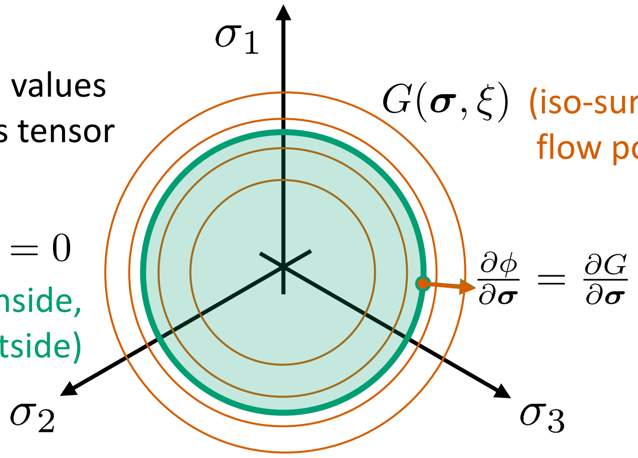
Review: Ideal plastic solids – Small strains

Constitutive law defined by yield function, flow potential and hardening rule

- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function $\phi(\boldsymbol{\sigma}, \xi) \leq 0$
  Internal variable (*for isotropic hardening*) $\xi = \int_0^t ||\dot{\boldsymbol{\epsilon}}^{(p)}|| \mathrm{d}t$
- Additive decomposition of (small) strain: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$
- Consistency condition $\dot{\phi} = 0 \rightarrow \text{solve for } \dot{\lambda}$
  (*determine* $\dot{\lambda}$ *under plastic loading;* $\dot{\lambda} = 0$, *otherwise*)
- Strain energy density: $W(\boldsymbol{\epsilon}, \xi) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(p)}(\xi)) + W^{(p)}(\xi)$
  (Lecture 11)
- Evolution equations (Lecture 11):
  - Flow rule (flow potential) $\dot{\boldsymbol{\epsilon}}^{(p)} = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma}, \xi) = \dot{\lambda}\frac{\partial G}{\partial \boldsymbol{\sigma}}$
  - Hardening rule $\dot{\xi} = \dot{\lambda}h(\boldsymbol{\sigma}, \xi)$
- Cauchy stress tensor: $\boldsymbol{\sigma} = \frac{\rho}{\rho_0}\frac{\partial W^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}} := \boldsymbol{c}^e : \boldsymbol{\epsilon}^{(e)}$

Isotropic plastic solids

Yield function and flow potential



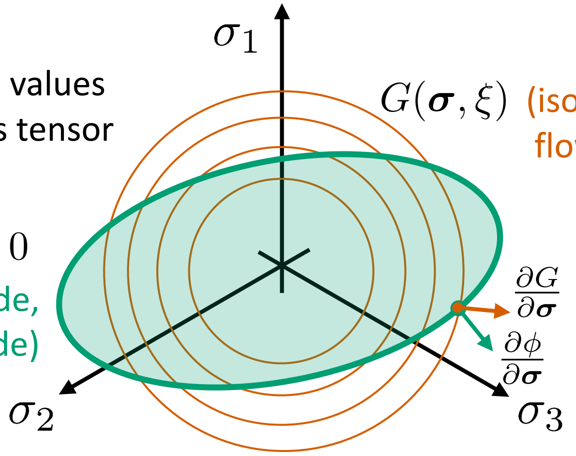
Space of principal values
of the Cauchy stress tensor

$G(\boldsymbol{\sigma}, \xi)$ (iso-surface of
flow potential)

$\phi(\boldsymbol{\sigma}, \xi) = 0$
(elastic behavior inside,
inadmissible behavior outside)

$$G(\boldsymbol{\sigma}, \xi) = \phi(\boldsymbol{\sigma}, \xi)$$

associated flow rule



Space of principal values
of the Cauchy stress tensor

$G(\boldsymbol{\sigma}, \xi)$ (iso-surface of
flow potential)

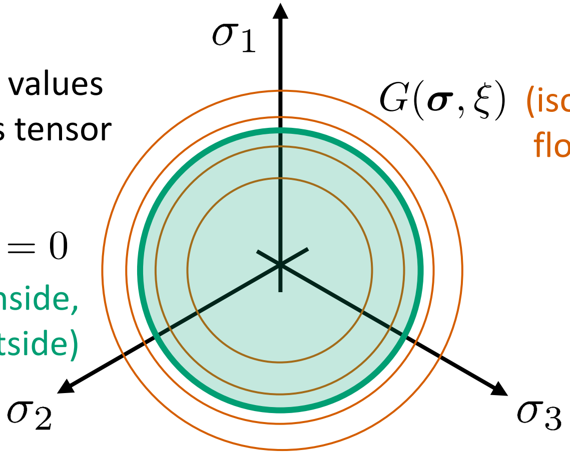
$\phi(\boldsymbol{\sigma}, \xi) = 0$
(elastic behavior inside,
inadmissible behavior outside)

$$G(\boldsymbol{\sigma}, \xi) \neq \phi(\boldsymbol{\sigma}, \xi)$$

non-associated flow rule

Isotropic plastic solids

Yield function and flow potential (*from your undergrad*)



Space of principal values
of the Cauchy stress tensor

$G(\boldsymbol{\sigma},\xi)$ (iso-surface of
flow potential)

$\phi(\boldsymbol{\sigma},\xi)=0$
(elastic behavior inside,
inadmissible behavior outside)

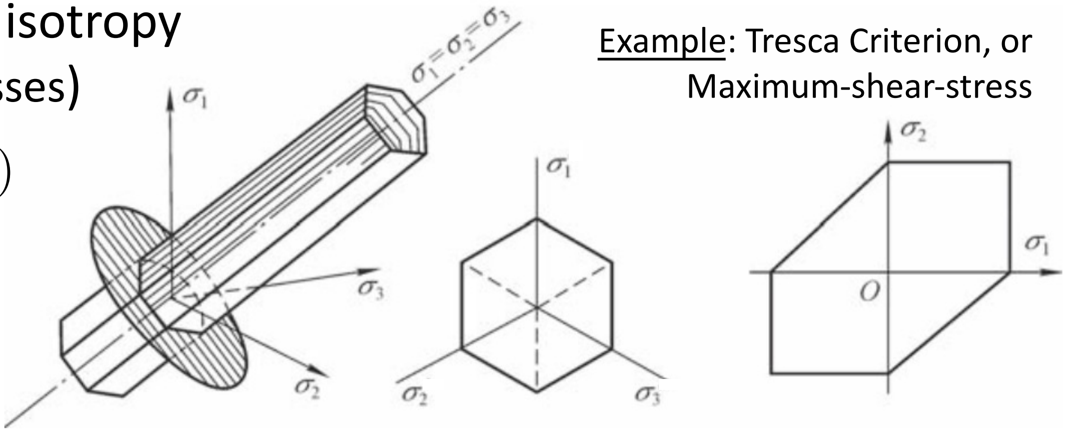
$$G(\boldsymbol{\sigma},\xi)=\phi(\boldsymbol{\sigma},\xi)$$

associated flow rule

- Frame indifference and isotropy (function of principal stresses)

$$\phi(\boldsymbol{\sigma},\xi)=\phi(\sigma_1,\sigma_2,\sigma_3,\xi)$$

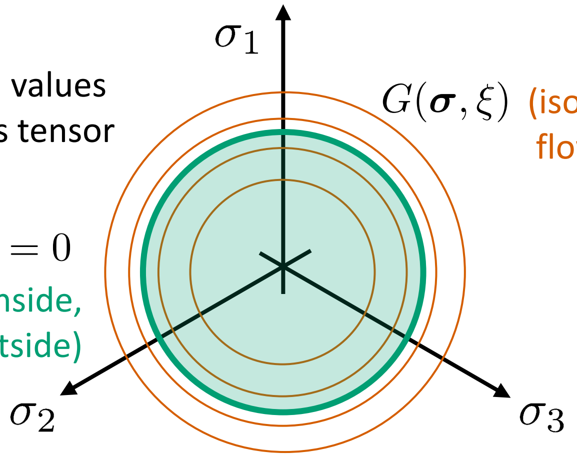
Yield function is independent
of hydrostatic pressure
$I_1=\mathrm{tr}(\boldsymbol{\sigma})=-3p$



Example: Tresca Criterion, or
Maximum-shear-stress

Isotropic plastic solids

Yield function and flow potential (*from your undergrad*)



Space of principal values
of the Cauchy stress tensor

$G(\boldsymbol{\sigma}, \xi)$ (iso-surface of
flow potential)

$\phi(\boldsymbol{\sigma}, \xi) = 0$

(elastic behavior inside,
inadmissible behavior outside)

$$G(\boldsymbol{\sigma}, \xi) = \phi(\boldsymbol{\sigma}, \xi)$$

associated flow rule

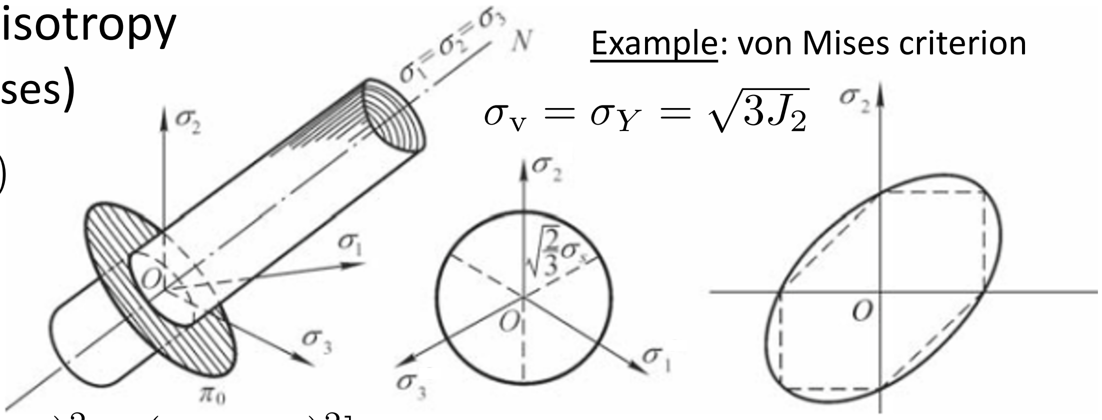
- Frame indifference and isotropy
  (function of principal stresses)

$$\phi(\boldsymbol{\sigma}, \xi) = \phi(\sigma_1, \sigma_2, \sigma_3, \xi)$$

Yield function is independent
of hydrostatic pressure

$$I_1 = \mathrm{tr}(\boldsymbol{\sigma}) = -3p$$

$$J_2 = \tfrac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]$$



<u>Example</u>: von Mises criterion

$$\sigma_{\mathrm{v}} = \sigma_Y = \sqrt{3J_2}$$

Isotropic plastic solids

J_2 or von Mises plasticity

- Plastic flow is incompressible.

  Yield function is isotropic and independent of hydrostatic pressure.

$$\boldsymbol{e} = \boldsymbol{\epsilon} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I}$$

$$\boldsymbol{s} = \boldsymbol{\sigma} + p\boldsymbol{I} = \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}$$

deviatoric parts of strain and stress

$$J_2 = \tfrac{1}{2}||\boldsymbol{s}||^2$$

second invariant of the deviatoric stress tensor

- Internal variable $e^p = \int_0^t \sqrt{2/3}||\dot{\boldsymbol{\epsilon}}^{(p)}||\mathrm{d}t$
- Yield function $\phi(\boldsymbol{\sigma}, e^p) = \sqrt{2J_2} - \sqrt{2/3}\sigma_Y(e^p) = ||\boldsymbol{s}|| - \sqrt{2/3}\sigma_Y(e^p)$

  power-law fit $\sigma_Y(e^p) = \sigma_{Y0} + C(e^p)^n$
- Consistency condition (under plastic loading) $\dot{\phi} = 0 \rightarrow \text{solve for } \dot{\lambda}$
- Evolution equations:
  - Flow rule (**associated flow rule**)
  - Hardening rule

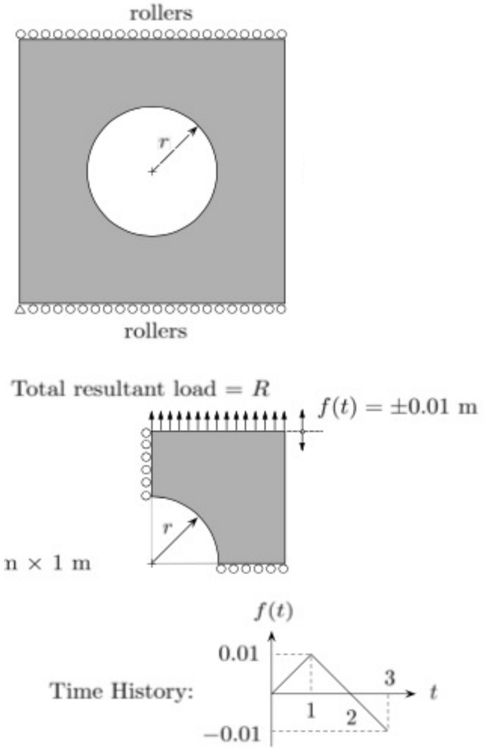
$$\dot{\boldsymbol{\epsilon}}^{(p)} = \dot{\lambda}\frac{\partial G(\boldsymbol{\sigma}, e^p)}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{\partial \phi(\boldsymbol{\sigma}, e^p)}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{3\boldsymbol{s}}{2||\boldsymbol{s}||}$$

$$\dot{e}^p = \dot{\lambda}h(\boldsymbol{\sigma}, e^p) = \sqrt{3/2}\dot{\lambda}$$

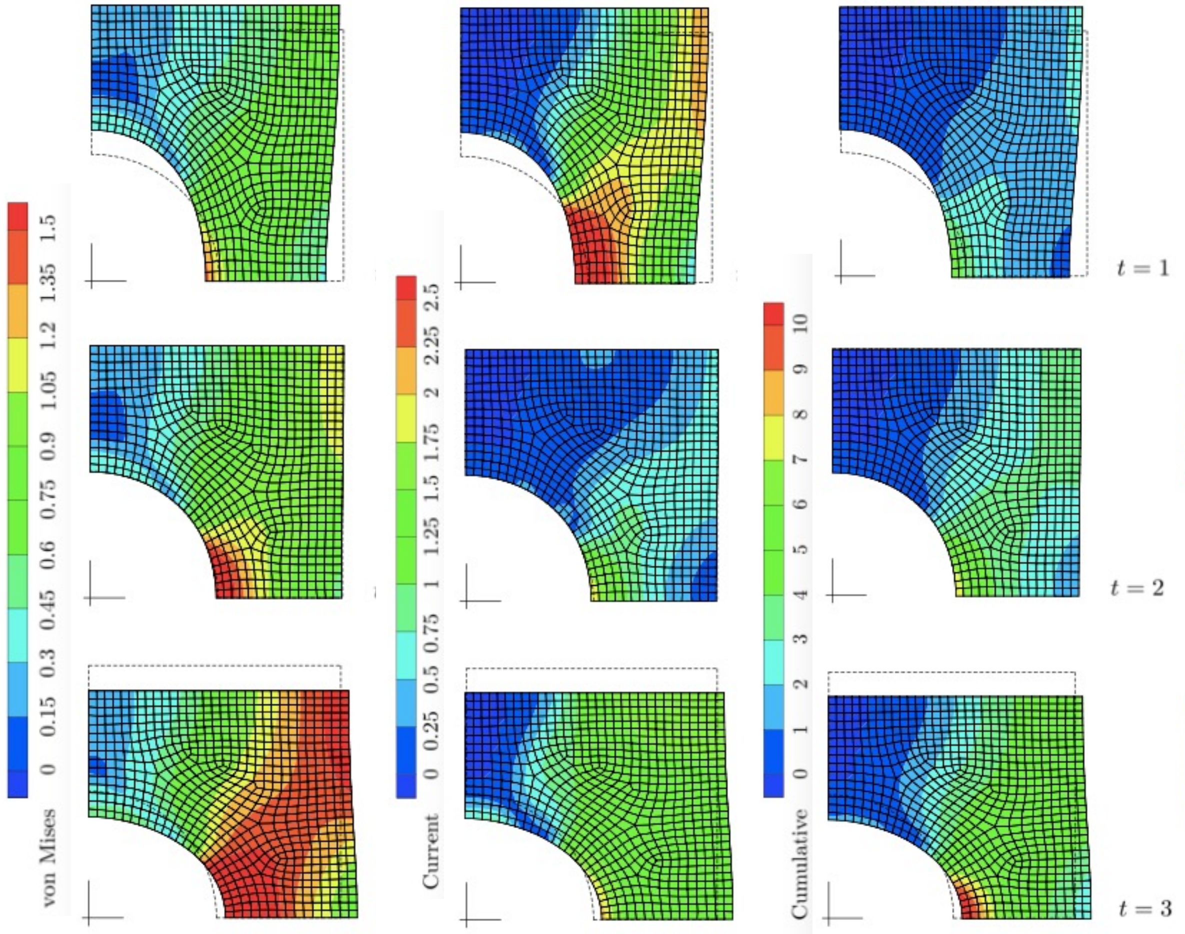
- Cauchy stress tensor: $\boldsymbol{\sigma} = \frac{\rho}{\rho_0}\frac{\partial W^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}} := \boldsymbol{c}^e : \boldsymbol{\epsilon}^{(e)}$

Isotropic plastic solids

Numerical example



displacements magnified 5×



$\sqrt{3/2}||\boldsymbol{s}||$ $\sqrt{2/3}||\boldsymbol{\epsilon}^{(p)}||$ $e^p = \int_0^t \sqrt{2/3}||\dot{\boldsymbol{\epsilon}}^{(p)}||\mathrm{d}t$

Isotropic plastic solids

Time integration must find admissible solutions!

- Time discretization: $t_n \rightarrow t_{n+1} = t_n + \Delta t$
  Straightforward before [Lecture 10], not so easy now …

- An acceptable algorithm should satisfy three basic requirements:
  (a) consistency with the constitutive relations
  (first-order accuracy; *second-order accuracy*);
  (b) numerical stability;
  (c) incremental plastic consistency.

convergence with respect to time step size

plastic consistency

- Generalized midpoint rule:

$$y_{n+1} = y_n + \Delta t f(t_{n+\alpha}, (1-\alpha)y_n + \alpha y_{n+1})$$

Aside: $\dot{y} = f(t, y)$

$$\frac{y_{n+1} - y_n}{\Delta t} = f(t_{n+\alpha}, (1-\alpha)y_n + \alpha y_{n+1})$$

Isotropic plastic solids

Time integration must find admissible solutions!

- Time discretization: $t_n \rightarrow t_{n+1} = t_n + \Delta t$

- Generalized midpoint rule: $\left\{\boldsymbol{\sigma}_n, \boldsymbol{\epsilon}^{(p)}_n, e^p_n; \boldsymbol{\epsilon}_{n+1}\right\} \rightarrow \left\{\boldsymbol{\sigma}_{n+1}, \boldsymbol{\epsilon}^{(p)}_{n+1}, e^p_{n+1}\right\}$

$$e^p_{n+1} = e^p_n + \Delta\lambda h_{n+\alpha}$$

with $h_{n+\alpha} = h((1-\alpha)\boldsymbol{\sigma}_n + \alpha\boldsymbol{\sigma}_{n+1}, (1-\alpha)e^p_n + \alpha e^p_{n+1})$

$$\boldsymbol{\epsilon}^{(p)}_{n+1} = \boldsymbol{\epsilon}^{(p)}_n + \Delta\lambda \boldsymbol{r}_{n+\alpha}$$

with $\boldsymbol{r}_{n+\alpha} = \boldsymbol{r}((1-\alpha)\boldsymbol{\sigma}_n + \alpha\boldsymbol{\sigma}_{n+1}, (1-\alpha)e^p_n + \alpha e^p_{n+1})$

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{c}^e : (\boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}^{(p)}_{n+1})$$

$$\phi_{n+1} = \phi(\boldsymbol{\sigma}_{n+1}, e^p_{n+1}) = 0$$

… but, is this ‘plastic loading’ ($\dot{\lambda} > 0$) or not?

$$\boldsymbol{\epsilon}^{(e)}_{n+1} = \underbrace{\left[\boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}^{(p)}_n\right]}_{\text{elastic strain which would result from a purely elastic step}} - \Delta\lambda \boldsymbol{r}_{n+\alpha}$$

Isotropic plastic solids

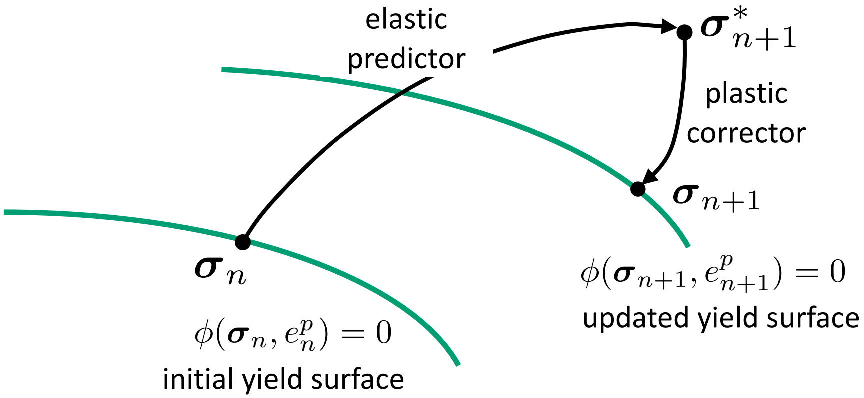
Return mapping algorithms: elastic predictor/plastic corrector

- Time discretization: $t_n \to t_{n+1} = t_n + \Delta t$

- **Elastic predictor**: $\left\{\boldsymbol{\sigma}_n, \boldsymbol{\epsilon}_n^{(p)}, e_n^p; \boldsymbol{\epsilon}_{n+1}\right\} \to \left\{\boldsymbol{\sigma}^*_{n+1}, \boldsymbol{\epsilon}_{n+1}^{(p)} = \boldsymbol{\epsilon}_n^{(p)}, e_{n+1}^p = e_n^p, \Delta\lambda = 0\right\}$

$$\boldsymbol{\epsilon}_{n+1}^{(e)*} = \boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}_n^{(p)}$$

$$\boldsymbol{\sigma}^*_{n+1} = \boldsymbol{c}^e : \boldsymbol{\epsilon}_{n+1}^{(e)*}$$



- Question: $\phi(\boldsymbol{\sigma}^*_{n+1}, e_n^p) \leq 0$?

… *Yes*: within the elastic domain.

Done

… *No*: solve a non-linear system of equations.

$$\boldsymbol{\epsilon}_{n+1}^{(e)} - \boldsymbol{\epsilon}_{n+1}^{(e)*} + \Delta\lambda \boldsymbol{r}_{n+\alpha} = \boldsymbol{0}$$
$$e_{n+1}^p - e_n^p - \Delta\lambda h_{n+\alpha} = 0$$
$$\phi(\boldsymbol{c}^e : \boldsymbol{\epsilon}_{n+1}^{(e)}, e_{n+1}^p) = 0$$

Newton-Raphson iterations to solve for

$\to \left\{\boldsymbol{\epsilon}_{n+1}^{(e)}, e_{n+1}^p, \Delta\lambda > 0\right\}$

Isotropic plastic solids

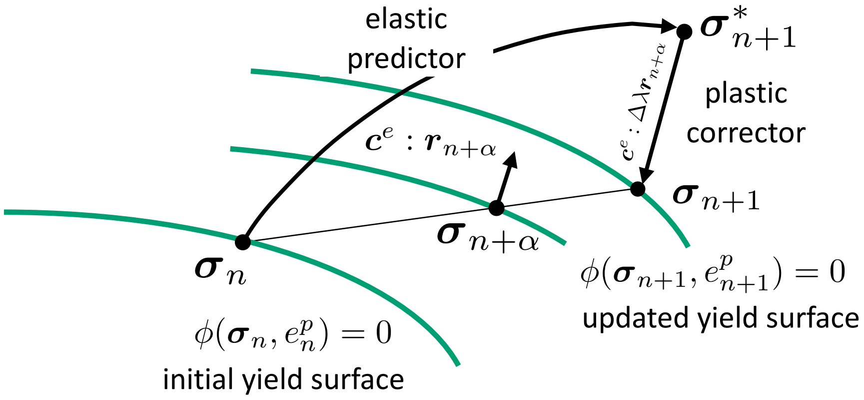
Return mapping algorithms: elastic predictor/plastic corrector

- Time discretization: $t_n \to t_{n+1} = t_n + \Delta t$

- **Elastic predictor**: $\left\{\boldsymbol{\sigma}_n, \boldsymbol{\epsilon}_n^{(p)}, e_n^p; \boldsymbol{\epsilon}_{n+1}\right\} \to \left\{\boldsymbol{\sigma}_{n+1}^*, \boldsymbol{\epsilon}_{n+1}^{(p)} = \boldsymbol{\epsilon}_n^{(p)}, e_{n+1}^p = e_n^p, \Delta\lambda = 0\right\}$

$$\boldsymbol{\epsilon}_{n+1}^{(e)*} = \boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}_n^{(p)}$$

$$\boldsymbol{\sigma}_{n+1}^* = \boldsymbol{c}^e : \boldsymbol{\epsilon}_{n+1}^{(e)*}$$



- Question: $\phi(\boldsymbol{\sigma}_{n+1}^*, e_n^p) \leq 0$?

  … *Yes*: within the elastic domain.

  Done

  … *No*: solve a non-linear system of equations.

  Plastic corrector

$$\boldsymbol{\epsilon}_{n+1}^{(p)} = \boldsymbol{\epsilon}_n^{(p)} + \Delta\lambda \boldsymbol{r}_{n+\alpha}$$

$$\to \left\{\boldsymbol{\sigma}_{n+1}, \boldsymbol{\epsilon}_{n+1}^{(p)}, e_{n+1}^p\right\}$$

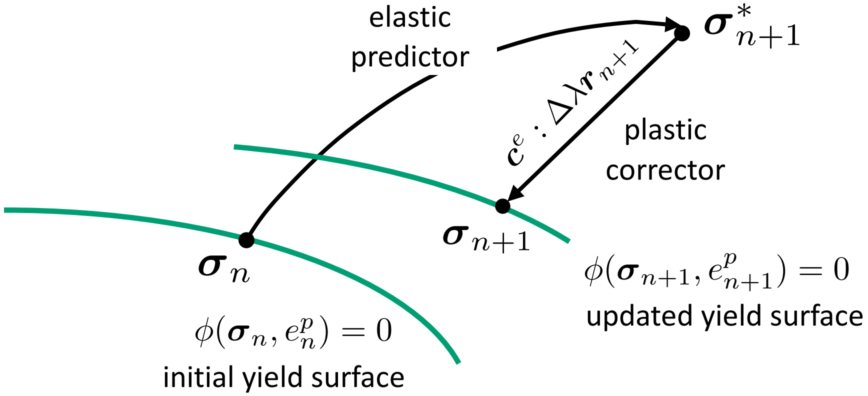
$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^* - \underbrace{\boldsymbol{c}^e : \Delta\lambda \boldsymbol{r}_{n+\alpha}}_{\text{plastic corrector}}$$

Isotropic plastic solids

Fully implicit return mapping and perfect plasticity

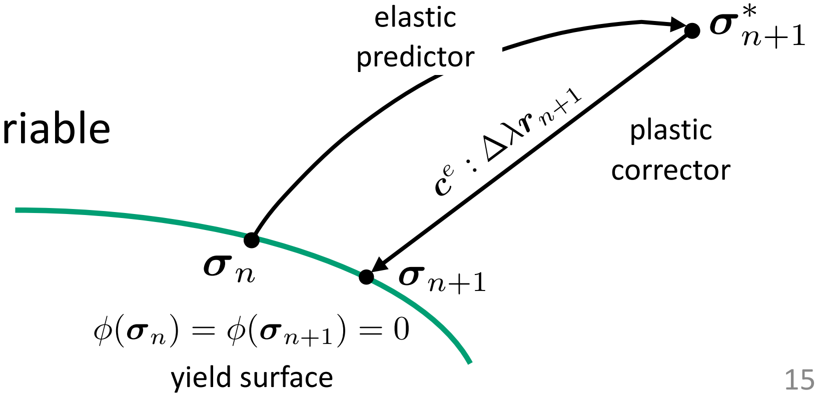
- Time discretization: $t_n \to t_{n+1} = t_n + \Delta t$

- Fully implicit algorithm ($\alpha = 1$)



- Perfect plasticity ($\alpha = 1$)

  + no hardening, thus no internal variable
  + no evolving yield surface



Isotropic plastic solids

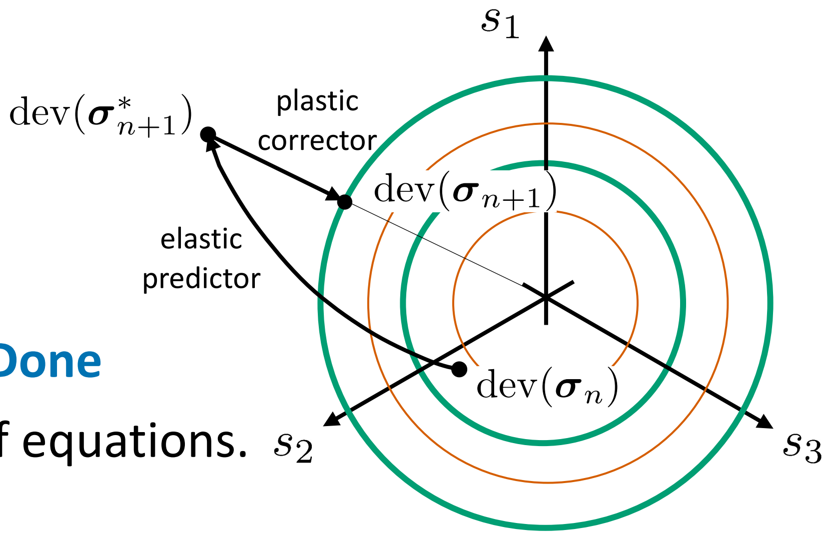
J_2 or von Mises plasticity – Radial return algorithm

- Time discretization: $t_n \rightarrow t_{n+1} = t_n + \Delta t$

- **Elastic predictor**: $\left\{ \boldsymbol{\sigma}_n, \boldsymbol{\epsilon}_n^{(p)}, e_n^p; \boldsymbol{\epsilon}_{n+1} \right\} \rightarrow \left\{ \boldsymbol{\sigma}_{n+1}^*, \boldsymbol{\epsilon}_{n+1}^{(p)} = \boldsymbol{\epsilon}_n^{(p)}, e_{n+1}^p = e_n^p, \Delta\lambda = 0 \right\}$

$$\boldsymbol{\epsilon}_{n+1}^{(e)*} = \boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}_n^{(p)}$$

$$\boldsymbol{\sigma}_{n+1}^* = \boldsymbol{c}^e : \boldsymbol{\epsilon}_{n+1}^{(e)*}$$



- Question: $\phi(\boldsymbol{\sigma}_{n+1}^*, e_n^p) \leq 0$?

… *Yes*: within the elastic domain. **Done**

… *No*: solve a non-linear system of equations.

Plastic corrector

Recall: $\boldsymbol{r}_{n+1} = \left.\frac{\partial \phi}{\partial \boldsymbol{\sigma}}\right|_{n+1} = \frac{3\boldsymbol{s}_{n+1}}{2||\boldsymbol{s}_{n+1}||}$

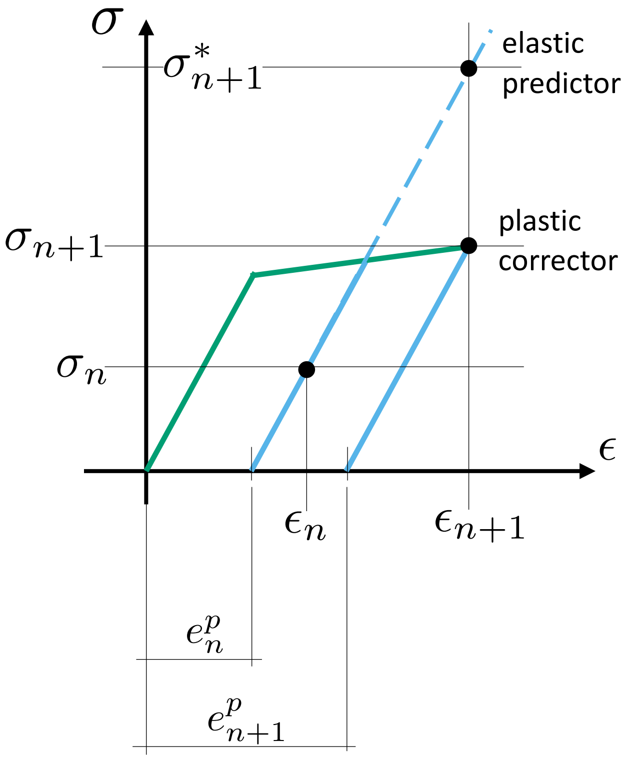
$$\rightarrow \left\{ \boldsymbol{\sigma}_{n+1}, \boldsymbol{\epsilon}_{n+1}^{(p)}, e_{n+1}^p \right\}$$

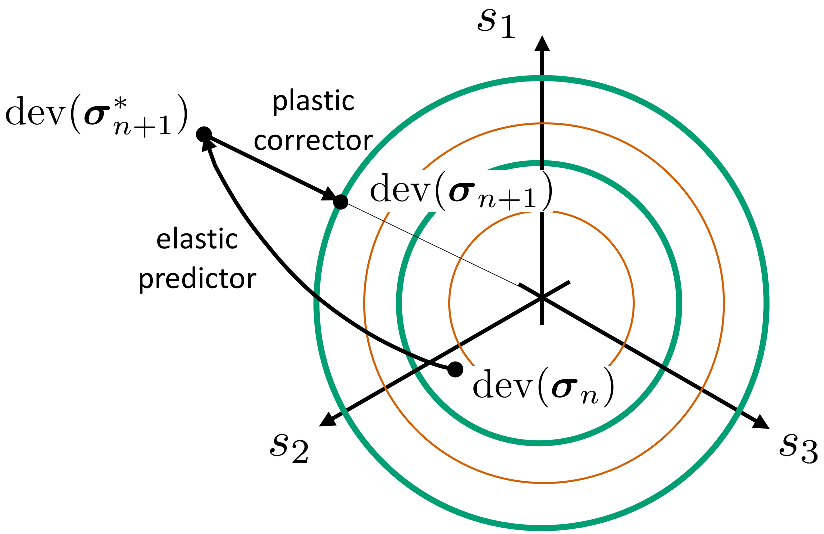
$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^* - \underbrace{3\mu\Delta\lambda \frac{\boldsymbol{s}_{n+1}}{||\boldsymbol{s}_{n+1}||}}_{\text{plastic corrector (\textit{radial return})}}$$

Isotropic plastic solids

J_2 or von Mises plasticity – Radial return algorithm

- Time discretization: $t_n \rightarrow t_{n+1} = t_n + \Delta t$

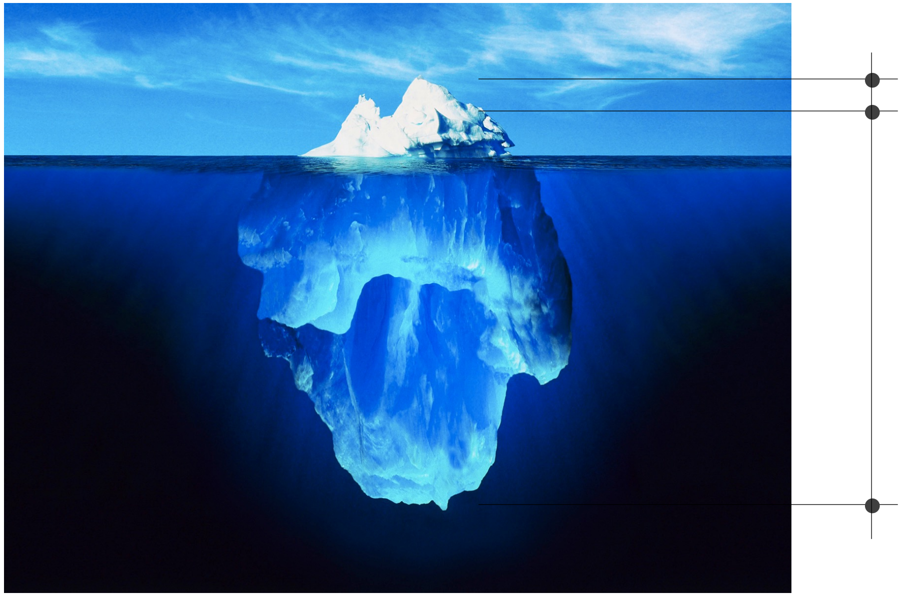




Space of principal values of the deviatoric part of the Cauchy stress tensor

Isotropic plastic solids

Any questions?



Lecture #13-#14

Theory
of plastic
solids