Lecture 15
Isotropic plastic solids
Mohr-Coulomb & Drucker-Prager

PURDUE UNIVERSITY | Mechanical Engineering
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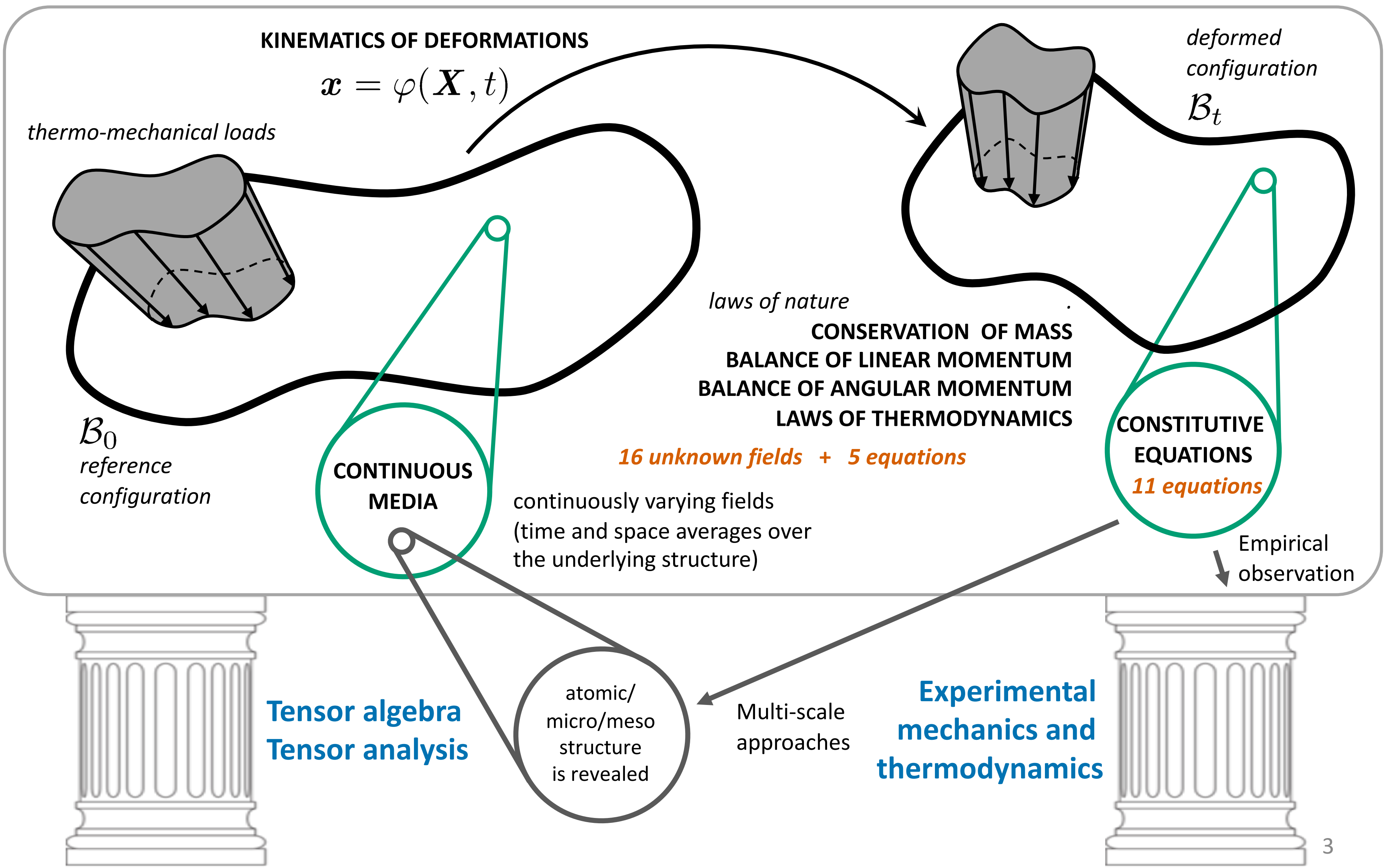
Last modified: 3/3/22 10:58:13 AM

Lecture 15 – Isotropic plastic solids



Isotropic plastic solids

Review: Ideal plastic solids – Small strains

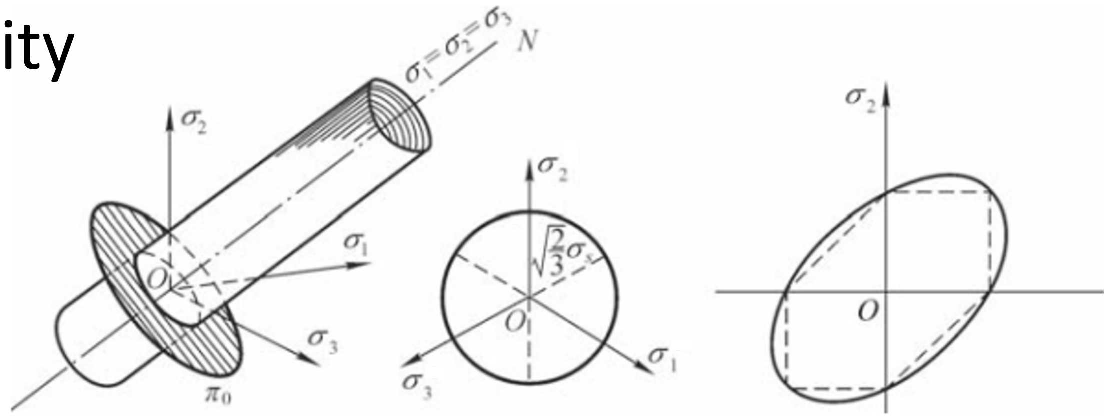
Constitutive law defined by yield function, flow potential and hardening rule

- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function $\phi(\boldsymbol{\sigma}, \xi) \leq 0$
  Internal variable (*for isotropic hardening*) $\xi = \int_0^t ||\dot{\boldsymbol{\epsilon}}^{(p)}||\mathrm{d}t$
- Additive decomposition of (small) strain: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$
- Consistency condition $\dot{\phi} = 0 \rightarrow \text{solve for } \dot{\lambda}$
  (*determine* $\dot{\lambda}$ *under plastic loading;* $\dot{\lambda} = 0$, *otherwise*)
- Strain energy density: $W(\boldsymbol{\epsilon}, \xi) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(p)}(\xi)) + W^{(p)}(\xi)$
  (Lecture 11)
- Evolution equations (Lecture 11):
  - Flow rule (flow potential) $\dot{\boldsymbol{\epsilon}}^{(p)} = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma}, \xi) = \dot{\lambda}\frac{\partial G}{\partial \boldsymbol{\sigma}}$
  - Hardening rule $\dot{\xi} = \dot{\lambda}h(\boldsymbol{\sigma}, \xi)$
- Cauchy stress tensor: $\boldsymbol{\sigma} = \frac{\rho}{\rho_0}\frac{\partial W^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}} := \boldsymbol{c}^e : \boldsymbol{\epsilon}^{(e)}$

Isotropic plastic solids

J_2 or von Mises plasticity

Yield function is independent of hydrostatic pressure
$I_1 = \mathrm{tr}(\boldsymbol{\sigma}) = -3p$



- Yield function (and isotropic hardening)

$$\phi(J_2, e^p) = \sqrt{3J_2} - \sigma_Y(e^p) = \sigma_{\mathrm{v}} - \sigma_Y(e^p) = q - \sigma_Y(e^p)$$

with $J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2] = \frac{1}{2}||\boldsymbol{s}||^2$

$$q = \sqrt{3/2}||\boldsymbol{s}||$$

- Flow potential (associative plasticity): $\phi(J_2, e^p) = G(J_2, e^p)$

Isotropic plastic solids

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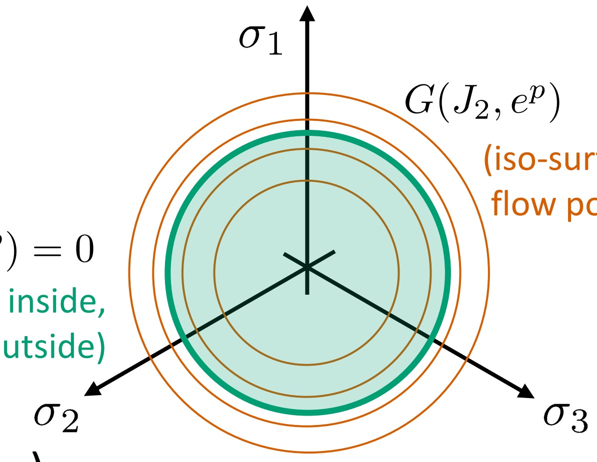

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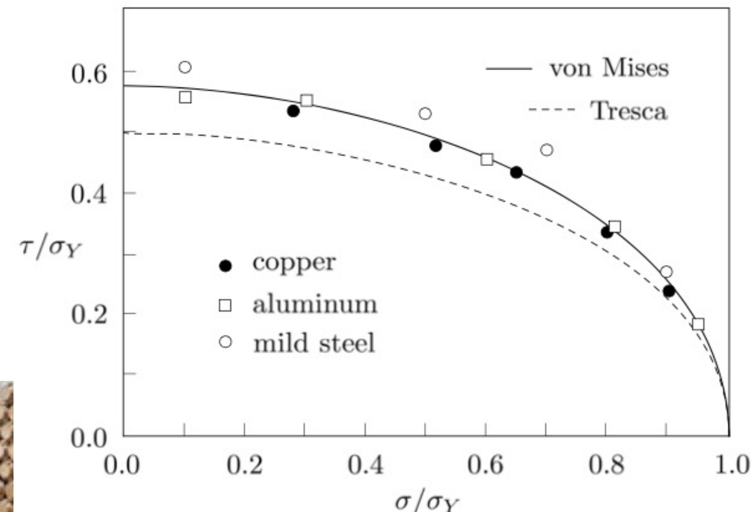
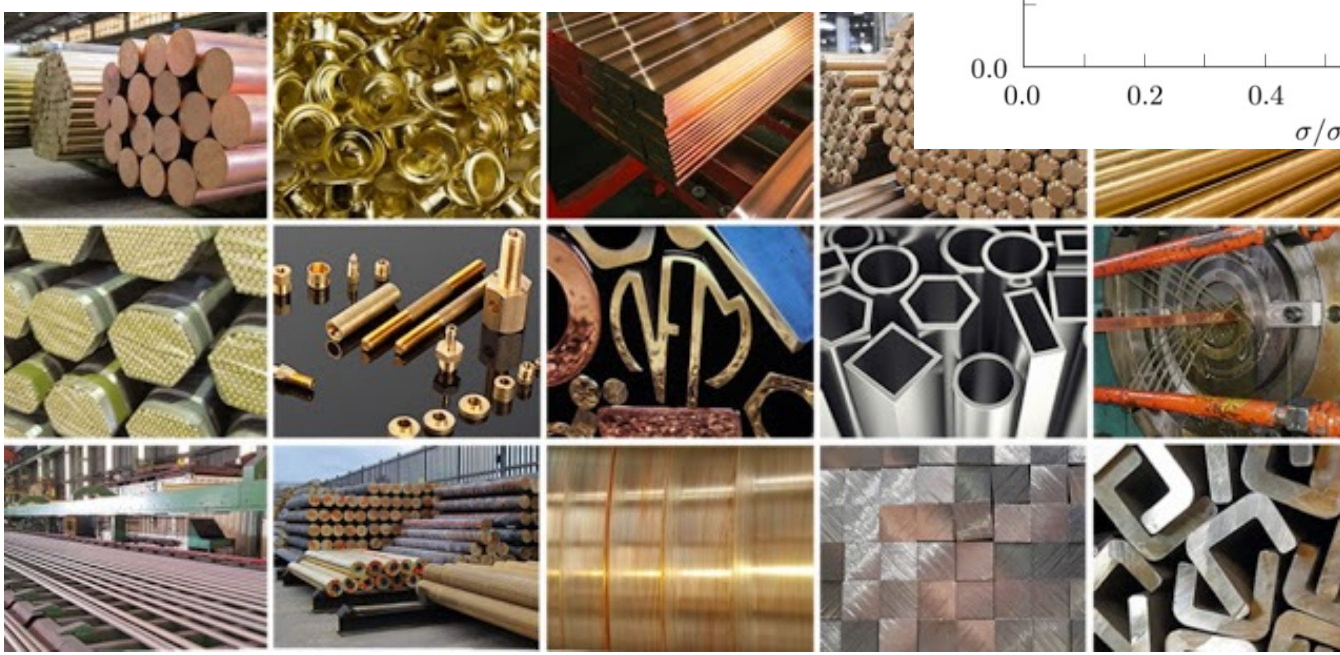
- Flow potential (associative plasticity): $\phi(J_2, e^p) = G(J_2, e^p)$
- Hardening rule: $\dot{e}^p = \sqrt{3/2}\dot{\lambda}$

Isotropic plastic solids

J_2 or von Mises plasticity

- Typically used for metals



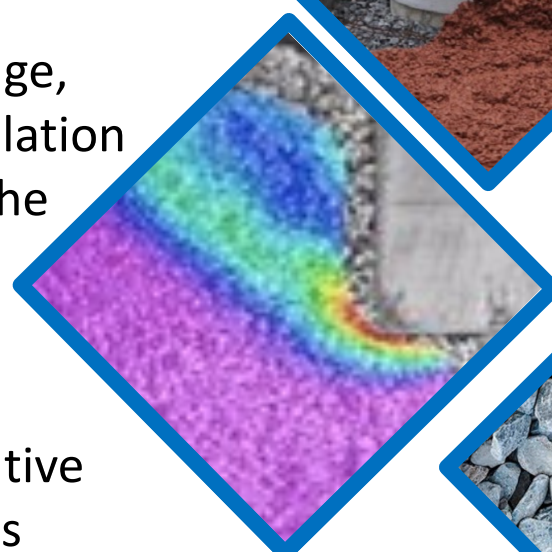


Isotropic plastic solids

Granular materials: from powders to soils

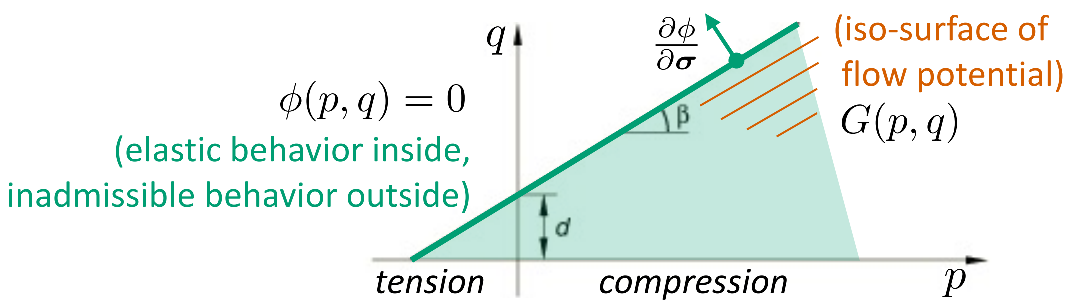
- Solid-like (a closely packed medium) … and fluid-like behavior.
- Significant difference in behavior of granular materials in tension and compression experiments.
  - Low accuracy of experimental measurements of phenomenological parameters of models
  - Formulation of constitutive laws is still inconclusive
- Exhibit inelastic volume change, whether by compaction or dilation (i.e., volume change due to the granular nature of the material when subjected to shear strain).
- Pressure-dependent constitutive models are used and failure is characterized by the angle of internal friction.

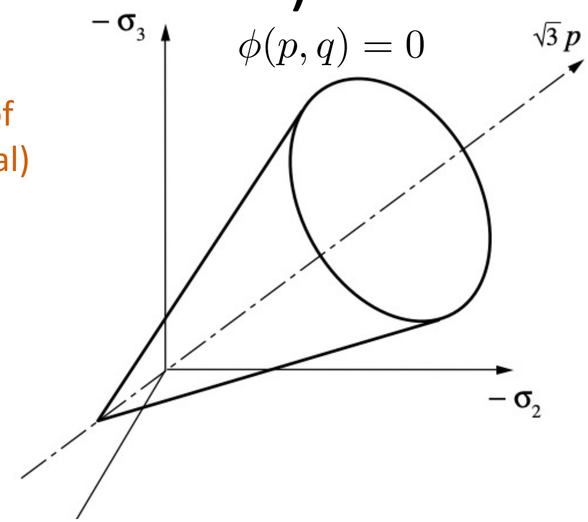





Isotropic plastic solids

Drucker-Prager plasticity (associated flow rule)





- Yield function (no hardening)

$$\phi(I_1, J_2) = \sqrt{3J_2} + \frac{I_1}{3}\tan\beta - d = q - p\tan\beta - d = \phi(p,q)$$

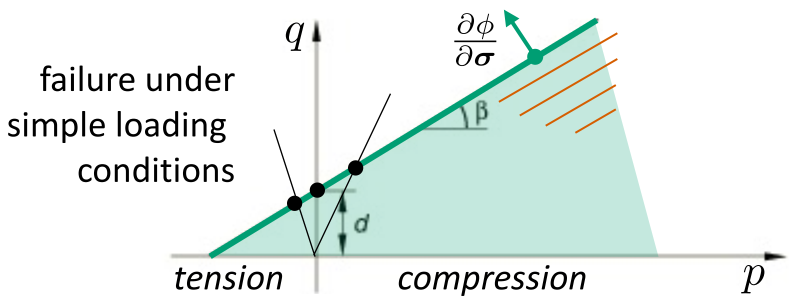
with $J_2 = \frac{1}{6}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2] = \frac{1}{2}||\boldsymbol{s}||^2$

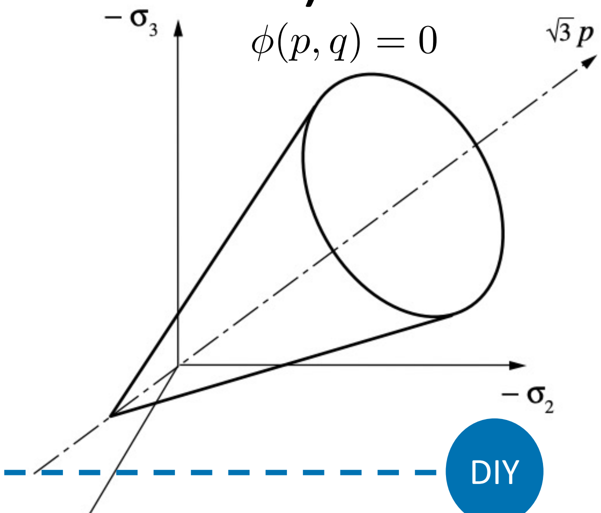
$$q = \sqrt{3/2}||\boldsymbol{s}|| \qquad\qquad I_1 = \mathrm{tr}(\boldsymbol{\sigma}) = -3p$$

- Flow potential (associative plasticity): $\phi(p,q) = G(p,q)$

Isotropic plastic solids

Drucker-Prager plasticity (associated flow rule)





DIY

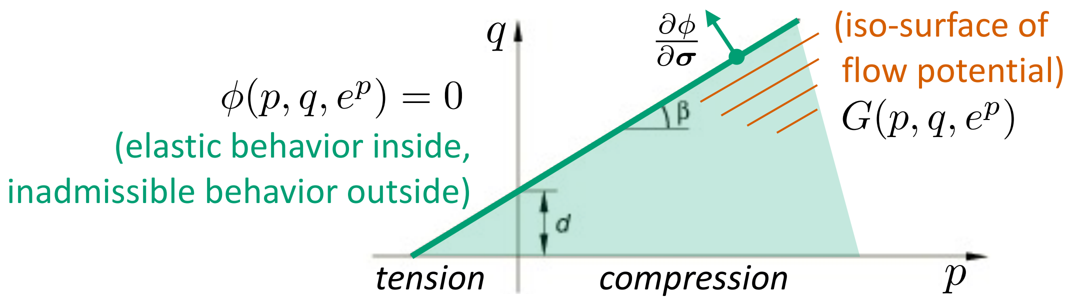
Uniaxial compression:

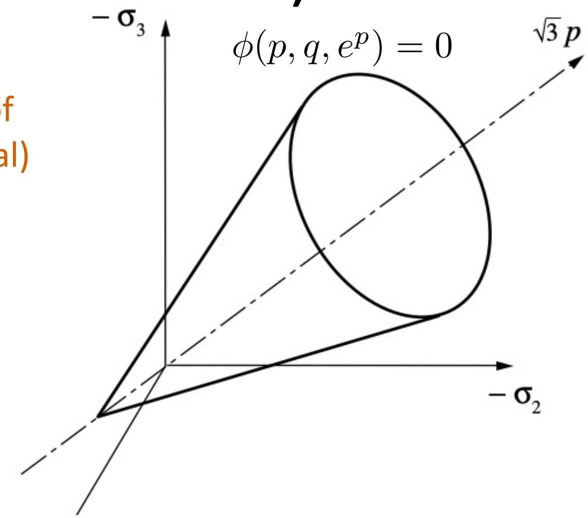
Uniaxial tension:

Shear:

Isotropic plastic solids

Drucker-Prager plasticity (associated flow rule)





- Yield function (with hardening)

$$\phi(p,q,e^p) = q - p\tan\beta - d(e^p)$$

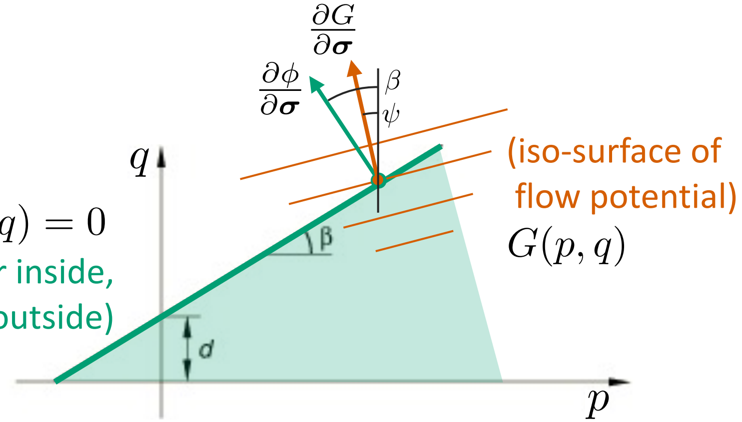
with $\quad q = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]^{1/2}$

$$p = -\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$$

- Flow potential (associative plasticity): $\phi(p,q,e^p) = G(p,q,e^p)$
- Hardening rule: $\dot{e}^p = \sqrt{3/2}\,\dot{\lambda}$ (note: the apex requires a different treatment)

Isotropic plastic solids

Drucker-Prager plasticity (non-associated flow rule)



d : cohesion

β : friction angle

ψ : dilation angle

- Yield function (no hardening)

$$\phi(p,q) = q - p\tan\beta - d$$

with $q = \frac{1}{\sqrt{2}}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2]^{1/2}$

$$p = -\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$$

- Flow potential (non-associative plasticity)

$$G(p,q) = q - p\tan\psi$$

$\psi = \beta$ (associated flow rule)

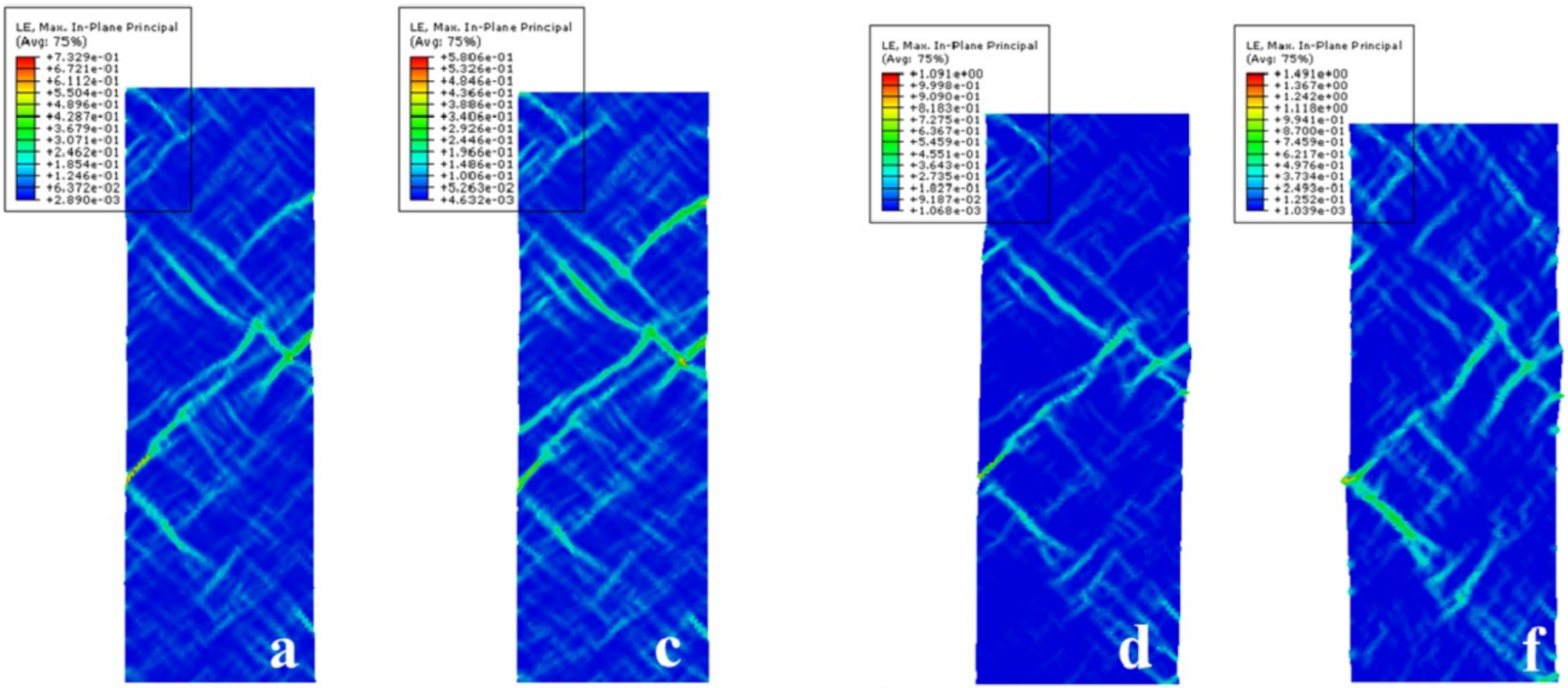
$0 < \psi < \beta$ (dilation)

$\psi = 0$ (no dilation)

Isotropic plastic solids

J2 (von Mises) and Drucker-Prager plasticity

Tension **Compression**

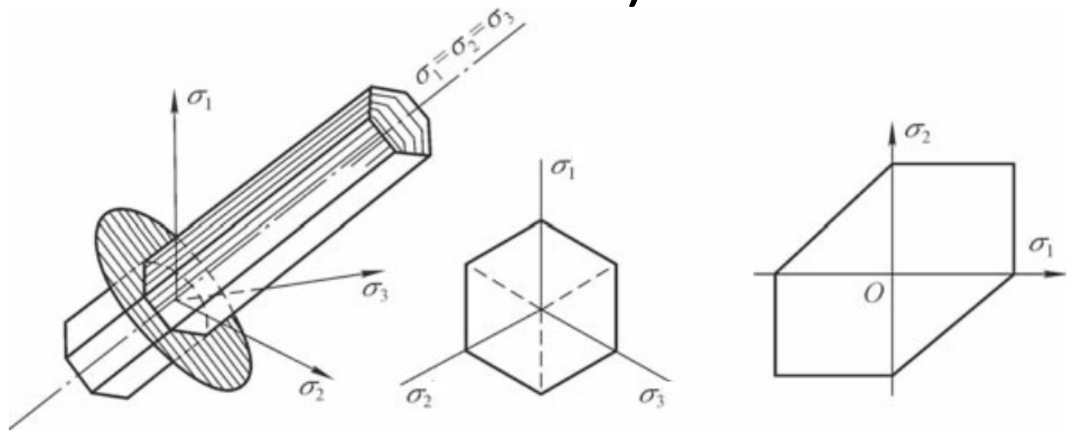


J2 plasticity **Drucker-Prager** **J2 plasticity** **Drucker-Prager**

Isotropic plastic solids

Tresca Criterion (or Maximum-shear-stress)

Yield function is independent of hydrostatic pressure
$I_1 = \mathrm{tr}(\boldsymbol{\sigma}) = -3p$



- Yield function (and isotropic hardening)

$$\phi(J_2, J_3, e^p) = 2\sqrt{J_2}\cos(\theta) - \sigma_Y(e^p)$$

with $J_2 = \frac{1}{6}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2] = \frac{1}{2}||\boldsymbol{s}||^2$

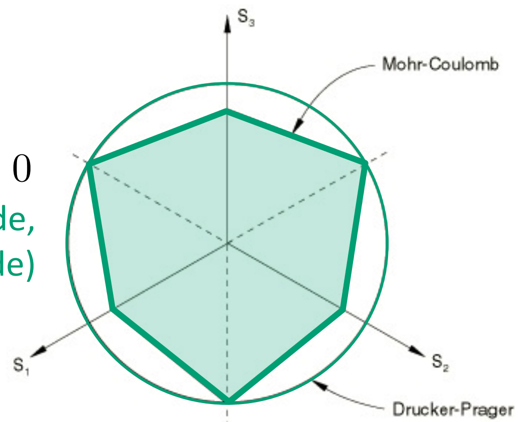
$$\theta = \frac{1}{3}\sin^{-1}\left(\frac{-3\sqrt{3}J_3}{2J_2^{3/2}}\right)$$

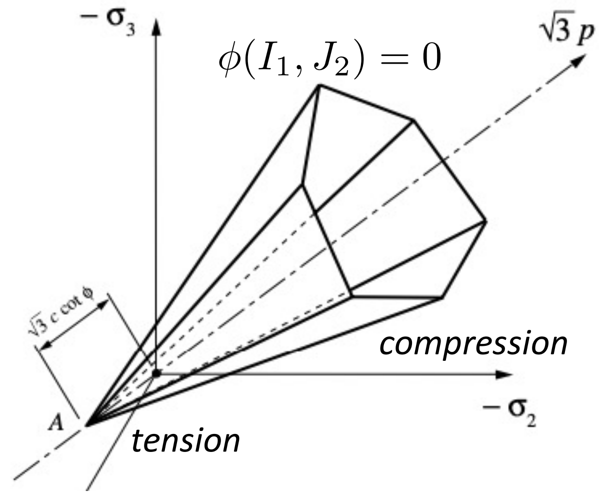
- Flow potential (associated plasticity): $\phi(J_2, J_3, e^p) = G(J_2, J_3, e^p)$
- Note: the multi-surface representation is preferred in the computational implementation of the model

Isotropic plastic solids

Mohr-Coulomb plasticity

$\phi(I_1, J_2) = 0$
(elastic behavior inside,
inadmissible behavior outside)





- Yield function (with and without hardening)

$$\phi(I_1, J_2) = \ldots$$

- Flow potential (associative and non-associative plasticity)
- Note: the multi-surface representation is preferred in the computational implementation of the model

Isotropic plastic solids

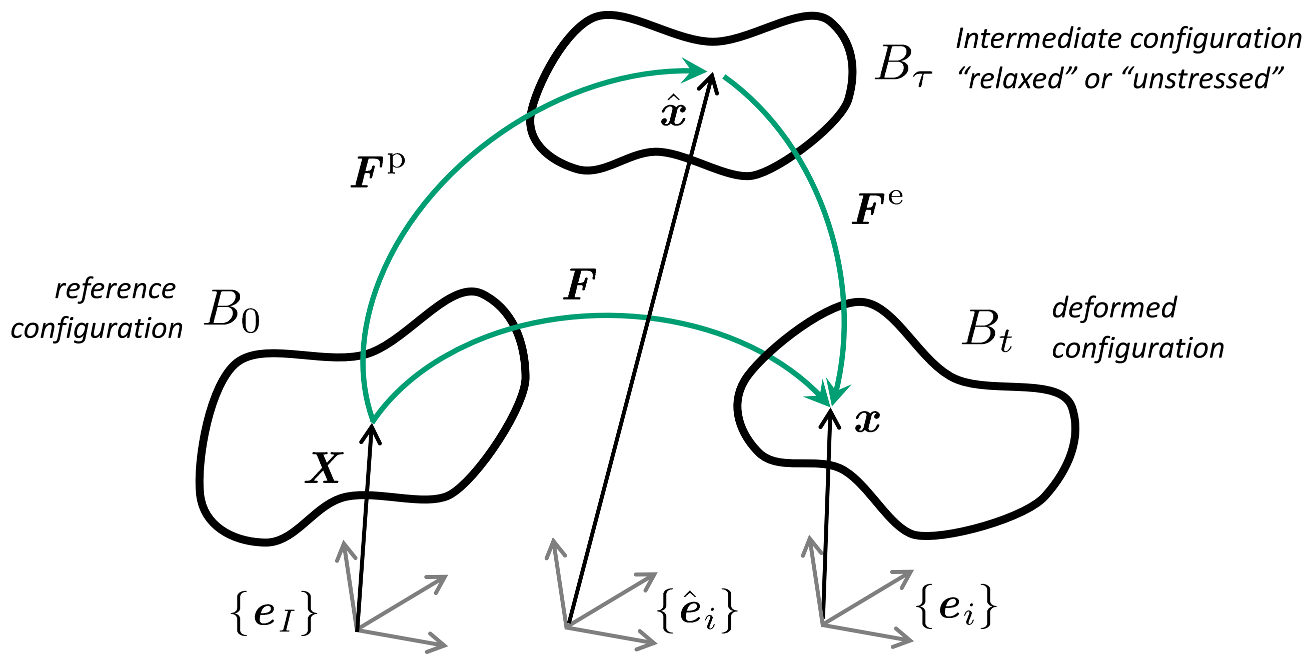
Elasto-plastic finite deformations

- Multiplicative decomposition of the deformation gradient

$$\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}}\boldsymbol{F}^{\mathrm{p}} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \qquad \det \boldsymbol{F}^{\mathrm{p}} = 1$$

plastic flow is incompressible

- The choice of stress and strain measure becomes very important! (they have to be work conjugate measures)



Isotropic plastic solids

Any questions?