ME 597 – Solid Mechanics II

Lecture 16
Solid-solid interactions
Contact mechanics

PURDUE UNIVERSITY | Mechanical Engineering

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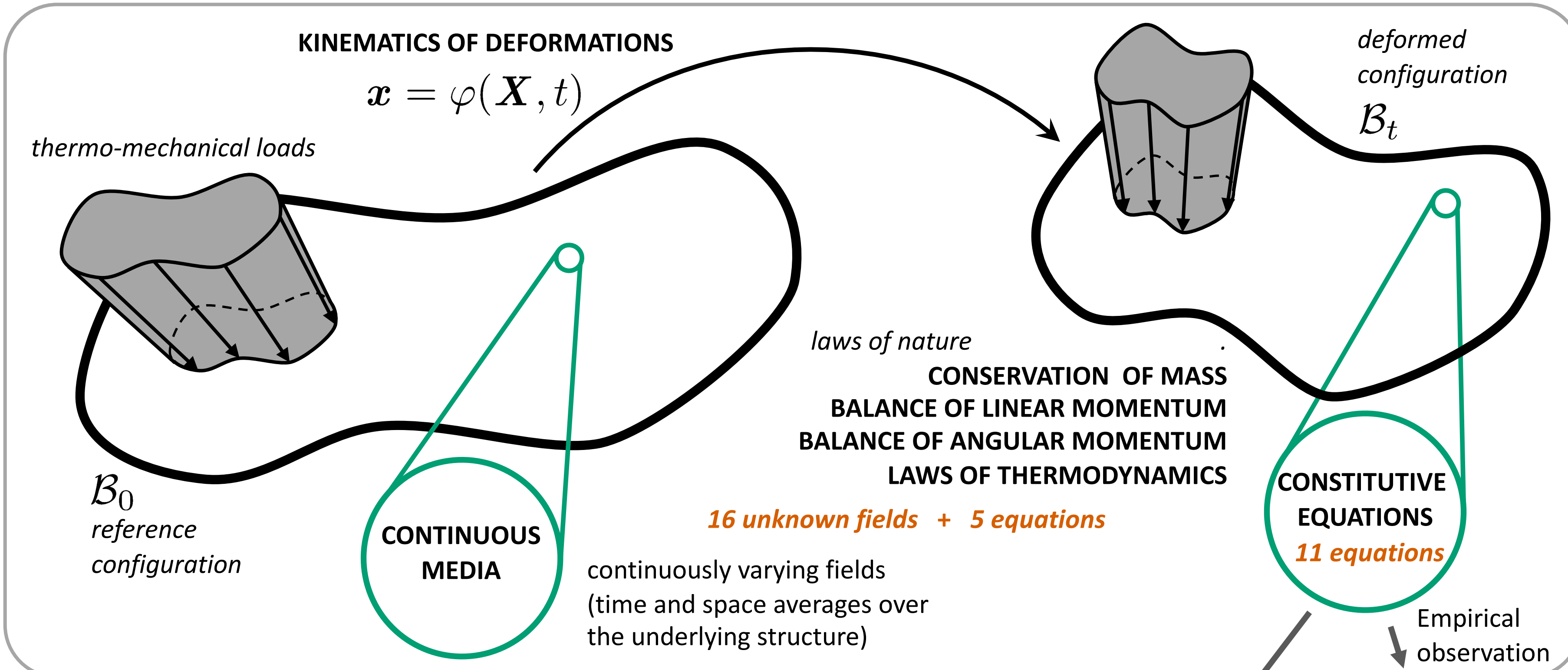
ME 597 – Solid Mechanics II … so far …

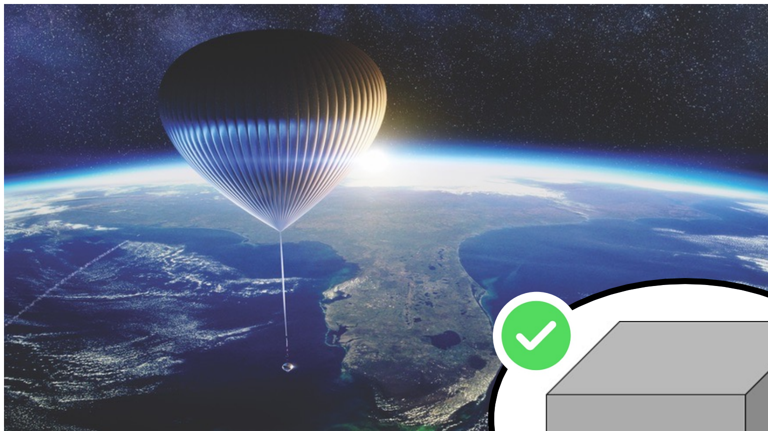


Tensor algebra
Tensor analysis

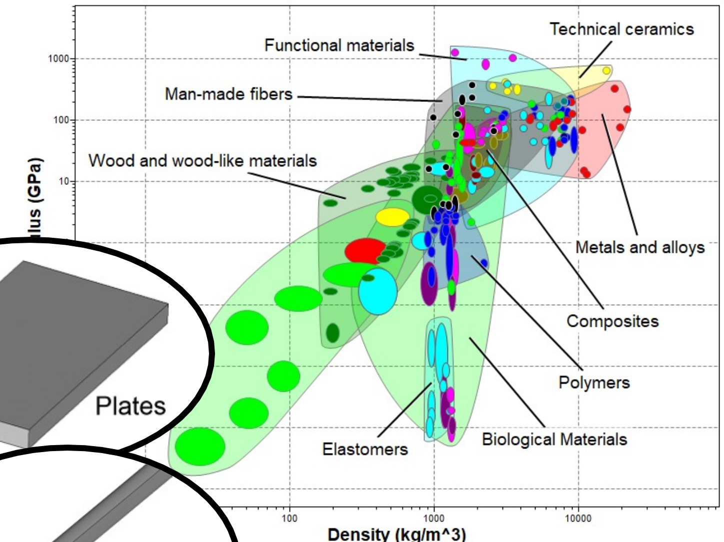
Hyperelastic isotropic solids ✓
Viscoelastic isotropic solids ✓
Plastic isotropic solids ✓
Material parameter estimation ✓

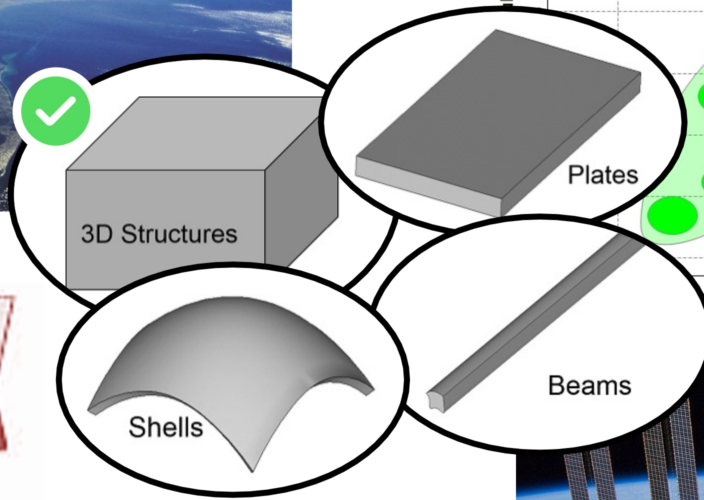
Structural elements: beams, plates, shells

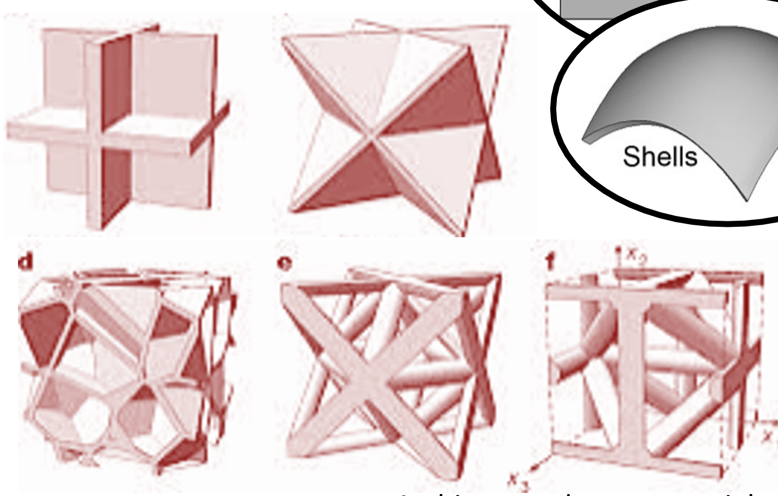
Formulation of structural elements (beams, plates, shells) as the analytical upscaling of continuum solids under kinematic assumptions.



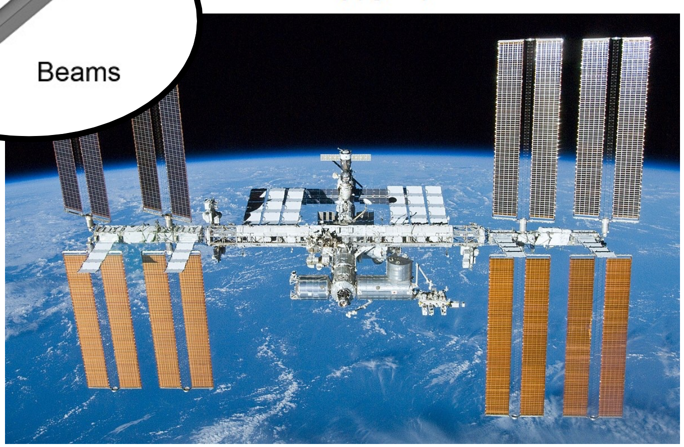
Stratospheric ballooning system







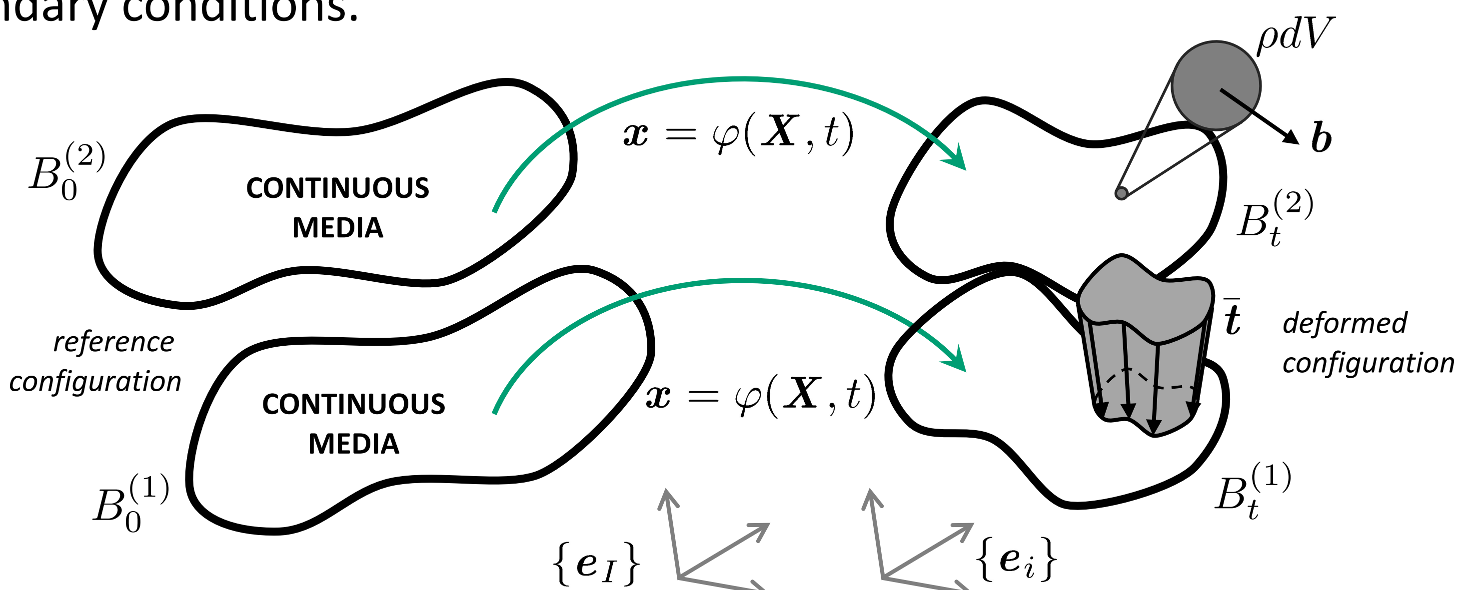
Architectured metamaterials



International space station

Lecture 16 – Solid-solid (… and fluid-solid) interactions

Formulation of **solid-solid interactions (contact mechanics)** as the analytical upscaling of continuum solids under kinematic assumptions and specific boundary conditions.



- Unilateral contact law in continuum mechanics (normal direction)

Hertz-Signorini conditions

$g(\partial B_t^{(1)}, \partial B_t^{(2)}) \geq 0$ no penetration (gap function)

$p(\partial B_t^{(1)}) \leq 0$ no tension (contact pressure)

$g(\partial B_t^{(1)}, \partial B_t^{(2)}) p(\partial B_t^{(1)}) = 0$ complementary conditions

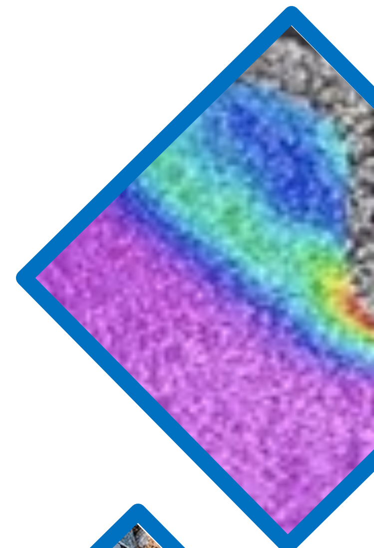
- Friction law (tangential direction)

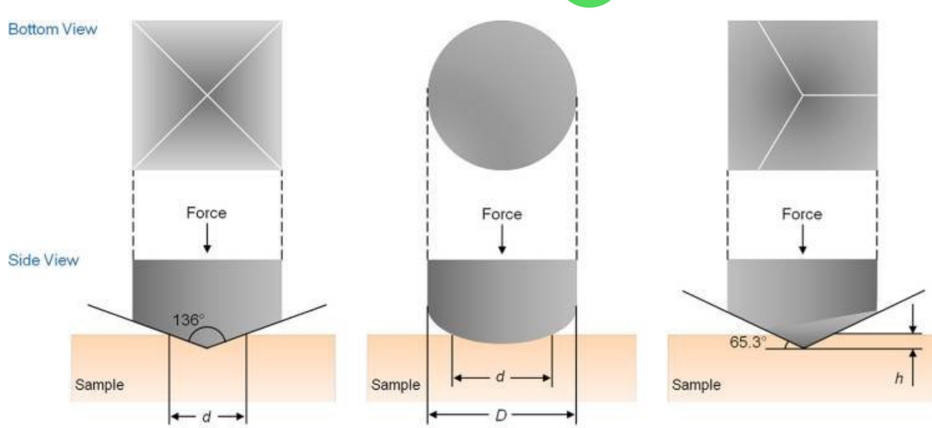
Lecture 16 – Solid-solid interactions

Formulation of **solid-solid interactions (contact mechanics)** as the analytical upscaling of continuum solids under kinematic assumptions and specific boundary conditions.

Engineering applications:

- Tribology, bearings (*lubrication, friction and wear*).
- Electrical contact resistance, locomotive wheel-rail contact, braking systems, tires, gasket seals, metal forming, ultrasonic welding (*in general, contact surface and contact pressure are coupled with each other*).
- At the core behavior of granular materials.
- Indentation hardness.

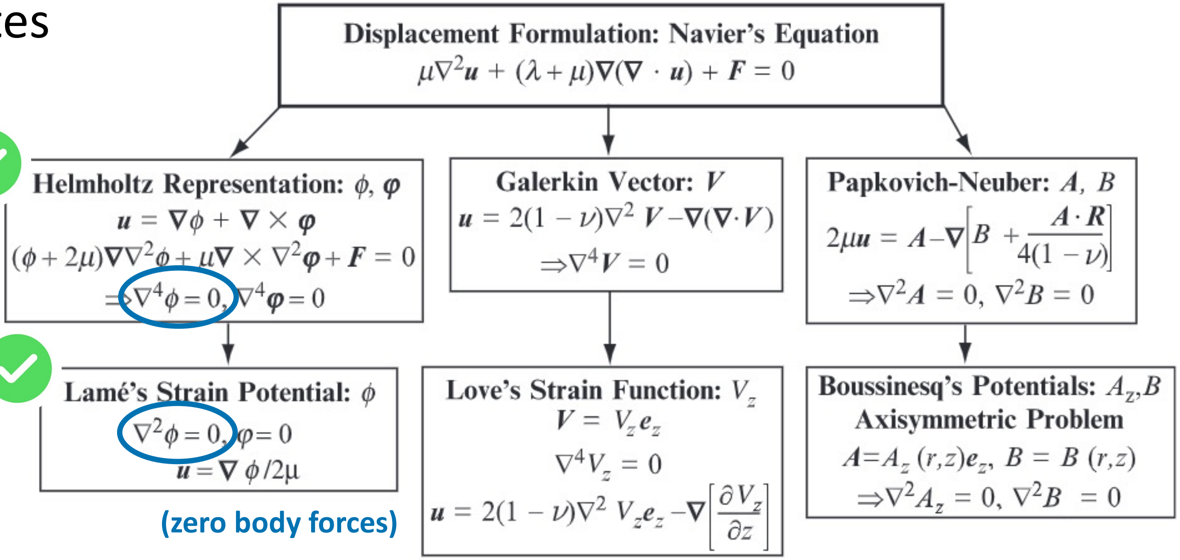


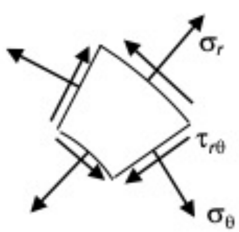


Solid-solid interactions: contact mechanics

Two-dimensional general solutions

- Cartesian coordinates (Airy's representation)
- Polar coordinates (Airy's representation)





$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

- Polar coordinates and axisymmetric

$$\phi = a_0 + a_1\log(r) + a_2 r^2 + a_3 r^2\log(r) \qquad \tau_{r\theta} = 0$$

SM1

Solid-solid interactions: contact mechanics

Two-dimensional general solutions

- Half-space under normal loads (concentrated and distributed)

$$\phi = (a_{12} r \log(r) + a_{15} r\theta)\cos(\theta) + (b_{12} r \log(r) + b_{15} r\theta)\sin(\theta)$$

Boundary conditions --- SM1

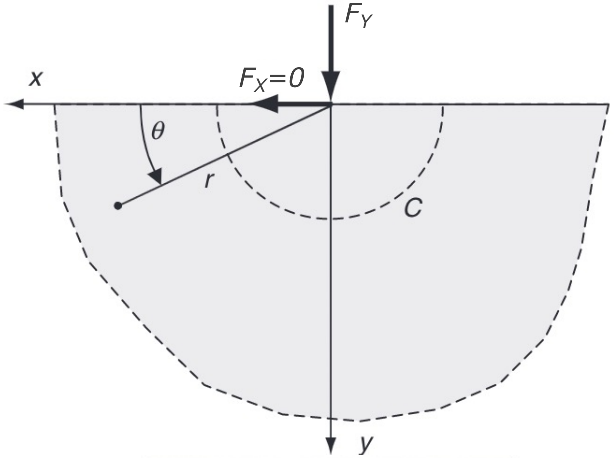
$$\tau_{r\theta}(r,\pi) = \tau_{r\theta}(r,0) = 0$$
$$\sigma_{\theta\theta}(r,\pi) = \sigma_{\theta\theta}(r,0) = 0$$
$$a_{12} = b_{12} = 0$$

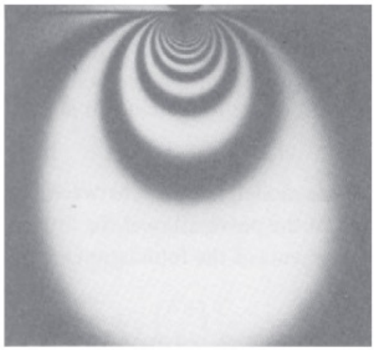
Equilibrium

$$b_{15} = -F_X/\pi = 0$$
$$a_{15} = F_Y/\pi$$

Resulting stress field

$$\sigma_{rr} = -\frac{2F_Y}{\pi r}\sin(\theta)$$
$$\sigma_{\theta\theta} = \tau_{r\theta} = 0$$





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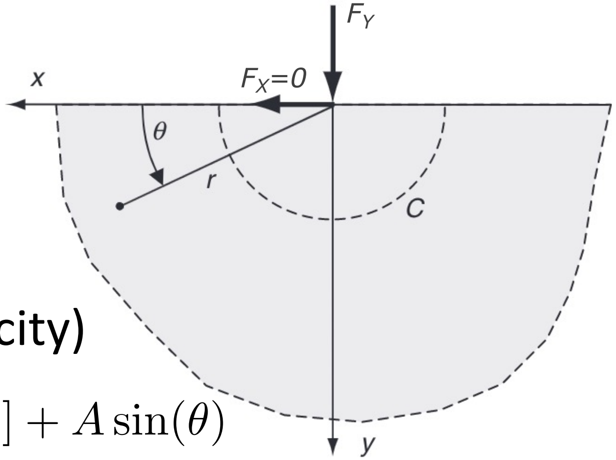
Solid-solid interactions: contact mechanics

Contact mechanics: two-dimensional problems

- Half-space under normal loads (concentrated and distributed)



Resulting stress field

$$\sigma_{rr} = -\frac{2F_Y}{\pi r}\sin(\theta)$$

$$\sigma_{\theta\theta} = \tau_{r\theta} = 0$$

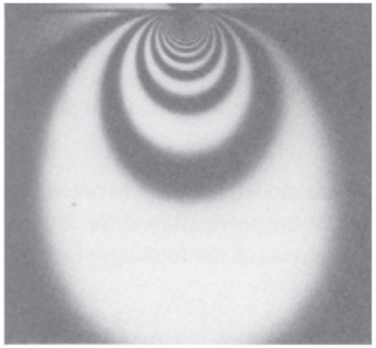
Resulting displacement field (linear elasticity)

$$u_r = \frac{F_Y}{\pi E}\left[(1-\nu)(\theta-\pi/2)\cos(\theta) - 2\log(r)\sin(\theta)\right] + A\sin(\theta)$$

$$u_\theta = \frac{F_Y}{\pi E}\left[-(1-\nu)(\theta-\pi/2)\sin(\theta) - 2\log(r)\cos(\theta) - (1+\nu)\cos(\theta)\right] + A\cos(\theta)$$

+ Rigid motion component cannot be solved for
+ Singular displacement/stress under the point load
+ Unbounded logarithmic terms lead to unrealistic predictions at infinity

SM1



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Solid-solid interactions: contact mechanics

Contact mechanics: two-dimensional problems

- Indentation of an elastic half-space by a frictionless punch

Contact radius: a

Boundary conditions:

+ surface displacement

$$\bar{u}_y(x) = \gamma \quad , \quad x \in [-a, a]$$

+ frictionless

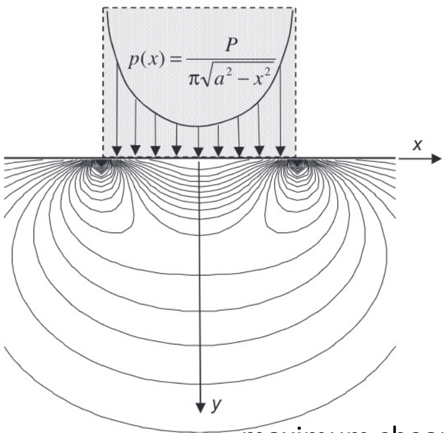
$$\mathrm{d}F_X = 0 \quad \mathrm{d}F_Y = p(x) = ?$$

Contact pressure:

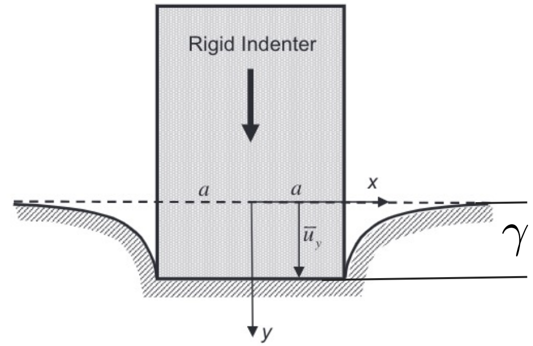
$$p(x) = \frac{P}{\pi\sqrt{a^2 - x^2}} \quad x \in [-a, a]$$

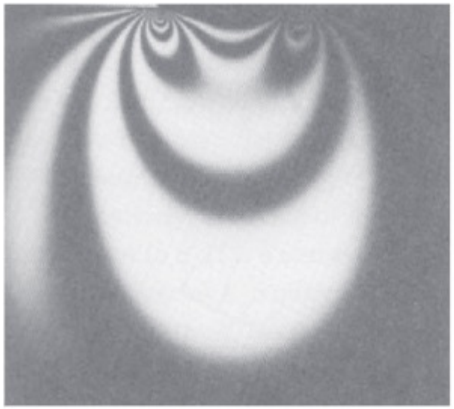
verification $\int_{-a}^{a} p(s)\mathrm{d}s = P$

+ Contact pressure is singular at the edges of the indenter
+ Unbounded logarithmic terms lead to unrealistic predictions at infinity



maximum shear stress contours




(Flat Punch Loading)

Solid-solid interactions: contact mechanics

Contact mechanics: two-dimensional problems

- Indentation of an elastic half-space by a frictionless punch

Contact radius: a

Boundary conditions:

+ surface displacement

$$\bar{u}_y(x) = \gamma \quad , \quad x \in [0, a]$$

+ frictionless

$$\mathrm{d}F_X = 0 \quad \mathrm{d}F_Y = p(x) = ?$$

Aside: contact pressure …

given $\bar{u}_y(x)$

find $p(x) = \ldots$

such that

$$\frac{\mathrm{d}\bar{u}_y}{\mathrm{d}x} = -\frac{2}{\pi E} \int_{-a}^{a} \frac{p(s)}{x-s} \mathrm{d}s$$

$$p(x) = \frac{P}{\pi\sqrt{a^2-x^2}} \quad x \in [-a, a]$$

$$\int_{-a}^{a} p(s)\mathrm{d}s = P$$

Aside

The vertical displacement of surface points is given by

$$\bar{u}_y(x) = -\frac{2}{\pi E} \int_{-a}^{a} p(s) \log(|x-s|)\mathrm{d}s + C$$

where the solution for a concentrated surface force $p(s)\mathrm{d}s$ acting on the free surface is used, and C is a constant of integration due to the undetermined rigid-body motion. This constant is eliminated from the analysis as follows

$$\frac{\mathrm{d}\bar{u}_y}{\mathrm{d}x} = -\frac{2}{\pi E} \int_{-a}^{a} \frac{p(s)}{x-s} \mathrm{d}s$$

where the Cauchy principal value integral is used to regularize the otherwise divergent integral.

For $p(x) \sim (a^2 - x^2)^{-1/2}$, the Cauchy principal value of the integral showed above is indeed zero, cf. $\bar{u}'_y = 0$.

Solid-solid interactions: contact mechanics

Contact mechanics: two-dimensional problems

- Indentation of an elastic half-space by a frictionless cylindrical punch

Boundary conditions:

+ surface displacement

$$\bar{u}_y(x) = \gamma - (R - \sqrt{R^2 - x^2})$$
$$\approx \gamma - \frac{1}{2R}x^2 \quad x \in [0, a]$$
$$\bar{u}_y(a) = \frac{\gamma}{2}$$

+ frictionless

$$\mathrm{d}F_X = 0 \quad \mathrm{d}F_Y = p(x) = ?$$

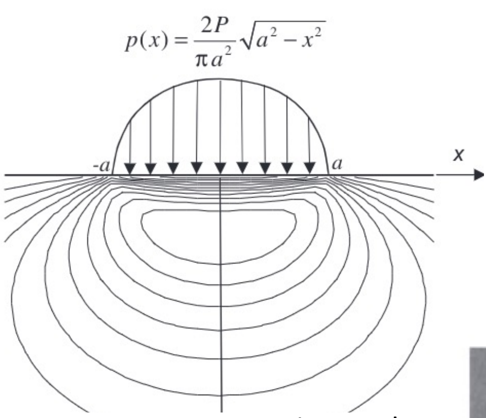
Contact radius: $a = \sqrt{\gamma R}$

Contact pressure:

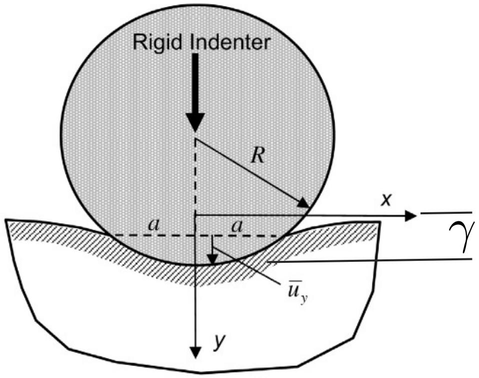
$$p(x) = \frac{2P}{\pi a^2}\sqrt{a^2 - x^2} \quad x \in [0, a]$$

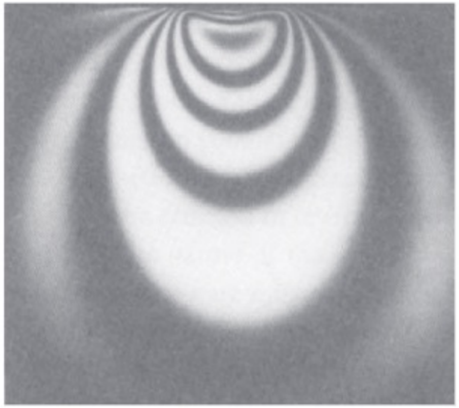
Contact law (force per unit of length):

$$a^2 = \frac{4PR}{\pi E} = \gamma R \implies \boxed{P(\gamma) = \frac{\pi E}{4}\gamma \quad a = \sqrt{\gamma R}}$$



maximum shear stress contours





(Cylinder Contact Loading)

Solid-solid interactions: contact mechanics

Contact mechanics: two-dimensional problems

- Indentation of an elastic half-space by a frictionless cylindrical punch

Boundary conditions:

+ surface displacement

$$\bar{u}_y(x) = \gamma - (R - \sqrt{R^2 - x^2})$$
$$\approx \gamma - \frac{1}{2R}x^2 \quad x \in [0, a]$$
$$\bar{u}_y(a) = \frac{\gamma}{2}$$

+ frictionless

$$\mathrm{d}F_X = 0 \quad \mathrm{d}F_Y = p(x) = ?$$

Contact radius: $a = \sqrt{\gamma R}$

Contact pressure:

$$p(x) = \frac{2P}{\pi a^2}\sqrt{a^2 - x^2} \quad x \in [0, a]$$

Aside

Similarly, for $p(x) = \frac{P}{\pi}\sqrt{a^2 - x^2}$, the Cauchy principal value of the integral simplifies to

$$-\frac{2}{\pi E}\int_{-a}^{a}\frac{p(s)}{x - s}\mathrm{d}s = -\frac{4Px}{a^2 E\pi}$$

and

$$\frac{\mathrm{d}\bar{u}_y}{\mathrm{d}x} = -\frac{x}{R}$$

Therefore, equating these two expressions

$$P = \frac{a^2 E\pi}{4R}$$

Due to the undetermined rigid-body motion in the two-dimensional solution, it is assumed that $\bar{u}_y(a) = \gamma/2$ and thus

$$\bar{u}_y(a) = \frac{\gamma}{2} = \gamma - \frac{1}{2R}a^2 \implies a^2 = \gamma R \ \text{ and } \ P = \frac{\pi E}{4}\gamma$$

Contact law (force per unit of length):

$$a^2 = \frac{4PR}{\pi E} = \gamma R \implies \boxed{P(\gamma) = \frac{\pi E}{4}\gamma \qquad a = \sqrt{\gamma R}}$$

Solid-solid interactions: contact mechanics

Three-dimensional solutions (applications are 3D!)

- Half-space under normal load: Boussinesq's problem
  Axisymmetric coordinate system

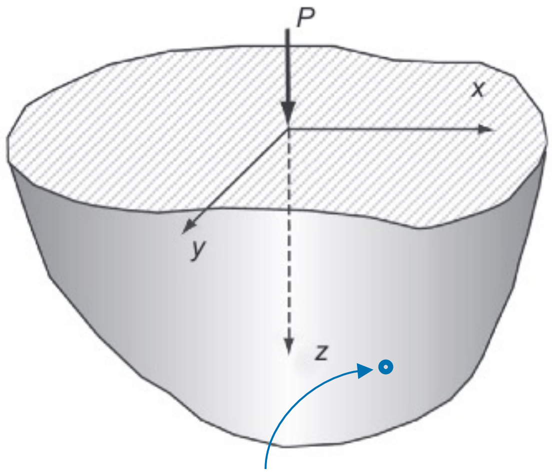
  Potentials: harmonic functions

$$A_z = \frac{C_1}{\rho} \quad , \quad B = C_2 \log(\rho + z)$$

  Displacement field:

$$u_r = \frac{P}{4\pi\mu}\left[\frac{rz}{\rho^3} - (1-2\nu)\frac{\rho - z}{\rho r}\right]$$

$$u_z = \frac{P}{4\pi\mu}\left[\frac{z^2}{\rho^3} + \frac{2(1-\nu)}{\rho}\right]$$



$$(x, y, z) \equiv (r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x), \rho = \sqrt{x^2 + y^2, z^2})$$

  Stress field (linear elasticity):

$$\sigma_{rr} = \frac{P}{2\pi}\left[(1-2\nu)\left(\frac{1}{r^2} - \frac{z}{\rho r^2}\right) - \frac{3zr^2}{\rho^5}\right]$$

$$\sigma_{\theta\theta} = -\frac{P}{2\pi}(1-2\nu)\left[\frac{1}{r^2} - \frac{z}{\rho r^2} - \frac{z}{\rho^3}\right]$$

$$\sigma_{zz} = -\frac{3P}{2\pi}\frac{z^3}{r^5} \qquad \tau_{rz} = -\frac{3P}{2\pi}\frac{rz^2}{r^5}$$

+ Singular displacement/stress under the point load
+ Displacements are bounded!

Solid-solid interactions: contact mechanics

Contact mechanics: three-dimensional problems

- Frictionless spherical indentation of an elastic half-space

Boundary conditions:

+ surface displacement

$$\bar{u}_z(r) = \gamma - (R - \sqrt{R^2 - r^2})$$
$$\approx \gamma - \frac{1}{R}r^2 \quad r \in [0, a]$$

+ frictionless

$$\mathrm{d}F_r = 0 \quad \mathrm{d}F_z = p(r) = ?$$

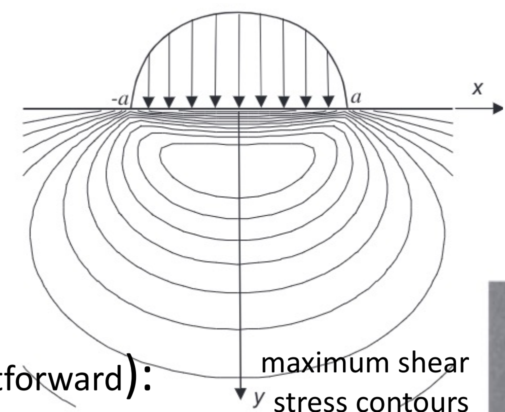
Contact radius: $a = \sqrt{\gamma R}$

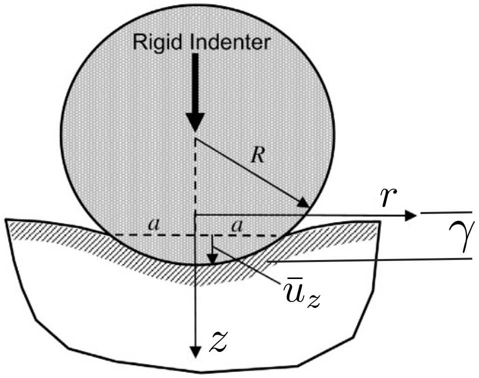
Contact pressure (not straightforward):

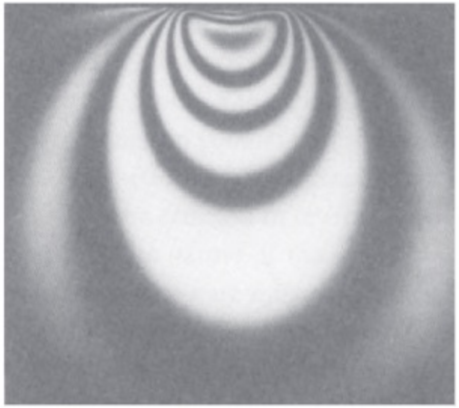
$$p(r) = \frac{3P}{2\pi a^3}\sqrt{a^2 - r^2} \quad , \quad r \in [0, a]$$

Contact law:

$$P(\gamma) = \frac{4}{3}\frac{E}{1-\nu^2}R^{1/2}\gamma^{3/2}$$
$$a = \sqrt{\gamma R}$$







Solid-solid interactions: contact mechanics

Any questions?