Lecture 17
Solid-solid interactions
Contact mechanics

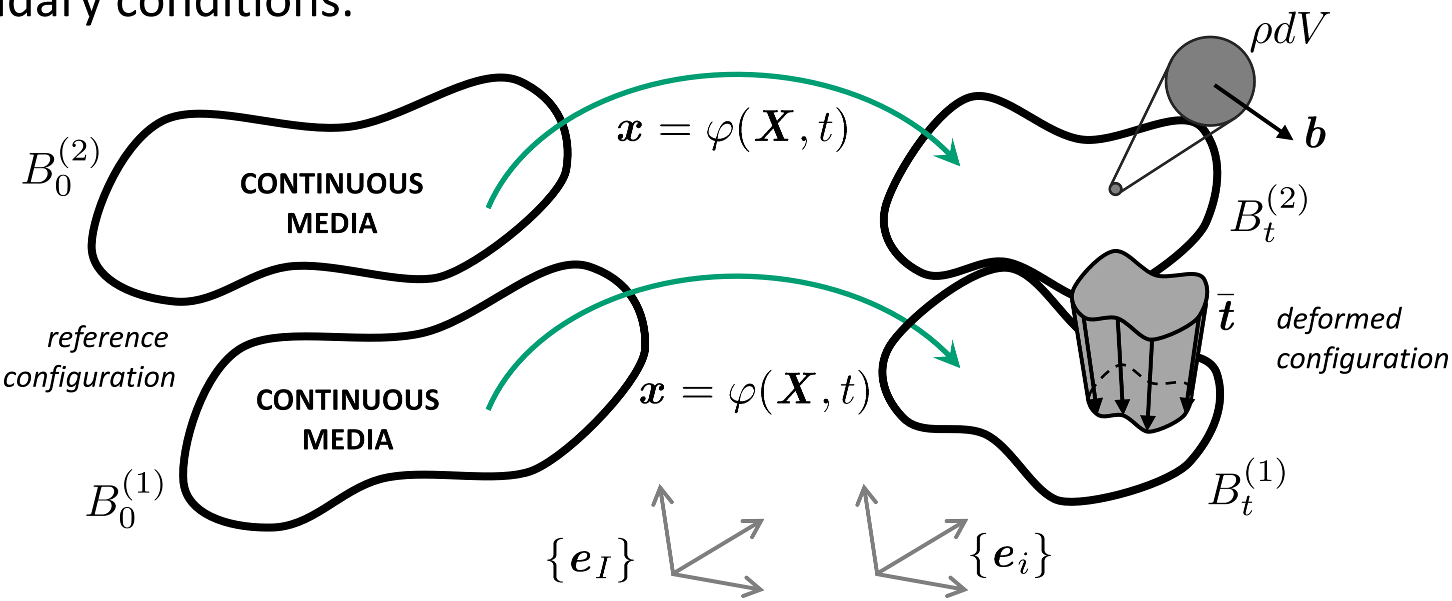
PURDUE UNIVERSITY | Mechanical Engineering

Instructor: Prof. Marcial Gonzalez

Last modified: 3/12/22 7:42:59 PM

Lecture 17 – Solid-solid interactions

Formulation of **solid-solid interactions (contact mechanics)** as the analytical upscaling of continuum solids under kinematic assumptions and specific boundary conditions.



- Unilateral contact law in continuum mechanics (normal direction)

Hertz-Signorini conditions

$g(\partial B_t^{(1)}, \partial B_t^{(2)}) \geq 0$ no penetration (gap function)

$p(\partial B_t^{(1)}) \leq 0$ no tension (contact pressure)

$g(\partial B_t^{(1)}, \partial B_t^{(2)}) p(\partial B_t^{(1)}) = 0$ complementary conditions

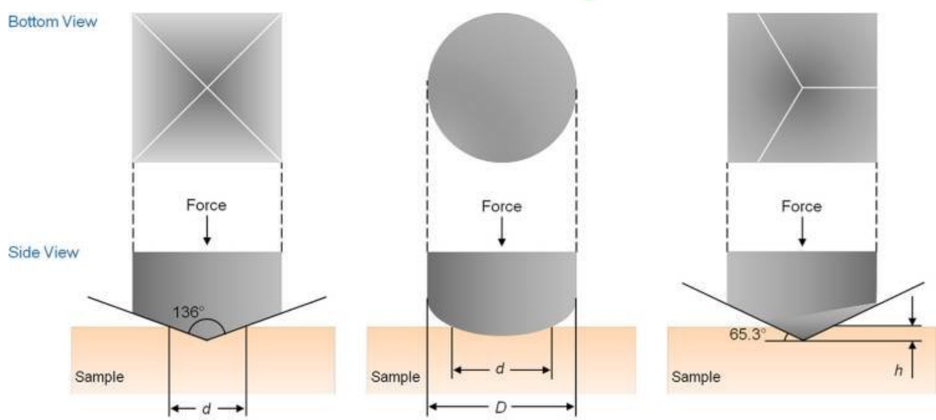
- Friction law (tangential direction)

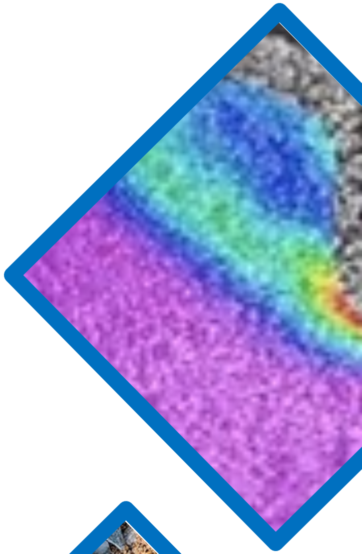
Lecture 17 – Solid-solid interactions

Formulation of **solid-solid interactions (contact mechanics)** as the analytical upscaling of continuum solids under kinematic assumptions and specific boundary conditions.

Engineering applications:

- Tribology, bearings (*lubrication, friction and wear*).
- Electrical contact resistance, locomotive wheel-rail contact, braking systems, tires, gasket seals, metal forming, ultrasonic welding (*in general, contact surface and contact pressure are coupled with each other*).
- At the core behavior of granular materials.
- Indentation hardness.

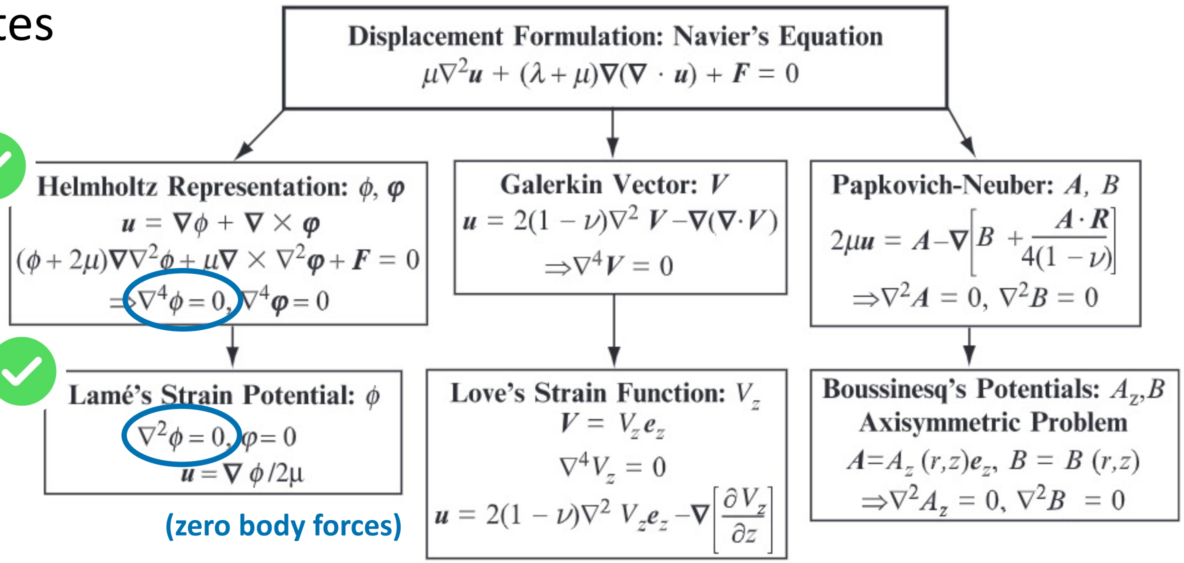




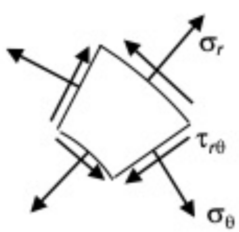
Solid-solid interactions: contact mechanics

Two-dimensional general solutions

- Cartesian coordinates (Airy's representation)
- Polar coordinates (Airy's representation)



(zero body forces)



$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

- Polar coordinates and axisymmetric

$$\phi = a_0 + a_1\log(r) + a_2 r^2 + a_3 r^2\log(r) \qquad \tau_{r\theta} = 0$$

SM1

Solid-solid interactions: contact mechanics

Two-dimensional general solutions

- Half-space under normal loads (concentrated and distributed)

$$\phi = (a_{12} r \log(r) + a_{15} r\theta)\cos(\theta) + (b_{12} r \log(r) + b_{15} r\theta)\sin(\theta)$$

Boundary conditions ---- SM1

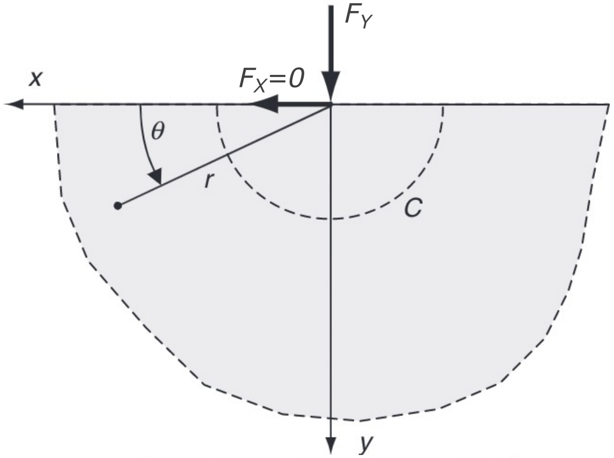
$$\tau_{r\theta}(r,\pi) = \tau_{r\theta}(r,0) = 0$$
$$\sigma_{\theta\theta}(r,\pi) = \sigma_{\theta\theta}(r,0) = 0$$
$$a_{12} = b_{12} = 0$$

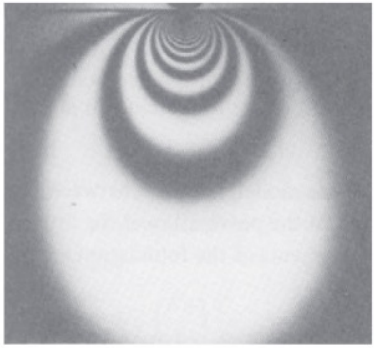
Equilibrium

$$b_{15} = -F_X/\pi = 0$$
$$a_{15} = F_Y/\pi$$

Resulting stress field

$$\sigma_{rr} = -\frac{2F_Y}{\pi r}\sin(\theta)$$
$$\sigma_{\theta\theta} = \tau_{r\theta} = 0$$





(Point Loading)

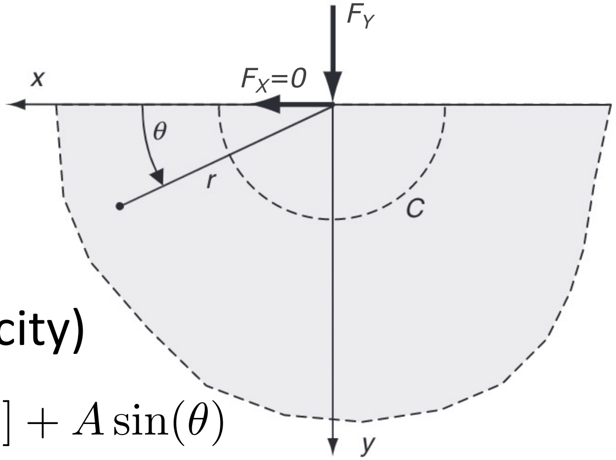
Solid-solid interactions: contact mechanics

Contact mechanics: two-dimensional problems

- Half-space under normal loads (concentrated and distributed)

Resulting stress field

$$\sigma_{rr} = -\frac{2F_Y}{\pi r}\sin(\theta)$$
$$\sigma_{\theta\theta} = \tau_{r\theta} = 0$$

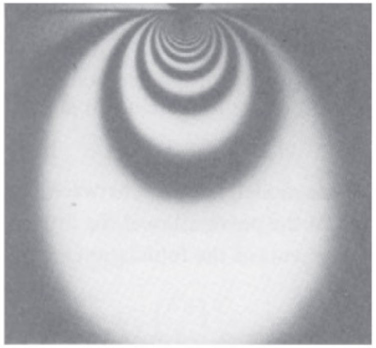


Resulting displacement field (linear elasticity)

$$u_r = \frac{F_Y}{\pi E}\left[(1-\nu)(\theta-\pi/2)\cos(\theta) - 2\log(r)\sin(\theta)\right] + A\sin(\theta)$$
$$u_\theta = \frac{F_Y}{\pi E}\left[-(1-\nu)(\theta-\pi/2)\sin(\theta) - 2\log(r)\cos(\theta) - (1+\nu)\cos(\theta)\right] + A\cos(\theta)$$

+ Rigid motion component cannot be solved for
+ Singular displacement/stress under the point load
+ Unbounded logarithmic terms lead to unrealistic predictions at infinity

SM1



(Point Loading)

Solid-solid interactions: contact mechanics

Contact mechanics: two-dimensional problems

- Indentation of an elastic half-space by a frictionless cylindrical punch

Boundary conditions:

+ surface displacement

$$\bar{u}_y(x) = \gamma - (R - \sqrt{R^2 - x^2})$$
$$\approx \gamma - \frac{1}{R}x^2 \quad x \in [0, a]$$
$$\bar{u}_y(a) = \frac{\gamma}{2}$$

+ frictionless

$$\mathrm{d}F_X = 0 \quad \mathrm{d}F_Y = p(x) =?$$

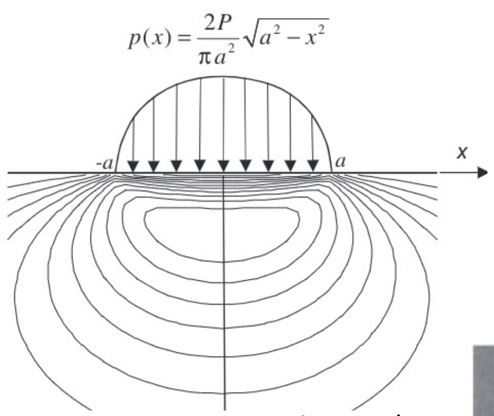
Contact radius: $a = \sqrt{\gamma R}$

Contact pressure:

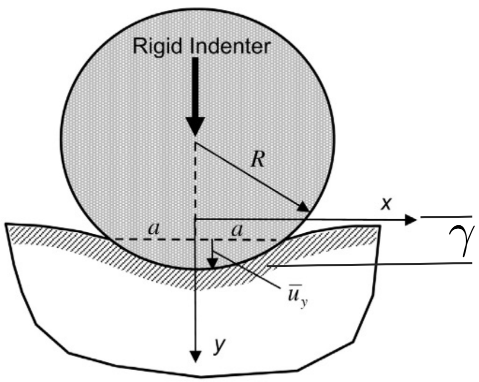
$$p(x) = \frac{2P}{\pi a^2}\sqrt{a^2 - x^2} \quad x \in [0, a]$$

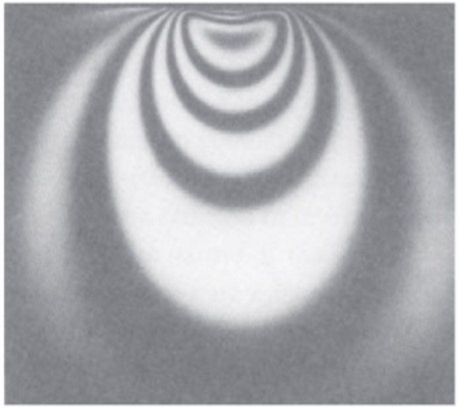
Contact law (force per unit of length):

$$a^2 = \frac{4PR}{\pi E} = \gamma R \implies \boxed{\begin{aligned} P(\gamma) &= \frac{\pi E}{4}\gamma \\ a &= \sqrt{\gamma R} \end{aligned}}$$



maximum shear stress contours

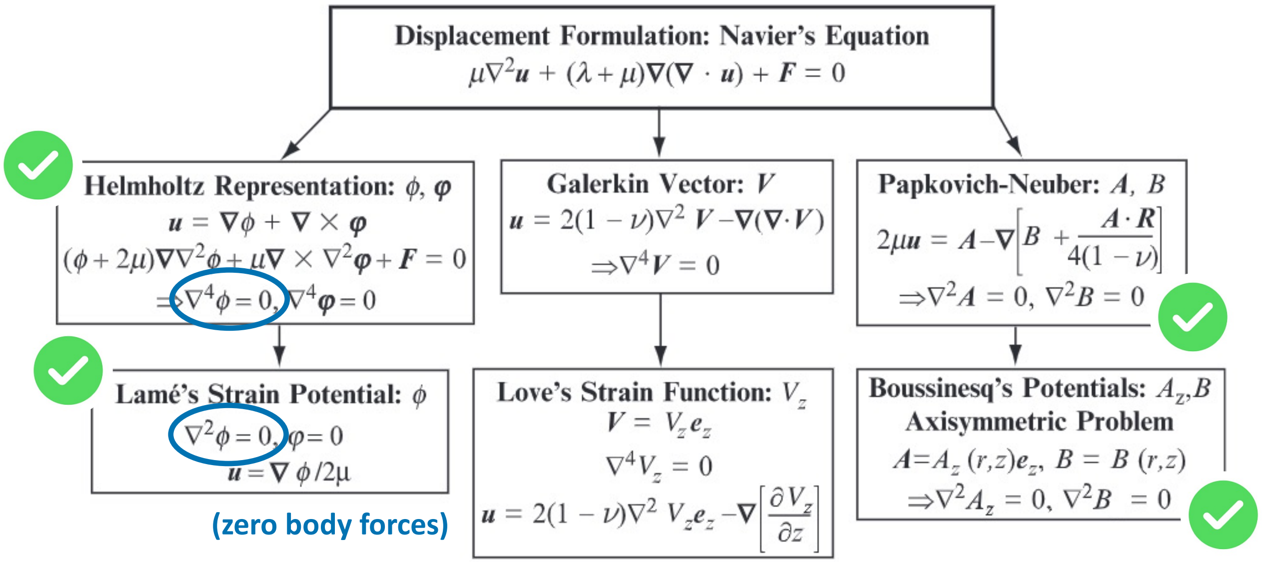




(Cylinder Contact Loading)

Solid-solid interactions: contact mechanics

Three-dimensional solutions (applications are 3D!)



Solid-solid interactions: contact mechanics

Three-dimensional solutions (applications are 3D!)

- Elastic half-space under normal load: Boussinesq's problem
  Axisymmetric coordinate system

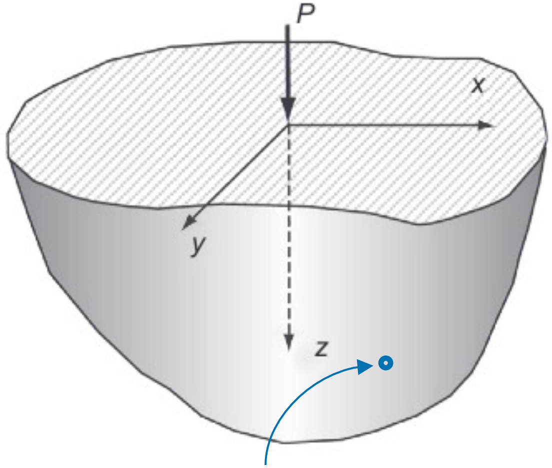
  Potentials: harmonic functions

$$A_z = \frac{C_1}{\rho} \quad , \quad B = C_2 \log(\rho + z)$$

  Displacement field:

$$u_r = \frac{P}{4\pi\mu}\left[\frac{rz}{\rho^3} - (1-2\nu)\frac{\rho-z}{\rho r}\right]$$

$$u_z = \frac{P}{4\pi\mu}\left[\frac{z^2}{\rho^3} + \frac{2(1-\nu)}{\rho}\right]$$



$$(x,y,z) \equiv (r = \sqrt{x^2+y^2}, \theta = \tan^{-1}(y/x), \rho = \sqrt{x^2+y^2,z^2})$$

  Stress field (linear elasticity):

$$\sigma_{rr} = \frac{P}{2\pi}\left[(1-2\nu)\left(\frac{1}{r^2} - \frac{z}{\rho r^2}\right) - \frac{3zr^2}{\rho^5}\right]$$

$$\sigma_{\theta\theta} = -\frac{P}{2\pi}(1-2\nu)\left[\frac{1}{r^2} - \frac{z}{\rho r^2} - \frac{z}{\rho^3}\right]$$

$$\sigma_{zz} = -\frac{3P}{2\pi}\frac{z^3}{r^5} \qquad \tau_{rz} = -\frac{3P}{2\pi}\frac{rz^2}{r^5}$$

+ Singular displacement/stress under the point load
+ Displacements are bounded!

Solid-solid interactions: contact mechanics

Frictionless spherical indentation of an elastic half-space

Boundary conditions:

+ surface displacement

$$\bar{u}_z(r) = \gamma - (R - \sqrt{R^2 - r^2})$$
$$\approx \gamma - \frac{1}{2R} r^2 \quad r \in [0, a]$$

+ frictionless

$$\mathrm{d}F_r = 0 \quad \mathrm{d}F_z = p(r) = ?$$

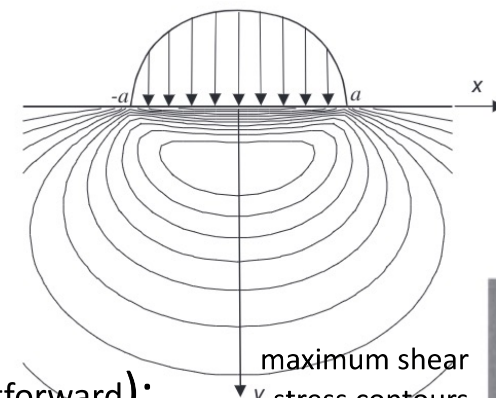
Contact radius: $a = \sqrt{\gamma R}$

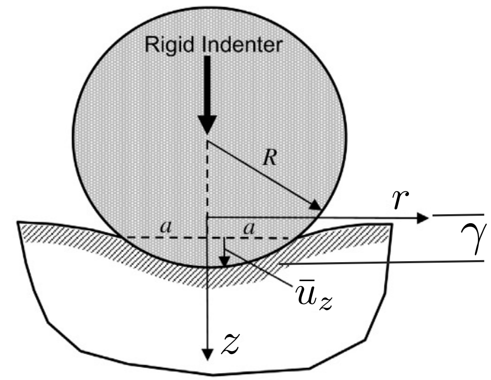
Contact pressure (not straightforward):

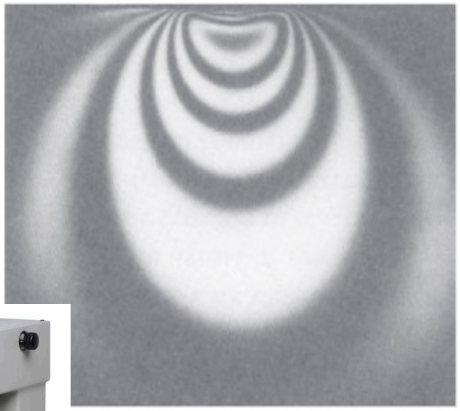
$$p(r) = \frac{3P}{2\pi a^3}\sqrt{a^2 - r^2} \quad , \quad r \in [0, a]$$

Contact law:

$$P(\gamma) = \frac{4}{3}\frac{E}{1-\nu^2} R^{1/2}\gamma^{3/2}$$
$$a = \sqrt{\gamma R}$$







Solid-solid interactions: contact mechanics

Frictionless spherical indentation of an elastic half-space

Boundary conditions:

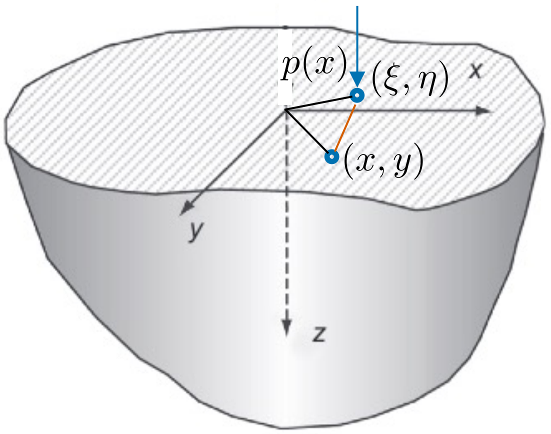
+ surface displacement

$$\bar{u}_z(r) = \gamma - (R - \sqrt{R^2 - r^2}) \approx \gamma - \frac{1}{2R} r^2 \quad r \in [0, a]$$

+ frictionless

$$\mathrm{d}F_r = 0 \quad \mathrm{d}F_z = p(r) = ?$$

$$\bar{u}_z(x, y) = \frac{1-\nu^2}{\pi E} \int \int_{\mathcal{S}} \frac{p(\xi, \eta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \mathrm{d}\xi \mathrm{d}\eta$$



Contact radius: $a = \sqrt{\gamma R}$

Contact pressure (not straightforward):

$$p(r) = \frac{3P}{2\pi a^3} \sqrt{a^2 - r^2} \quad , \quad r \in [0, a]$$

DIY

$$u_z = \frac{P}{4\pi\mu} \left[\frac{z^2}{\rho^3} + \frac{2(1-\nu)}{\rho} \right]$$

Solid-solid interactions: contact mechanics

Contact mechanics: three-dimensional problems

- Frictionless spherical indentation of an elastic half-space

Boundary conditions:

+ surface displacement

$$\bar{u}_z(r) = \gamma - (R - \sqrt{R^2 - r^2})$$
$$\approx \gamma - \frac{1}{R} r^2 \quad r \in [0, a]$$
$$\bar{u}_y(a) = \frac{\gamma}{2}$$

+ frictionless

$$\mathrm{d}F_r = 0 \quad \mathrm{d}F_z = p(r) = ?$$

Contact radius: $a = \sqrt{\gamma R}$

Contact pressure:

$$p(r) = \frac{3P}{2\pi a^3}\sqrt{a^2 - r^2} \quad , \quad r \in [0, a]$$

Contact law:

$$P(\gamma) = \frac{4}{3}\frac{E}{1-\nu^2} R^{1/2}\gamma^{3/2}$$
$$a = \sqrt{\gamma R}$$

Aside

The vertical displacement of surface points is given by

$$\bar{u}_z(x, y) = \frac{1-\nu^2}{\pi E}\int\int_S \frac{p(\xi, \eta)}{\sqrt{(x-\xi)^2 + (x-\eta)^2}}\mathrm{d}\xi\mathrm{d}\eta$$

where the Boussinesq's solution for vertical displacement of surface point (x, y) due to a concentrated force $p\mathrm{d}\xi\mathrm{d}\eta$ acting on (ξ, η) was used. Using polar coordinates (s, ϕ), with origin at a distance r from the center of the circular contact surface and a pressure $ps\mathrm{d}s\mathrm{d}\phi$ with distribution $p(r) = \frac{3P}{2\pi a^3}\sqrt{a^2 - r^2}$, the above integral simplifies to

$$\bar{u}_z(r) = \frac{1-\nu^2}{\pi E}\frac{3P}{2\pi a^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^{s_1}\sqrt{a^2 - r^2 - 2rs\cos\phi - s^2}\mathrm{d}s$$

with s_1 given by the positive root of $a^2 - r^2 - 2rs\cos\phi - s^2 = 0$. Therefore, the vertical displacement of points r within the contact surface $(r < a)$ is given by

$$\bar{u}_z(r) = \frac{1-\nu^2}{E}\frac{3P}{8a^3}(2a^2 - r^2)$$

Lastly, compatibility of surface displacement yields

$$\bar{u}_z = \gamma - \frac{1}{2R}r^2 = \frac{1-\nu^2}{E}\frac{3P}{8a^3}(2a^2 - r^2)$$

and, equating coefficients of the second order polynomial of r, expressions for $P(\gamma)$ and for $a(\gamma)$ are obtained as follows

$$\left.\begin{aligned} \frac{1}{2R} &= \frac{1-\nu^2}{E}\frac{3P}{8a^3} \\ \gamma &= \frac{1-\nu^2}{E}\frac{3P}{4a} \end{aligned}\right\} \implies \begin{aligned} a^2 &= \gamma R \\ P &= \frac{4}{3}\frac{E}{1-\nu^2}R^{1/2}\gamma^{3/2} \end{aligned}$$

Solid-solid interactions: contact mechanics

Frictionless indentation of an elastic half-space

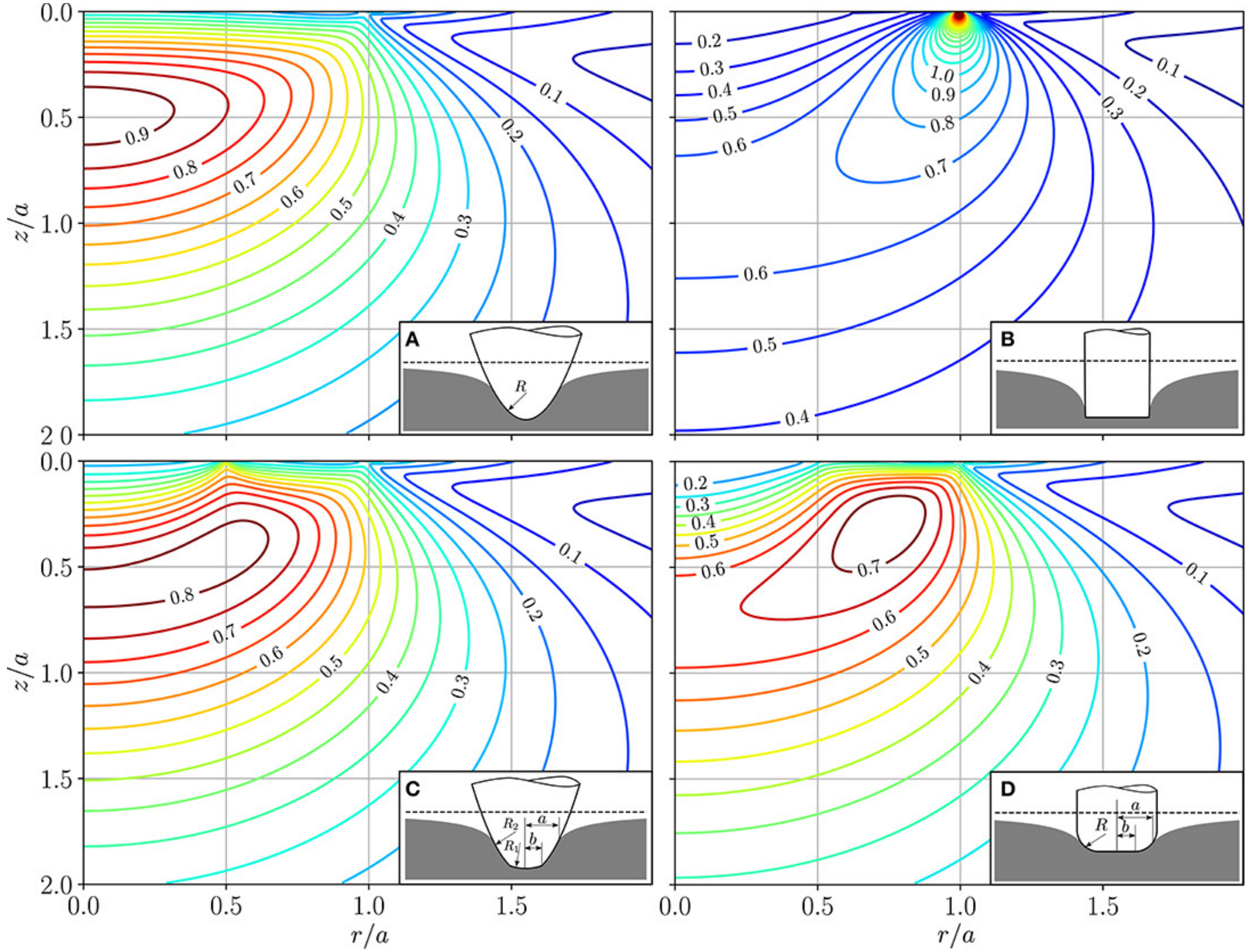


fig.: von Mises stress

+ complex stress field under the indenter
+ most critical state of stress not necessarily at the contact interface
+ not all cases are amenable of analytical solution
+ cylindrical indenter but not an elastic material:

$$P(\gamma) = ?$$

Solid-solid interactions: contact mechanics

Contact between two elastic spheres

Boundary conditions:

+ surface displacement

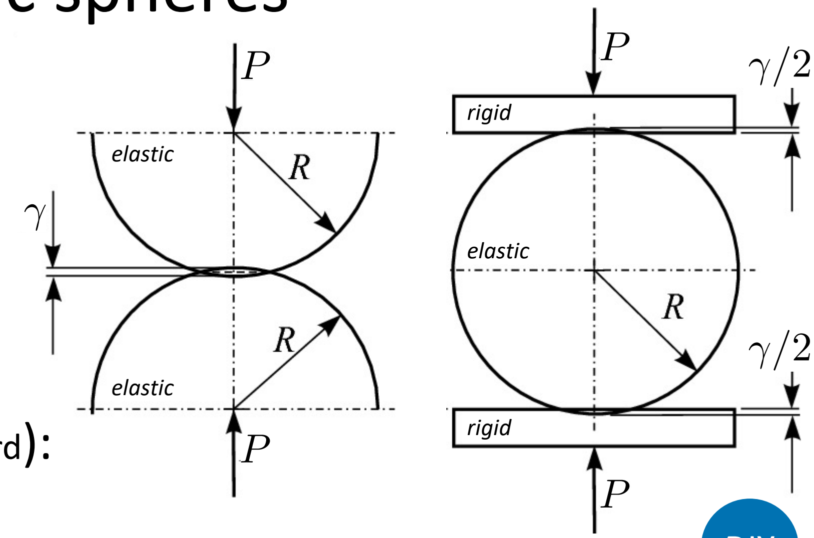
+ frictionless

DIY

$$\mathrm{d}F_r = 0 \quad \mathrm{d}F_z = p(r) = ?$$

Contact radius: $a = \sqrt{\gamma R/2}$

Contact pressure (not straightforward):



DIY

$$\bar{u}_{z,1}(r) + \bar{u}_{z,1}(r) = \gamma - (R_1 - \sqrt{R_1^2 - r^2}) - (R_2 - \sqrt{R_2^2 - r^2})$$

Solid-solid interactions: contact mechanics

Contact between two elastic spheres

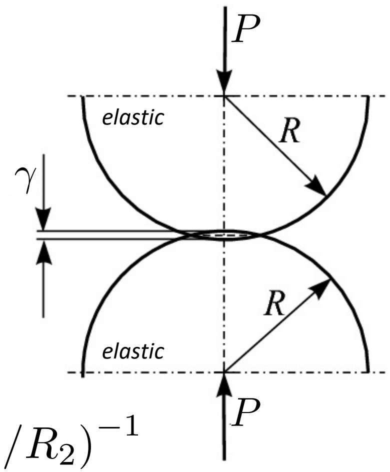
Boundary conditions:

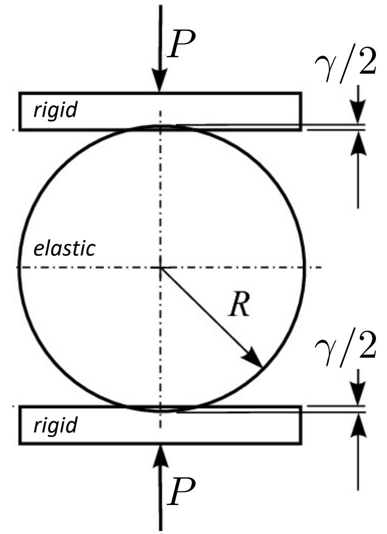
+ surface displacement

$$\bar{u}_{z,1}(r) + \bar{u}_{z,2}(r) \approx \gamma - \frac{1}{2R_1}r^2 - \frac{1}{2R_2}r^2 \quad , \; r \in [0,a]$$

+ frictionless

$$\mathrm{d}F_r = 0 \quad \mathrm{d}F_z = p(r) = ?$$





Contact radius: $a^2 = \gamma\,(1/R_1 + 1/R_2)^{-1}$

Contact pressure (not straightforward):

$$p(r) = \frac{3P}{2\pi a^3}\sqrt{a^2 - r^2} \quad , \quad r \in [0,a]$$

Contact law:

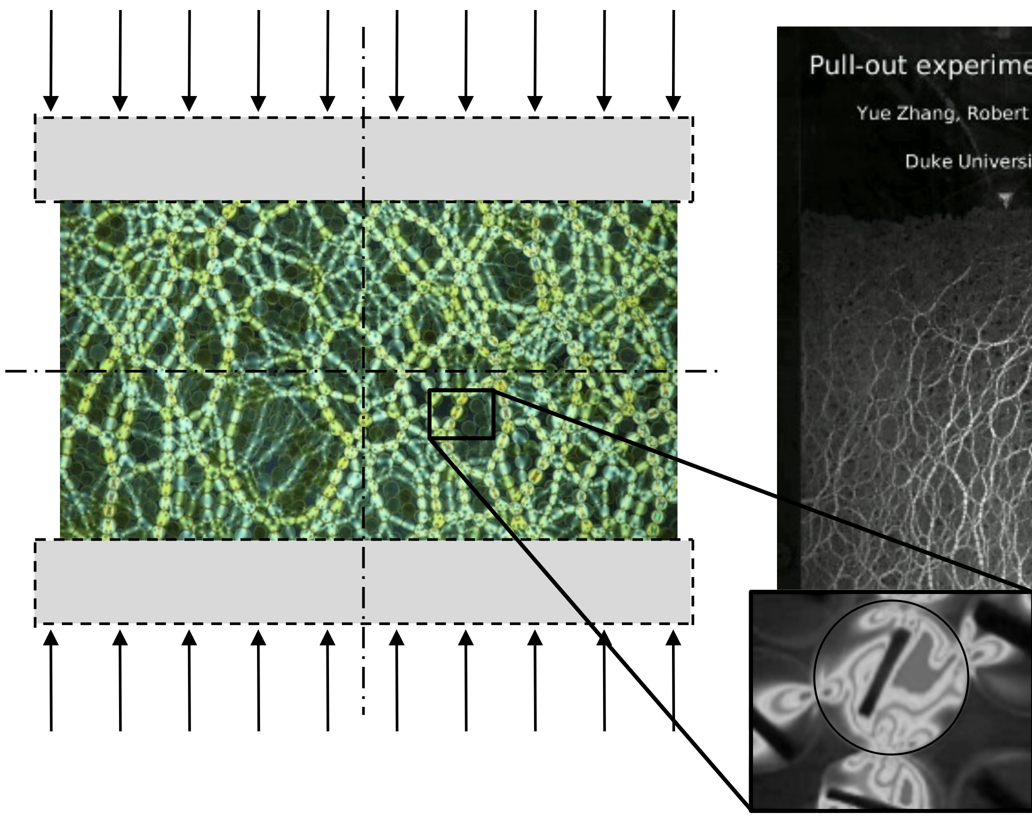
$$P(\gamma) = \frac{4}{3}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)^{-1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1/2}\gamma^{3/2}$$

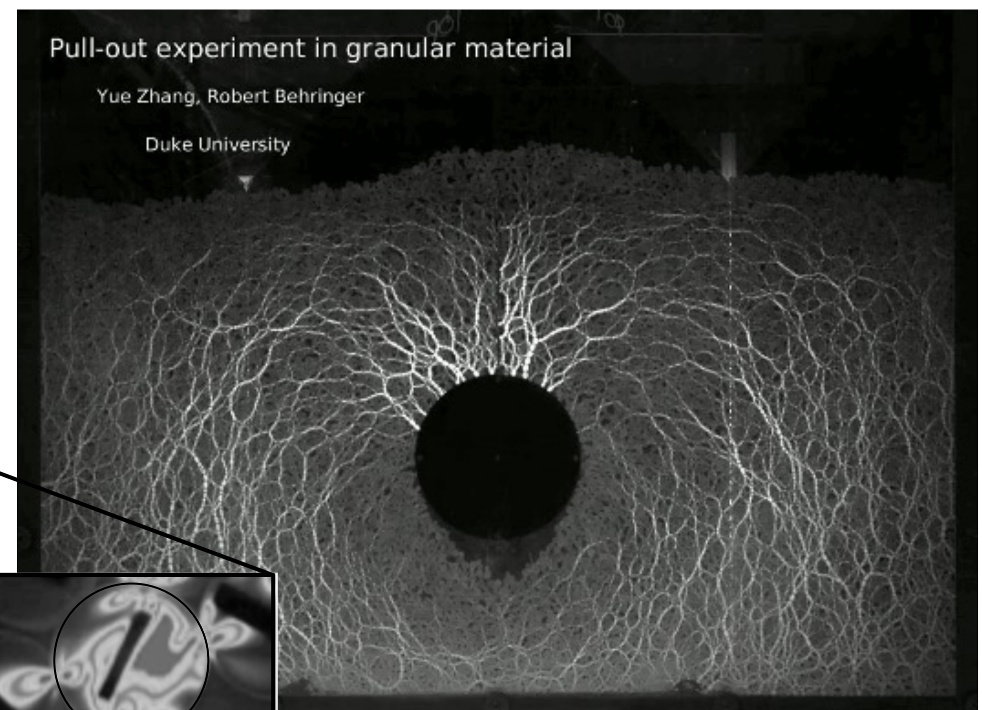
$$a^2 = \gamma\,(1/R_1 + 1/R_2)^{-1}$$

Solid-solid interactions: contact mechanics

Contact mechanics at the core behavior of granular media

- How amorphous solids support stress?



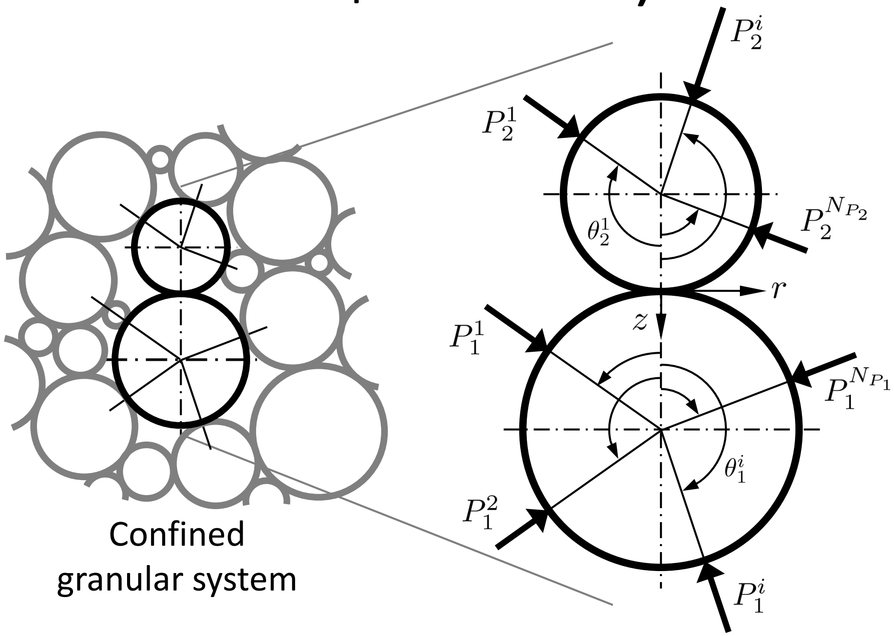


Prof. Behringer. Duke University

Solid-solid interactions: contact mechanics

Dominant mechanisms (a mechano-chemical point of view)

- Elastic deformations
- Plastic deformations
- Bonding/Solid bridge
- Strain-rate mechanisms
- Fracture
- Transport phenomena



Confined granular system

Bridging length-scales

- Analytical up-scaling of continuum contact mechanics
  (geometric nonlinearities and material nonlinearities)

Underlying assumption

- Contacts between particles are independent
  (interaction between particles is not affected by neighboring particles)

Solid-solid interactions: contact mechanics

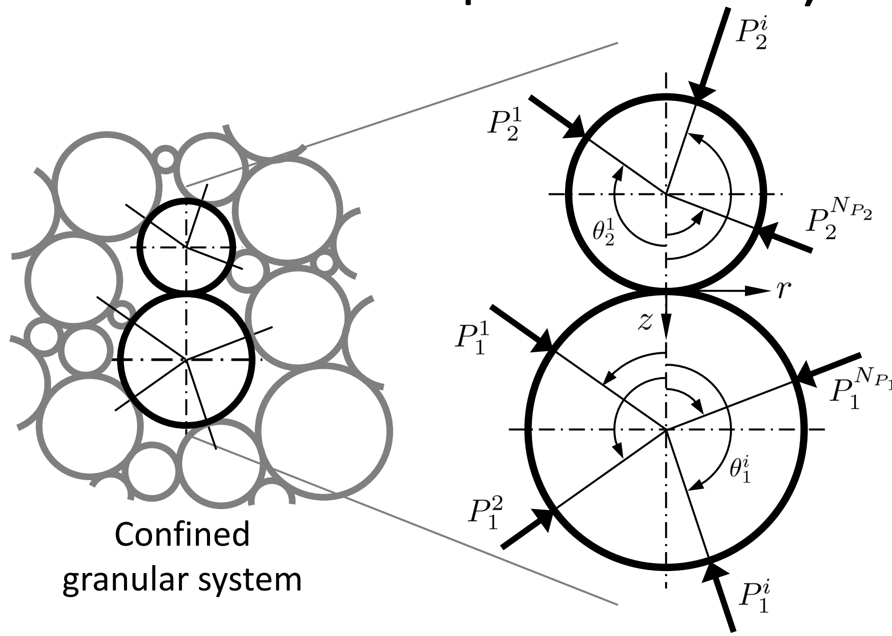
Any questions?

Appendix: Granular systems at high levels of confinement

Granular systems at high levels of confinement

Dominant mechanisms (a mechano-chemical point of view)

- Elastic deformations
- Plastic deformations
- Bonding/Solid bridge
- Strain-rate mechanisms
- Fracture
- Transport phenomena



Bridging length-scales

- Analytical up-scaling of continuum contact mechanics
  (geometric nonlinearities and material nonlinearities)

Underlying assumption

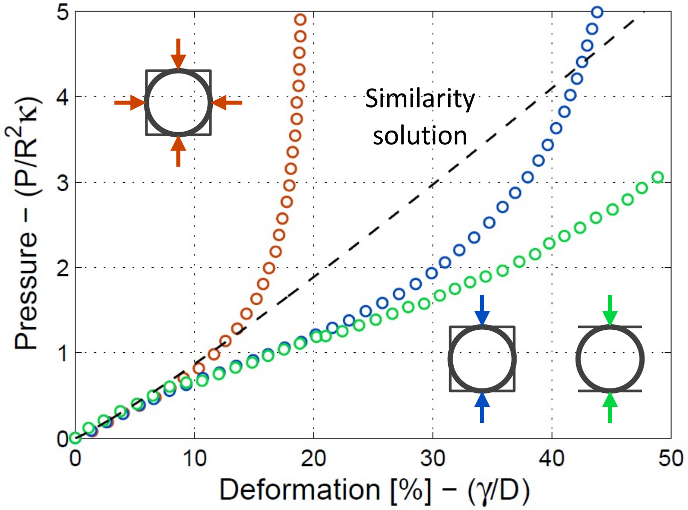
- Contacts between particles are independent
  (interaction between particles is not affected by neighboring particles)

Granular systems at high levels of confinement

Technological/Scientific impact

Not an issue until recent years

(e.g., there is a need to be predictive at high relative densities)



Elastoplastic spherical particles

Mesarovic & Fleck (2000)
Harthong et al. (2009)

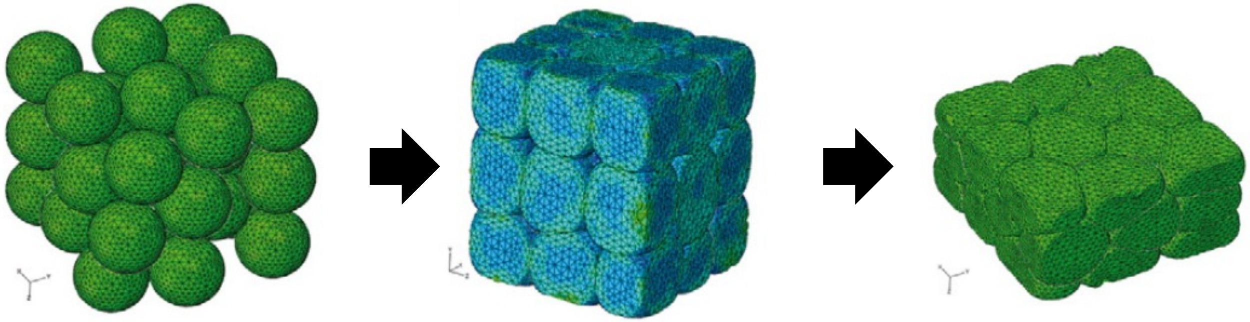
Case-by-case …

… empirical scaling of analytical solutions (e.g. Storåkers similarity solution)

… curve-fitting of finite-element solutions

Elastoplastic spherical particles

Jerier et al. (2011)



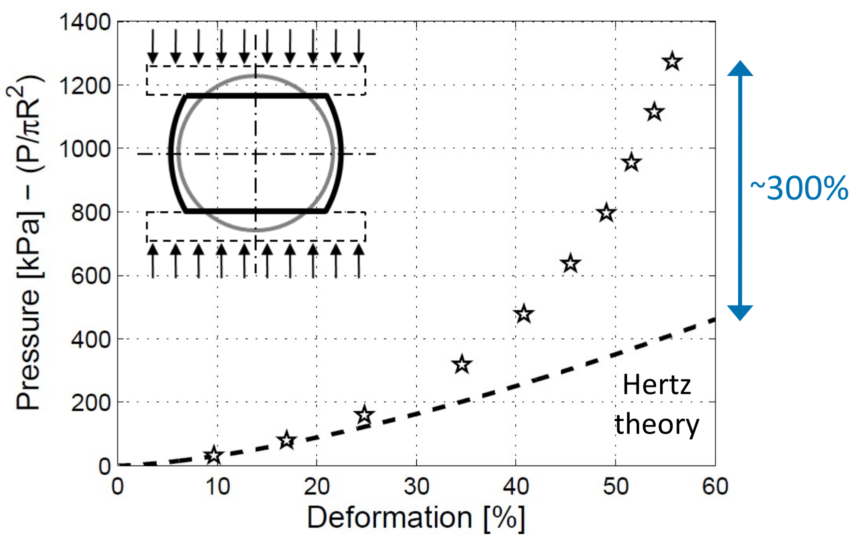
Granular systems at high levels of confinement

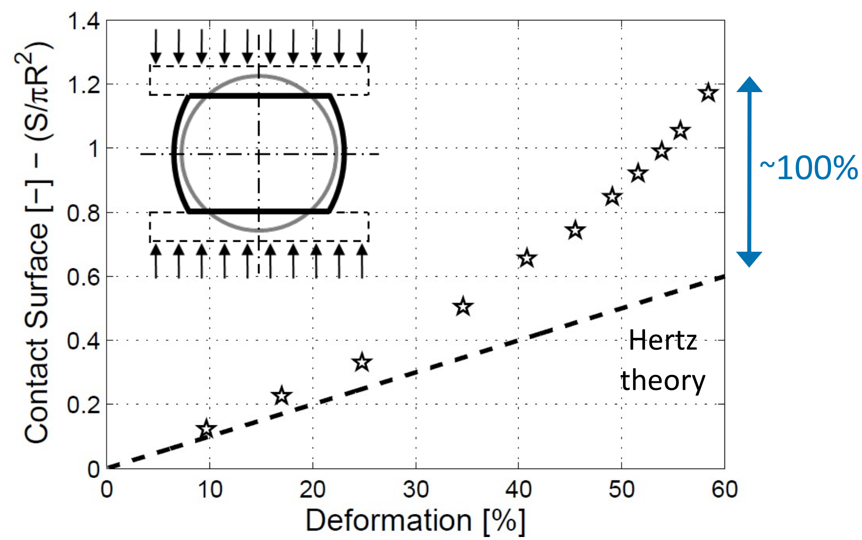
Question:

To what extent contacts between particles are independent?

Restrict attention to:

- Elastic spheres
- Absence of gravitational forces, adhesion and friction





☆ **Tatara (1991)**: experimental data, rubber sphere of radius 10 mm, no hysteresis, no permanent deformations, E = 1.85 MPa, v = 0.46

Granular systems at high levels of confinement

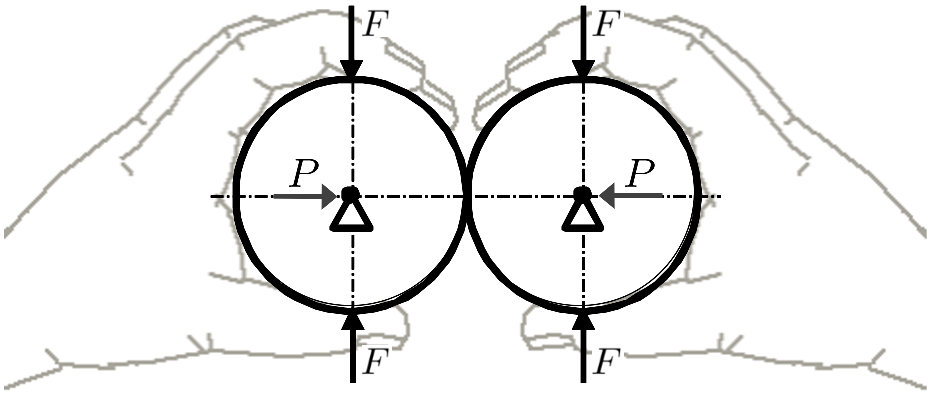
Question:

To what extent contacts between particles are independent?

Restrict attention to:

- Elastic spheres
- Absence of gravitational forces, adhesion and friction

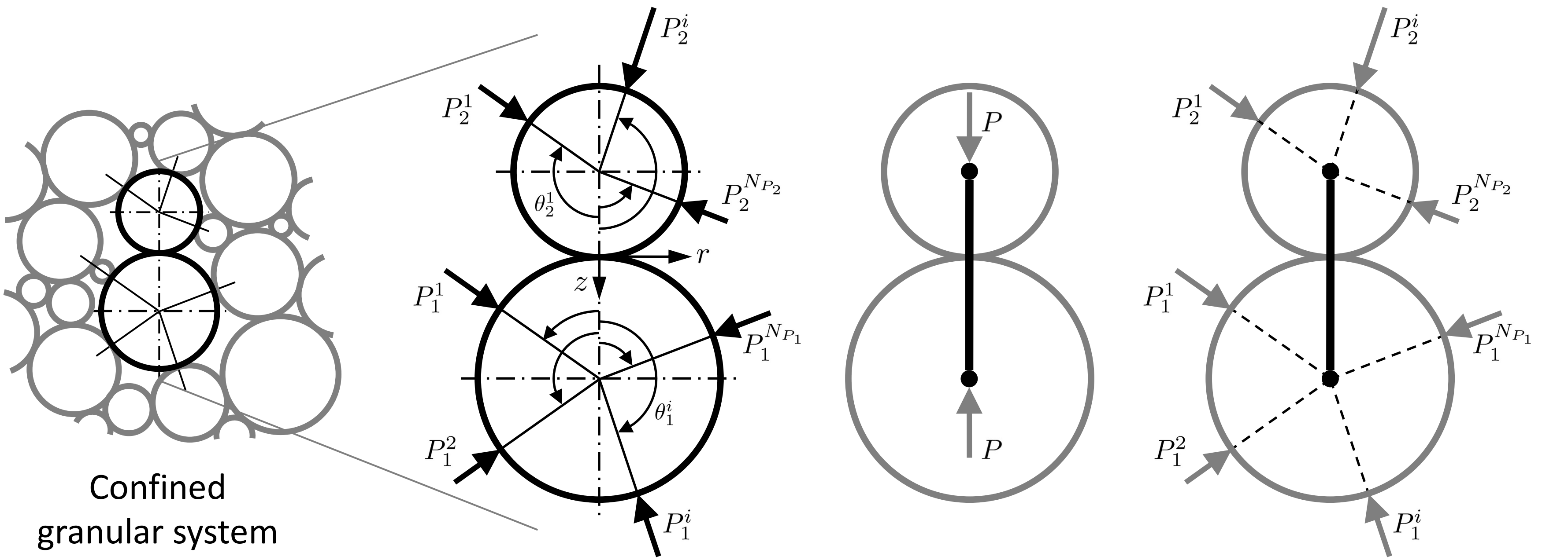
Intuition and observation suggest a gap in knowledge



Any contact law $P = 0$

... we know that $P \neq 0$

Elastic interactions beyond Hertz theory



Confined granular system

Schematic of loading configuration

Hertz theory approach

Nonlocal theory approach

Hertz theory (1881)

$$P(\gamma) = n^{\mathrm{H}} \gamma^{3/2} \qquad \frac{1}{n^{\mathrm{H}}} = \frac{3}{4}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{1/2}$$

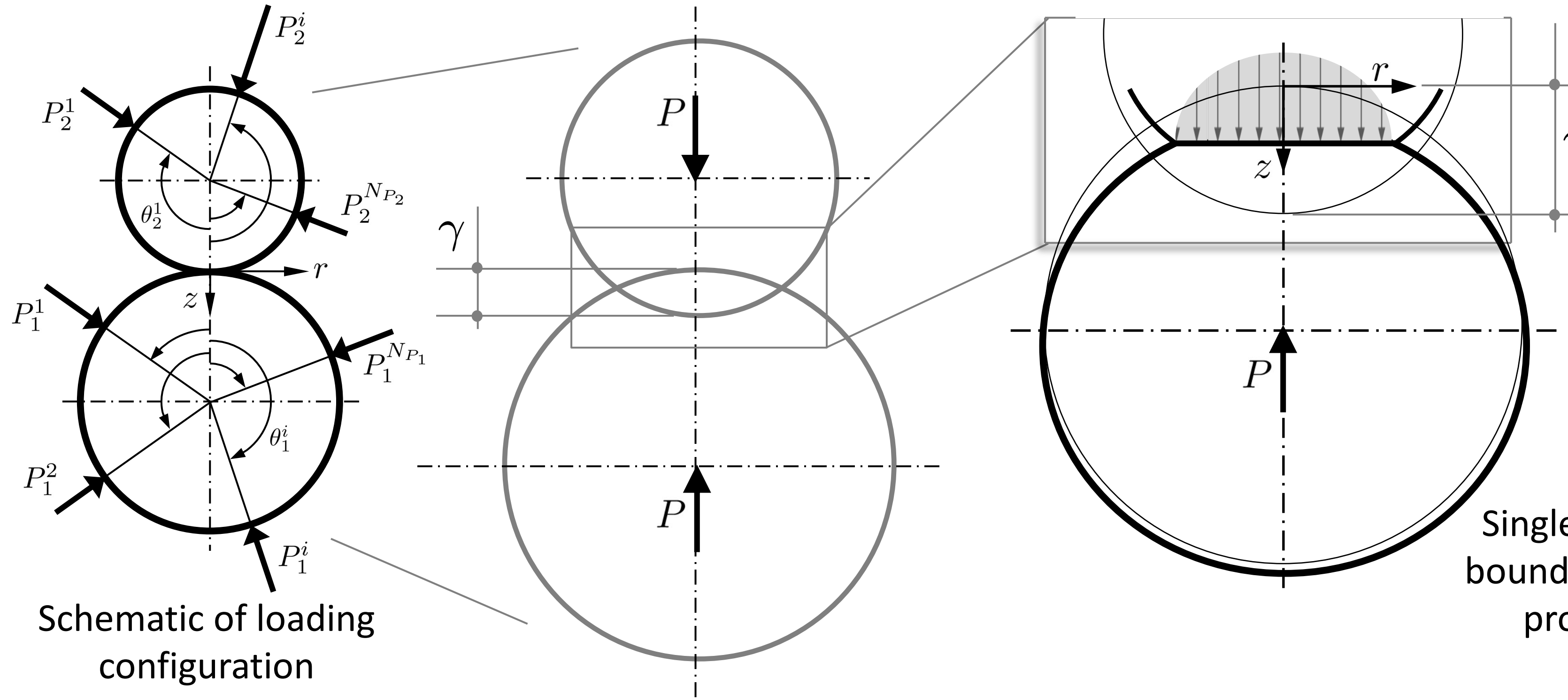
Geometry and elastic properties

Relative position between particles

$$\gamma = R_1 + R_2 - \|x_1 - x_2\|$$

Elastic interactions beyond Hertz theory

Hertz theory:



Schematic of loading configuration

Single-loaded boundary-value problem

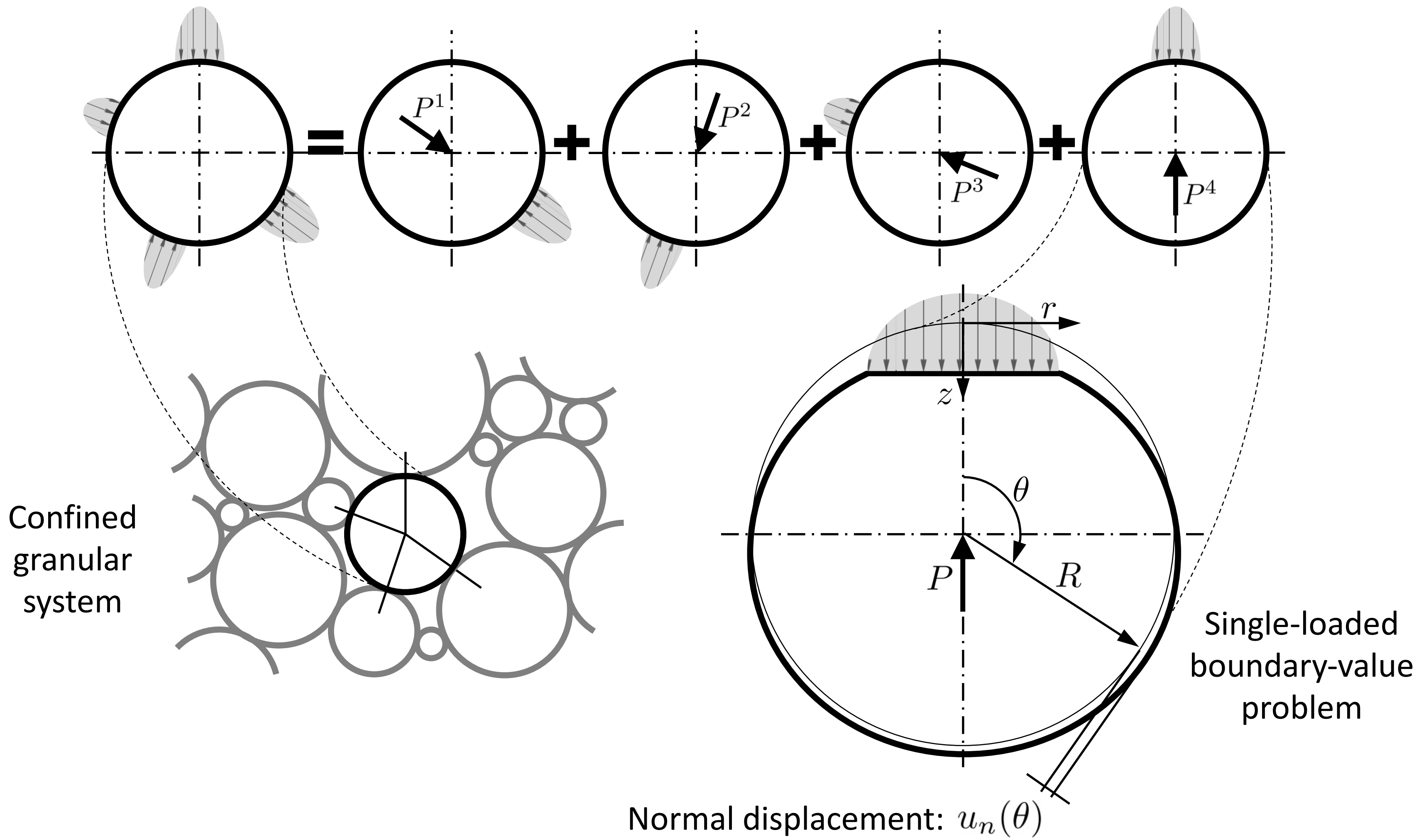
Compatibility condition: $\gamma = R_1 - \sqrt{R_1^2 - r^2} + R_2 - \sqrt{R_2^2 - r^2} + w_1(r) + w_2(r)$

Flat circular contact surface
Spherically-distributed contact pressure
(based on Boussinesq's solution)

$$w_1(r) = \frac{3P(1-\nu_1^2)}{8a^3 E_1}(2a^2 - r^2)$$

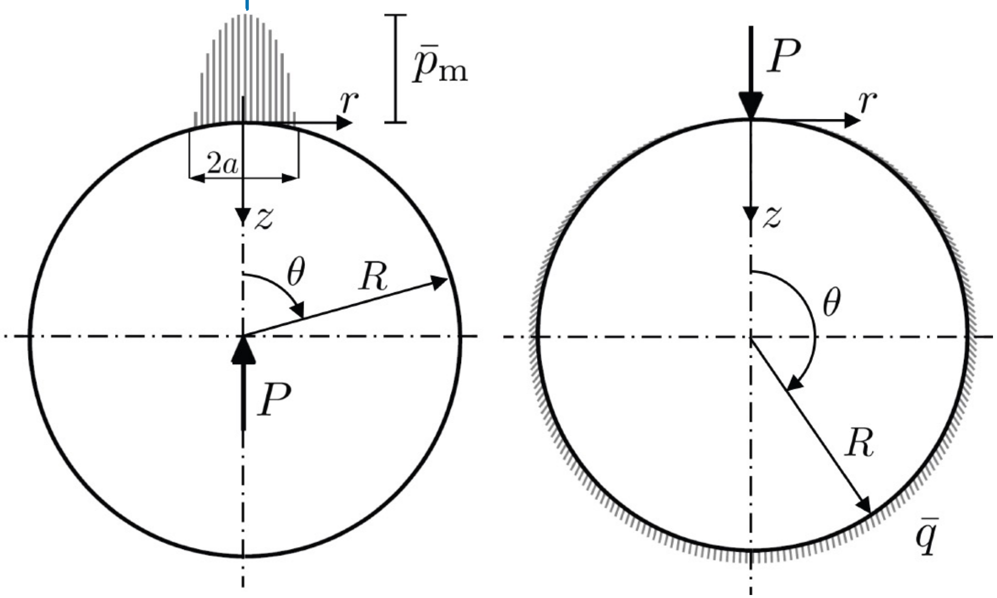
Elastic interactions beyond Hertz theory

Principle of superposition:



Elastic interactions beyond Hertz theory

Single-loaded boundary-value problem:



Exact boundary-value problem

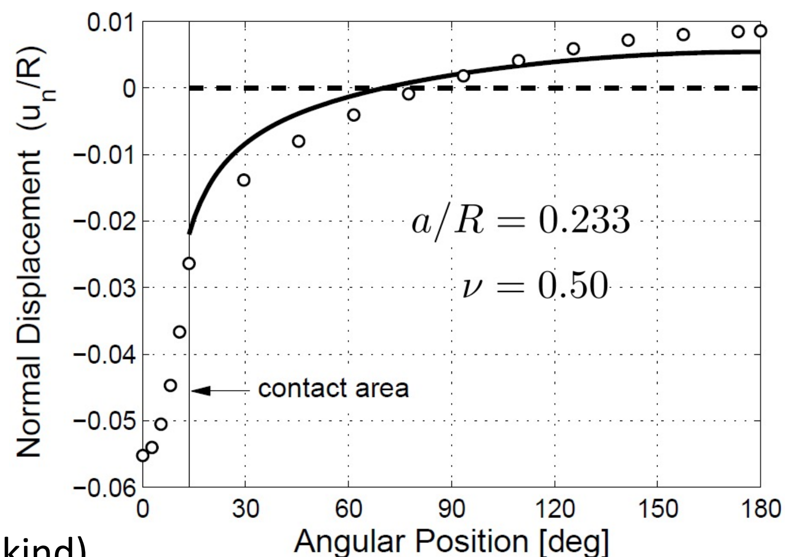
Approximate problem

Contact surface (Hertz approximation):

$$u_z(r) = \frac{3P(1-\nu^2)}{8a^3E}\left(2a^2 - r^2\right)$$

Traction free surface:

$$u_n(\theta) = \ldots$$

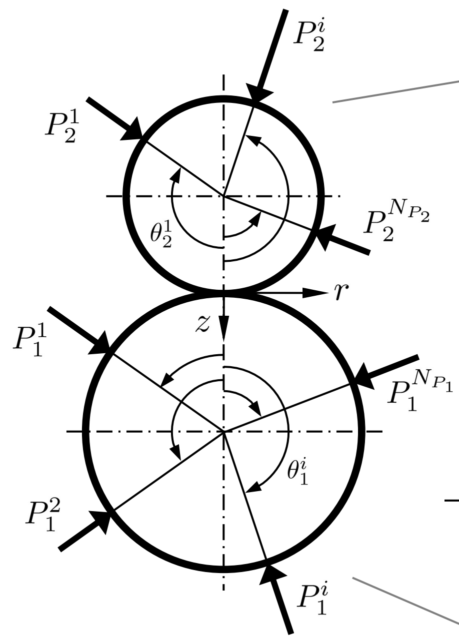


Exact solution : Zhupanska (2011)
(reduces to a Fredholm integral equation of the second kind)

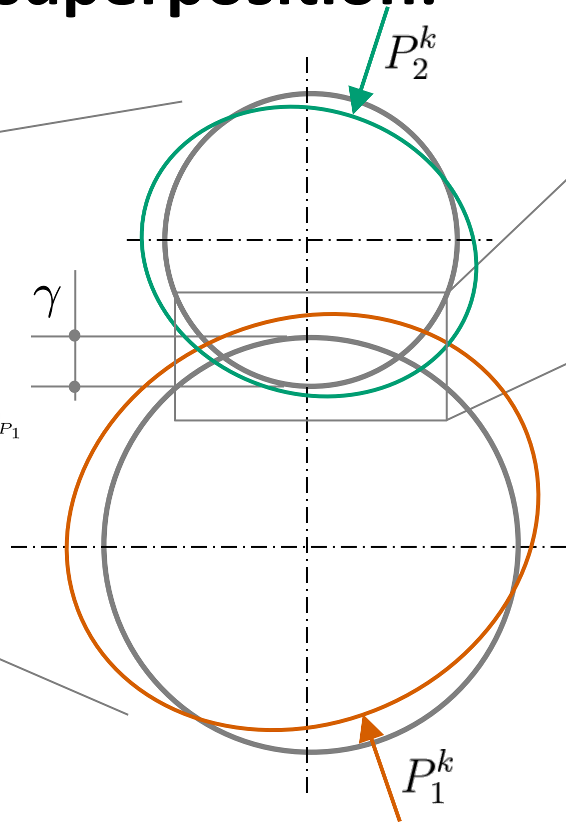
Approximate problem (based on Boussinesq's solution): $\|\bar{q}\|/\bar{p}_{\mathrm{m}} = \mathcal{O}(a^2/R^2)$

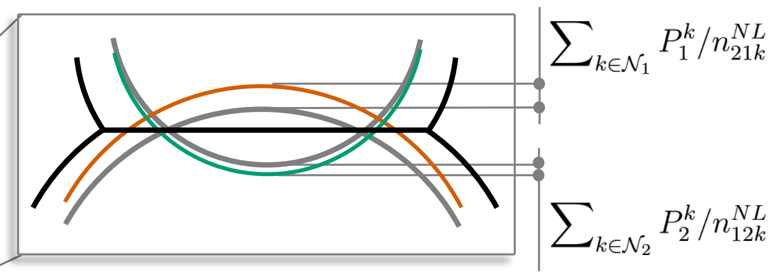
Nonlocal contact formulation

Principle of superposition:



Schematic of loading configuration





Hertz's hypotheses:

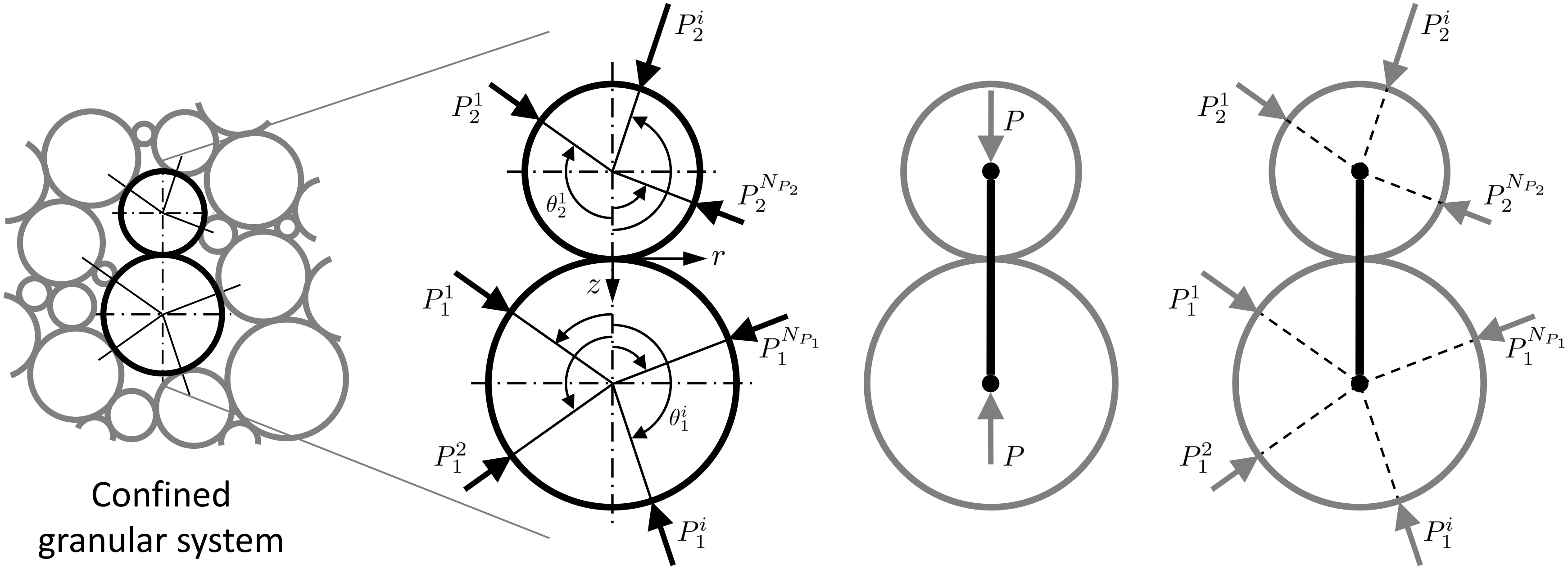
- Independent contacts (**RELAXED**)
- Flat circular contact surface
- Linear elasticity
- Spherically-distributed contact pressure

Amenable to analytical integration!

$$n_{jik}^{NL} = \frac{4\pi R_i E_i \sin(\theta_{jik}/2)}{(1+\nu_i)\left[-2(1-\nu_i) - 2(1-2\nu_i)\sin(\theta_{jik}/2) + (7-8\nu_i)\sin(\theta_{jik}/2)^2\right]}$$

Nonlocal contributions: $\gamma_{ij}^{NL} = \sum_{k\in\mathcal{N}_i} P_i^k/n_{jik}^{NL} + \sum_{k\in\mathcal{N}_j} P_j^k/n_{ijk}^{NL}$

Nonlocal contact formulation



Confined granular system

Schematic of loading configuration

Hertz theory approach

Nonlocal theory approach

Nonlocal contact formulation

$$P(\gamma) = n_H \left(\gamma + \gamma_{NL}\right)^{3/2}$$

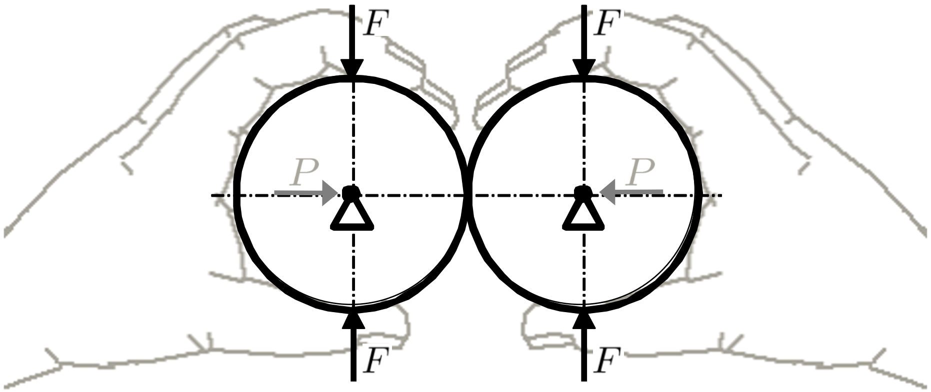
Geometry, elastic properties and loading configuration

Nonlocal contributions (analytical expression)

Gonzalez M. and Cuitiño A.M., "A nonlocal contact formulation for confined granular systems", *Journal of the Mechanics and Physics of Solids* 2012; 60:333-350.

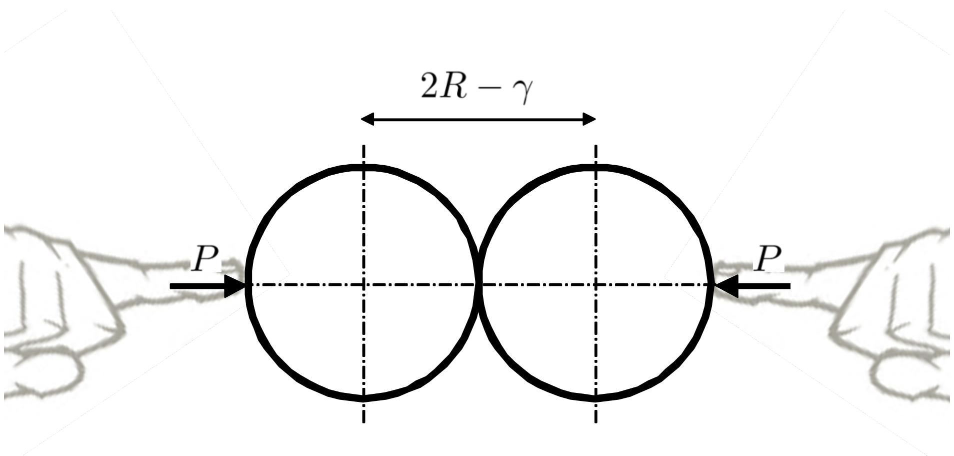
Nonlocal contact formulation

Two intuitive examples:



Hertz theory: $P = 0$

Nonlocal formulation: $P = \frac{n_H}{n_{NL}^{3/2}} F^{3/2}$



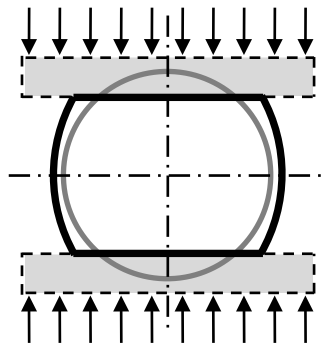
Hertz theory: $P = n_H \gamma^{3/2}$

Nonlocal formulation:

$$P = n_H \gamma^{3/2} + \mathcal{O}(\gamma^2)$$

$$P = n_H \gamma^{3/2} + \frac{3n_H^2}{2n_{NL}} \gamma^2 + \mathcal{O}(\gamma^{5/2})$$

Nonlocal contact formulation - Validation



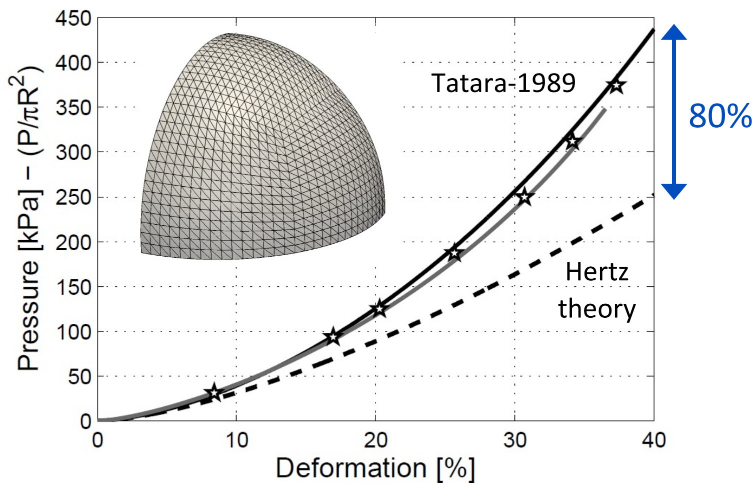
Experimental setup

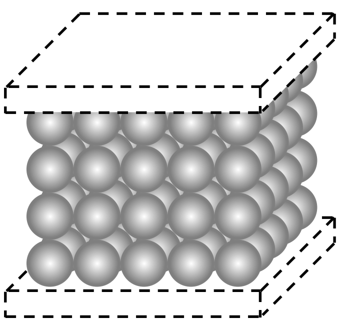
Finite-element model.

- Elements: ~1.00.000
- Nodes: ~1.500.000
- CPU-time: hours

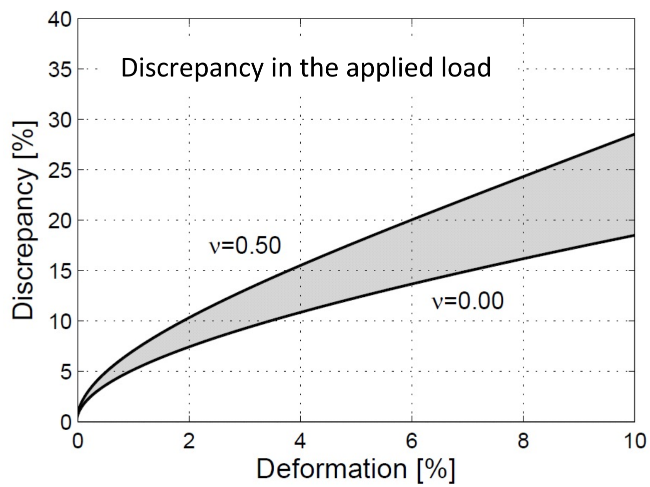
Nonlocal formulation.

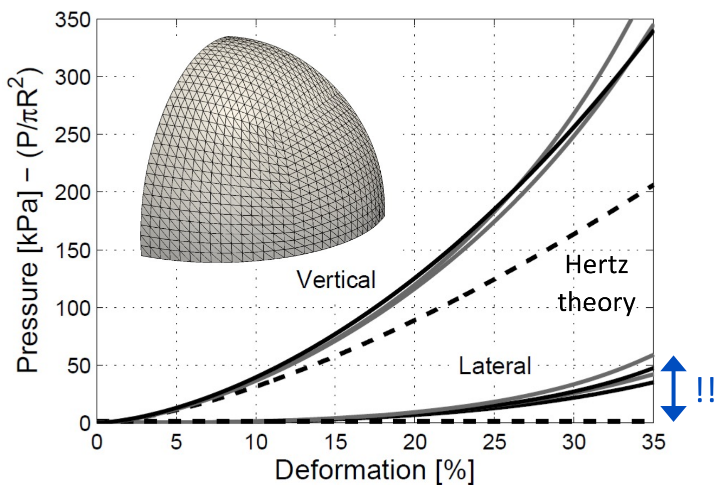
- Memory requirements: none
- CPU-time: few seconds





SC granular crystal





Difference depends only on Poisson's ratio

$$\mathcal{D} = \frac{P(\gamma) - n_H \gamma^{3/2}}{n_H \gamma^{3/2}} = \frac{1}{2\pi}\left(\frac{3-2\nu}{1-\nu}\right)\epsilon^{1/2} + \frac{7}{24\pi^2}\left(\frac{3-2\nu}{1-\nu}\right)^2 \epsilon + \mathcal{O}(\epsilon^{5/4})$$

Nonlocal contact formulation – Granular crystals

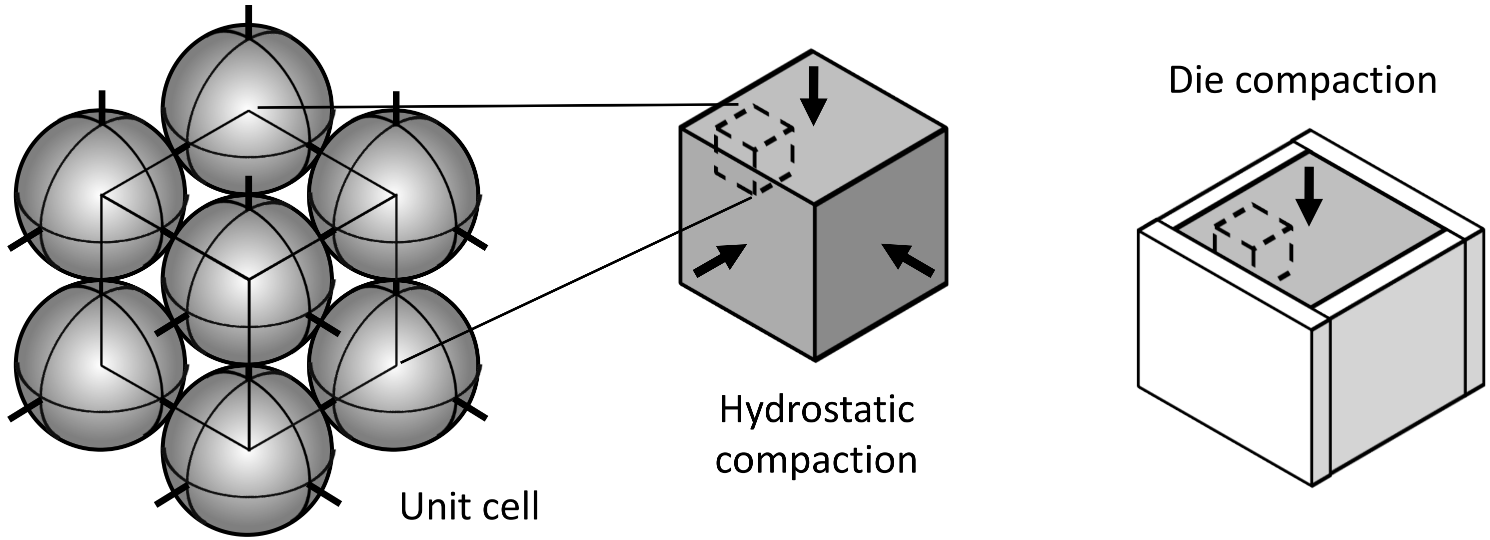
Question:

To what extent contacts between particles are independent?

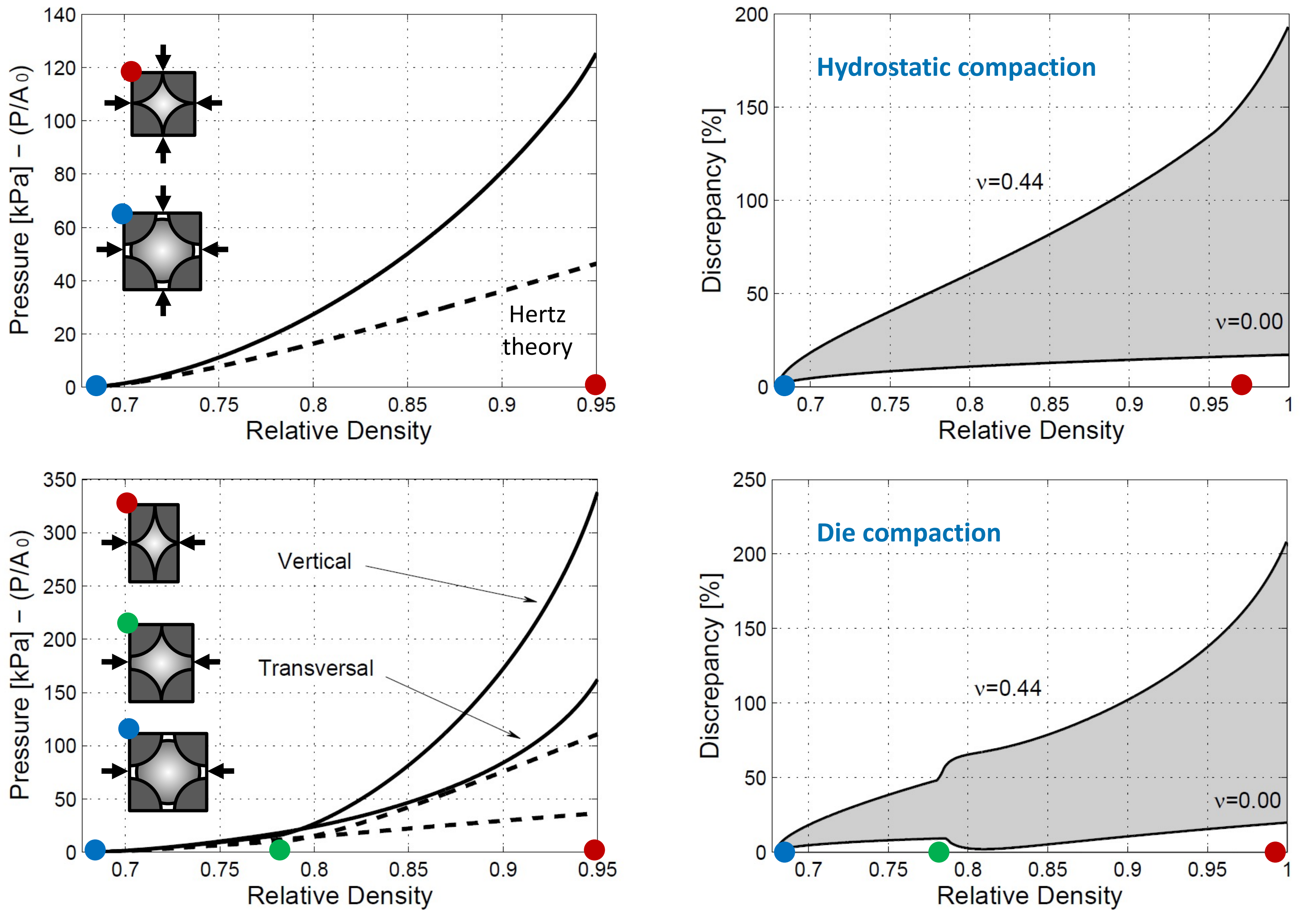
Compaction of body centered cubic packing:

Two different loading conditions: *Hydrostatic compaction*
Die compaction

Difference between Hertz theory and the nonlocal contact formulation.

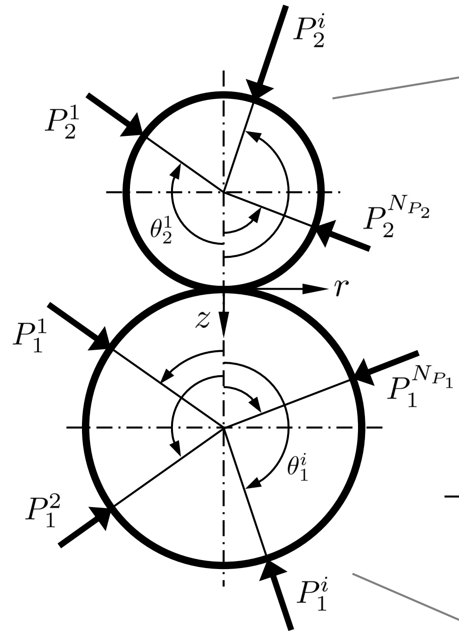


Nonlocal contact formulation – Granular crystals

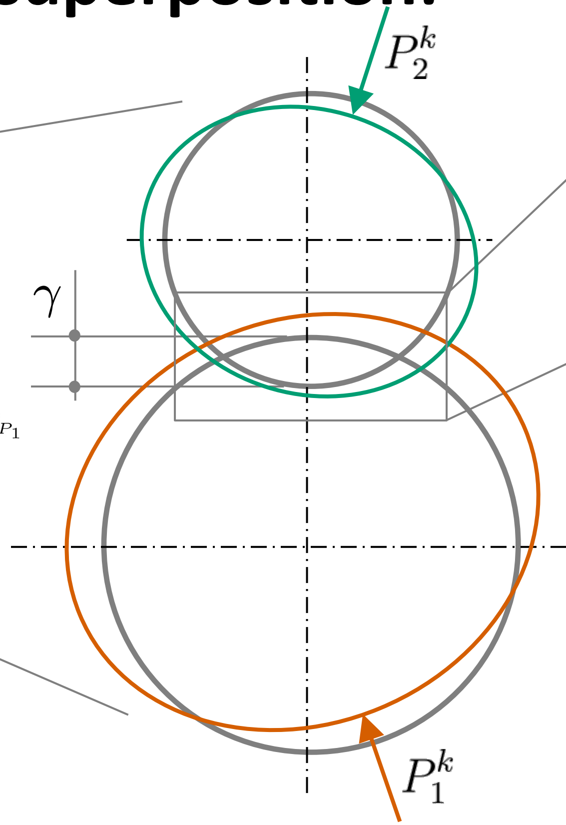


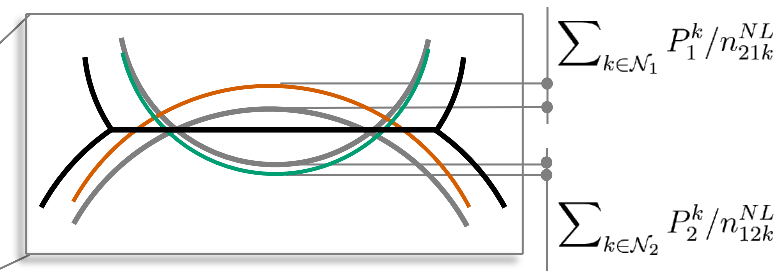
Nonlocal contact formulation – Can we do better?

Principle of superposition:



Schematic of loading configuration





Hertz's hypotheses:

- Independent contacts (**RELAXED**)
- Flat circular contact surface
- Linear elasticity
- Spherically-distributed contact pressure

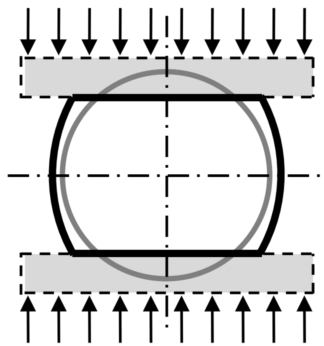
Amenable to analytical integration!

Curvature correction: $\gamma = R_1 - \sqrt{R_1^2 - r^2} + R_2 - \sqrt{R_2^2 - r^2} + w_1(r) + w_2(r) - \gamma_{NL}$

$\approx \frac{r^2}{2R_2^2} + \frac{r^4}{8R_2^3} + \frac{r^6}{16R_2^5}$

Contact radius correction: $a + a_{NL}$

Nonlocal contact formulation - Validation



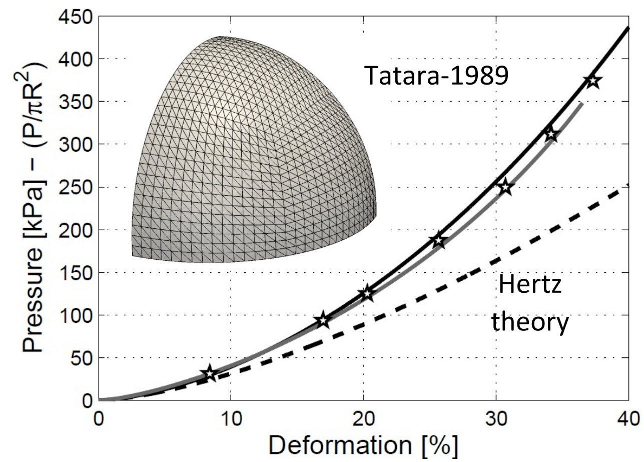
Experimental setup

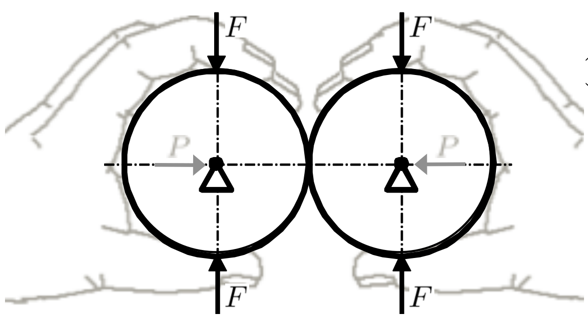
Finite-element model.

- Elements: ~1.00.000
- Nodes: ~1.500.000
- CPU-time: hours

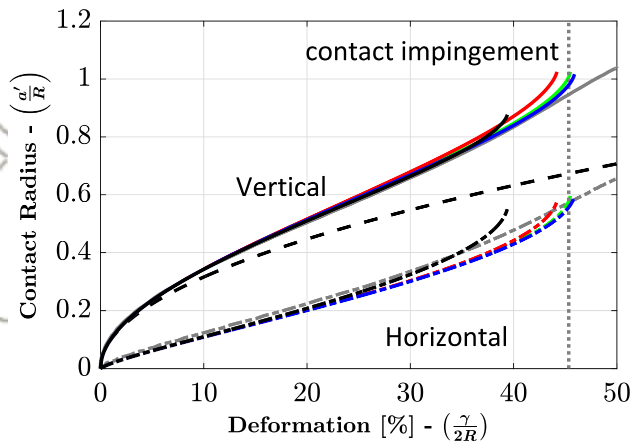
Nonlocal formulation.

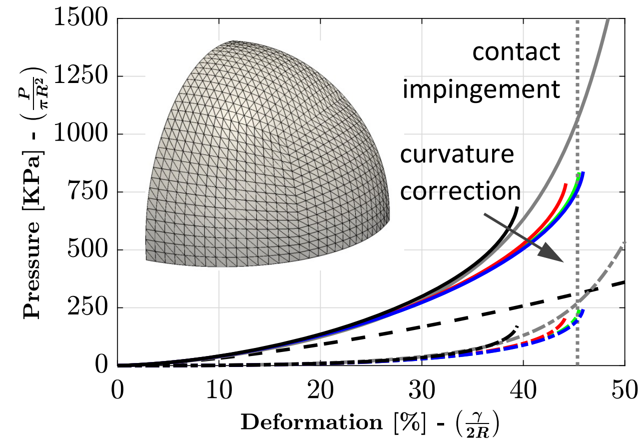
- Memory requirements: none
- CPU-time: few seconds





Die Compaction





Milestone: Development and validation of a nonlocal contact formulation

Insight: The discrepancy with Hertzian prediction depends on Poisson's ratio

Agarwal A. and Gonzalez M., "Contact radius and curvature corrections to the nonlocal contact formulation …", *International Journal of Engineering Science* 2018; 133, 26-46.