Spring, 2022

ME 597 – Solid Mechanics II

Lecture 19
Review for the exam

News: Ready for the exam?

PURDUE UNIVERSITY | Mechanical Engineering

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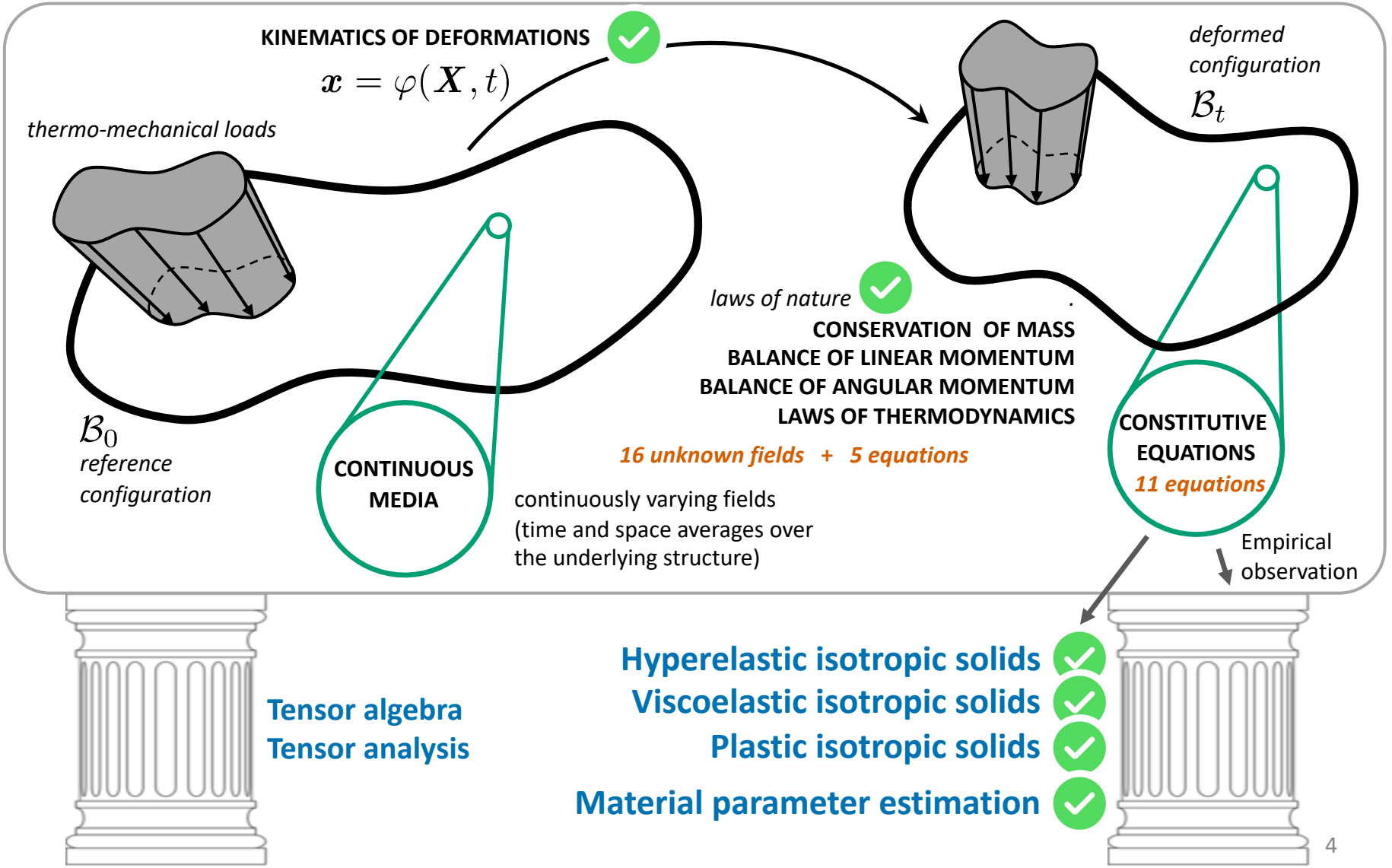
ME 597 – Course outline - Syllabus

- Tensor algebra and tensor analysis (Lecture 2)
- Kinematics of deformations (Lecture 3-4)
- Mechanical conservation and balance laws (Lecture 5)
- Thermodynamics (Lecture 6)
- Constitutive relations (Lecture 7)
- Hyperelastic solids (Lecture 8-9)
- Viscoelastic solids (Lecture 10-11)
- Material parameter estimation (Lecture 12)
- Plasticity (Lecture 13-15)
- Homework set 1, 2, 3 & 4

ME 597 – Course outline - Syllabus



Introduction to tensors and vectors

Tensor algebra and analysis in Cartesian coordinates

$$B_{ij} = A_{ji} \Longleftrightarrow \boldsymbol{B} = \boldsymbol{A}^T$$

$$D_{ijk} = A_{ij} v_k \Longleftrightarrow \boldsymbol{D} = \boldsymbol{A} \otimes \boldsymbol{v} \qquad \boldsymbol{A} : \boldsymbol{B} = A_{ij} B_{ij}$$

$$v_i = A_{ik} u_k \Longleftrightarrow \boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}$$

1-order tensor (vector)

$$v_i \equiv \boldsymbol{v}[\boldsymbol{e}_i] \quad \boldsymbol{v} = v_i \boldsymbol{e}_i$$

2-order tensor

$$A_{ij} \equiv \boldsymbol{T}[\boldsymbol{e}_i, \boldsymbol{e}_j] \quad \boldsymbol{A} = A_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$$

Symmetric, positive-definite 2-order tensor $\quad \lambda_\alpha^{\boldsymbol{S}} \in \mathbb{R} \;\; , \;\; \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \cdot \boldsymbol{\Lambda}_\beta^{\boldsymbol{S}} = \delta_{\alpha\beta}$

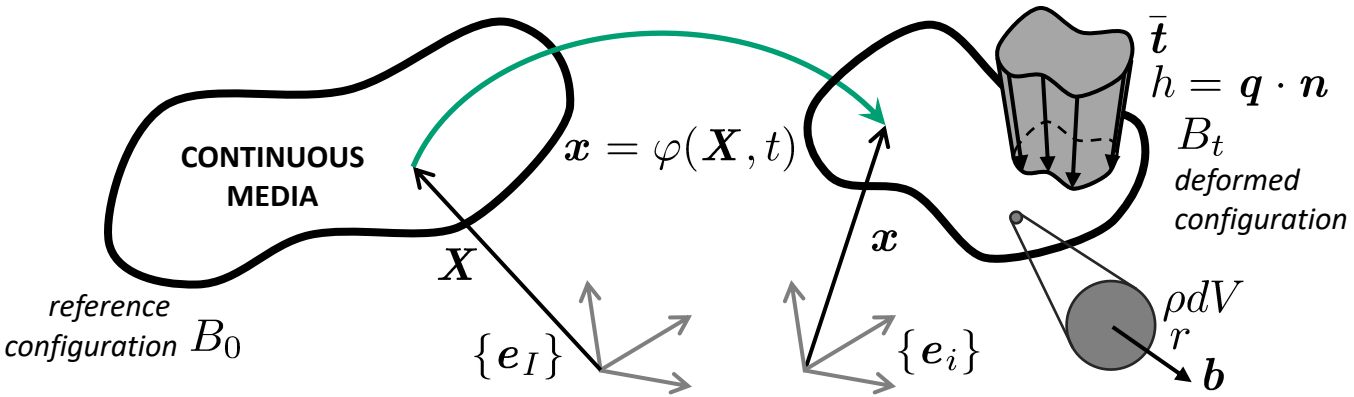
$$\boldsymbol{S} = \sum_{\alpha=1}^{3} \lambda_\alpha^{\boldsymbol{S}} \, \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \otimes \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} = S_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \qquad \sqrt{\boldsymbol{S}} = \sum_{\alpha=1}^{3} \sqrt{\lambda_\alpha^{\boldsymbol{S}}} (\boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \otimes \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}})$$

- Gradient $\quad \nabla \boldsymbol{B} = \dfrac{\partial \boldsymbol{B}(\boldsymbol{x})}{\partial x_i} \otimes \boldsymbol{e}_i \qquad \nabla s = \dfrac{\partial s(\boldsymbol{x})}{\partial x_i} \boldsymbol{e}_i$

- Divergence $\quad \operatorname{div} \boldsymbol{B} = \dfrac{\partial \boldsymbol{B}(\boldsymbol{x})}{\partial x_i} \boldsymbol{e}_i \qquad \operatorname{div} \boldsymbol{v} = \dfrac{\partial \boldsymbol{v}(\boldsymbol{x})}{\partial x_i} \cdot \boldsymbol{e}_i$

Kinematics of deformations

- Deformation mapping $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$
- Deformation gradient $\boldsymbol{F} = F_{iJ}(\boldsymbol{e}_i \otimes \boldsymbol{e}_J) = \dfrac{\partial x_i}{\partial X_J}(\boldsymbol{e}_i \otimes \boldsymbol{e}_J)$
- Right Cauchy-Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$
- Left Cauchy-Green deformation tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$
- Jacobian $J = \det \boldsymbol{F}$ - Nanson's eqn. $\boldsymbol{n}dA = J\boldsymbol{F}^{-T}\boldsymbol{N}dA_0$
- Lagrangian strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$
- Material tensors, spatial tensors, two-point tensors.



Kinematics of deformations

Right and left polar decomposition $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$ $\qquad \boldsymbol{U} = \sqrt{\boldsymbol{C}}$

Q: How to calculate $\boldsymbol{U}$ or $\boldsymbol{V}$?
A: Using the spectral decomposition…

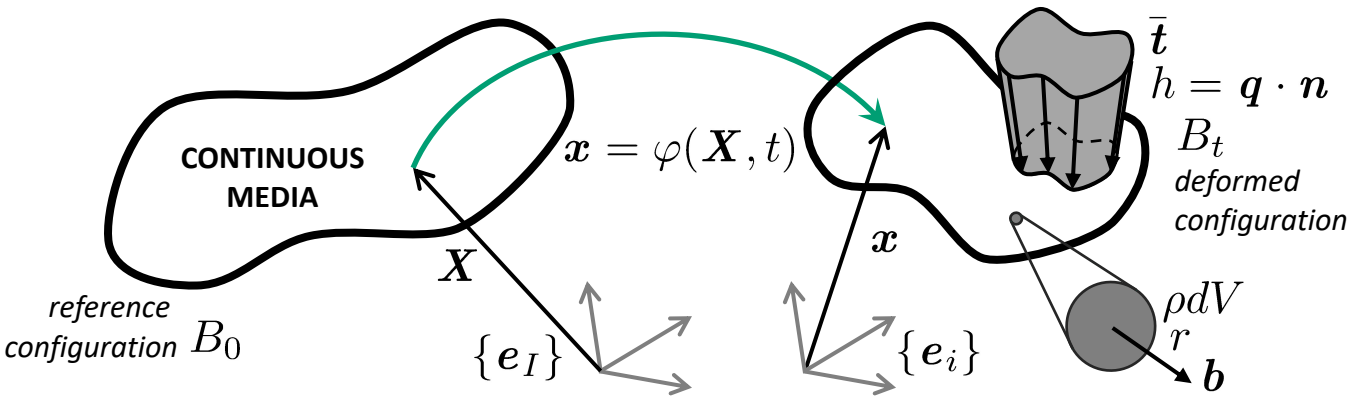
Q: How to calculate $\boldsymbol{R}$? A: Using $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$

Linear parts of kinematic fields

- Examples (for $\boldsymbol{X} \to \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$, i.e., $\boldsymbol{F} = \boldsymbol{I},\ \nabla_0 \boldsymbol{u} = \nabla \boldsymbol{u}$):
  [the linearization is evaluated in the undeformed configuration]

$$\langle \nabla_{\varphi} \boldsymbol{E}; \boldsymbol{u} \rangle = \tfrac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] = \boldsymbol{\epsilon}$$

small-strain tensor
(*employed in elasticity theory*)



Conservation and balance laws

Stress tensors

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$$ **Cauchy stress tensor**

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$ **first Piola-Kirchhoff stress tensor**

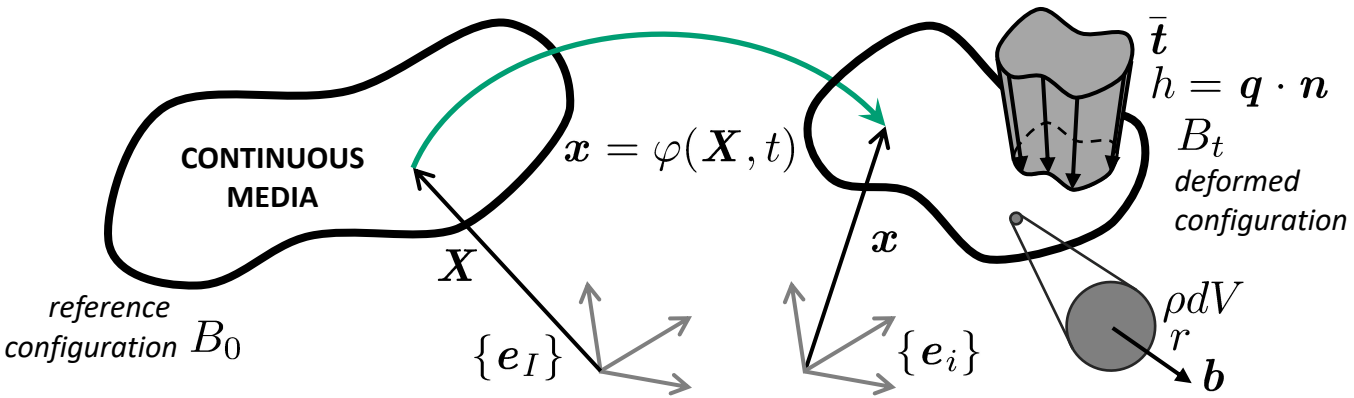
$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$ **second Piola-Kirchhoff stress tensor**

Additive decomposition of the Cauchy stress tensor

$$\sigma_{ij} = s_{ij} - p\delta_{ij} \iff \boldsymbol{\sigma} = \boldsymbol{s} - p\boldsymbol{I}$$

$p = -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}$ is the hydrostatic stress or pressure

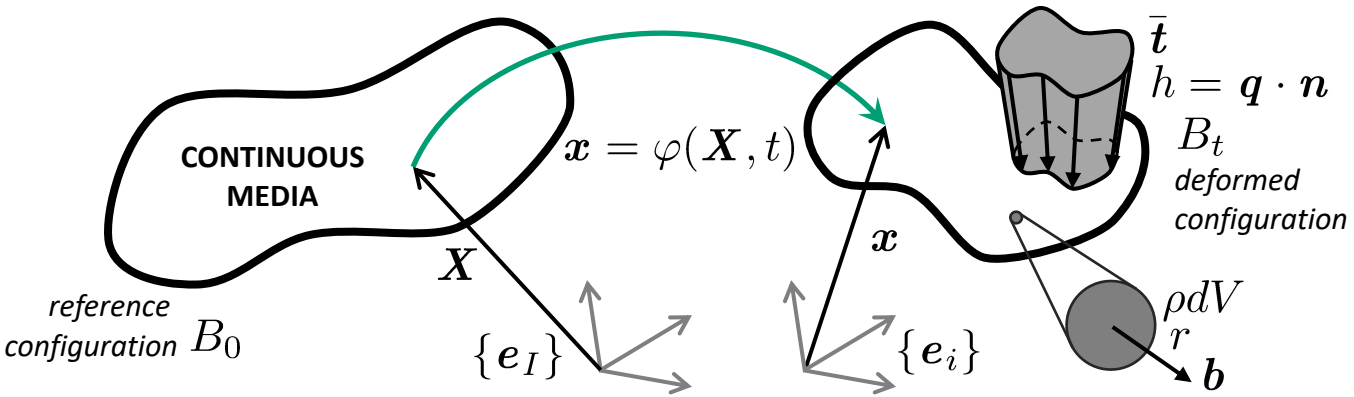
$\boldsymbol{s}$ is the deviatoric part of the stress tensor
(only includes information on shear stress)



Constitutive relations

Conservation and balance laws

$J\rho = \rho_0$	$\forall \boldsymbol{X} \in B_0$	**conservation of mass**	(1 equation)
$\mathrm{div}\boldsymbol{\sigma} + \rho\boldsymbol{b} = \rho\boldsymbol{a}$	$\forall \boldsymbol{x} \in B$	**balance of linear momentum**	(3 equations)
$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$	$\forall \boldsymbol{x} \in B$	**balance of angular momentum**	(constraint)
$\rho\dot{u} = \boldsymbol{\sigma} : \boldsymbol{d} + \rho r - \mathrm{div}\boldsymbol{q}$	$\forall \boldsymbol{x} \in B$	**conservation of energy**	(1 equation)
$\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\frac{\boldsymbol{q}}{T}$	$\forall \boldsymbol{x} \in B$	**Clausius-Duhem inequality**	(constraint)
		$\rho,\ \boldsymbol{x},\ \boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ s,\ T$	(16 unknowns)



Constitutive relations

Constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g., $\boldsymbol{\sigma},\ \boldsymbol{q},\ W,\ s$ (11 constitutive equations)

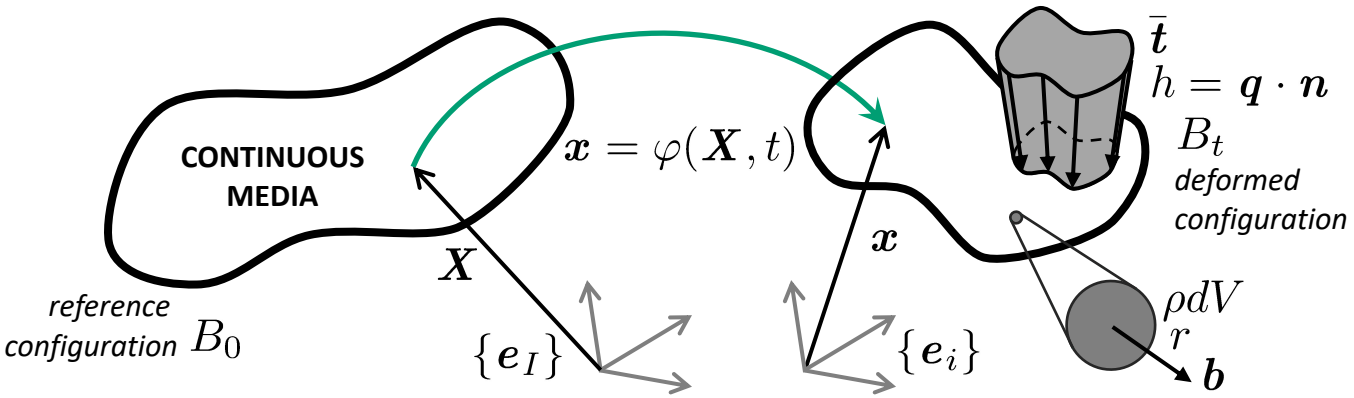
Simple elastic material

… based on the Helmholtz free energy $W = \overline{W}\left(\boldsymbol{F}, T\right)$

entropy $s = \overline{s}\left(\boldsymbol{F}, T, \ldots\right)$

heat flux $\boldsymbol{q} = \overline{\boldsymbol{q}}\left(\boldsymbol{F}, T, \ldots\right)$

Cauchy's stress tensor $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}\left(\boldsymbol{F}, T, \ldots\right)$



Constitutive relations

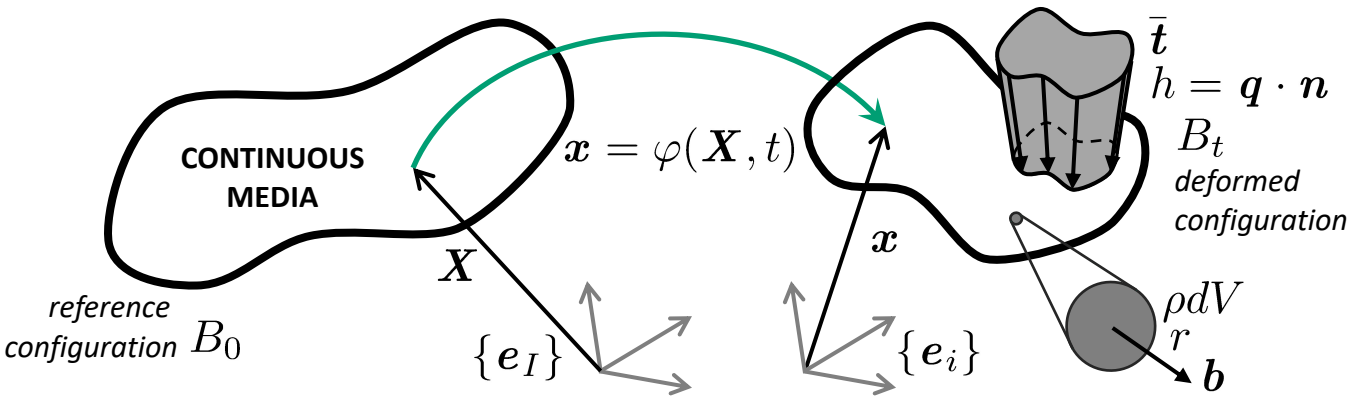
Coleman-Noll procedure

Using conservation of energy, the Clausius-Duhem inequality can be written as

$$\rho T \dot{s}^{\text{int}} = -\rho \left[\frac{1}{\rho_0} \frac{\partial W}{\partial T} + s \right] \dot{T} + \left[\boldsymbol{\sigma} \boldsymbol{F}^{-T} - \frac{\rho}{\rho_0} \frac{\partial W}{\partial \boldsymbol{F}} \right] : \dot{\boldsymbol{F}} - \frac{1}{T} \boldsymbol{q} \nabla T \geq 0$$

Coleman and Noll made the argument that this inequality must be satisfied for <u>every</u> admissible process.

The application of the Clausius-Duhem inequality to constitutive equations is known as the Coleman-Noll procedure (1963)



Constitutive relations

Constraints on constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g., $\boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ T$ (11 constitutive equations)

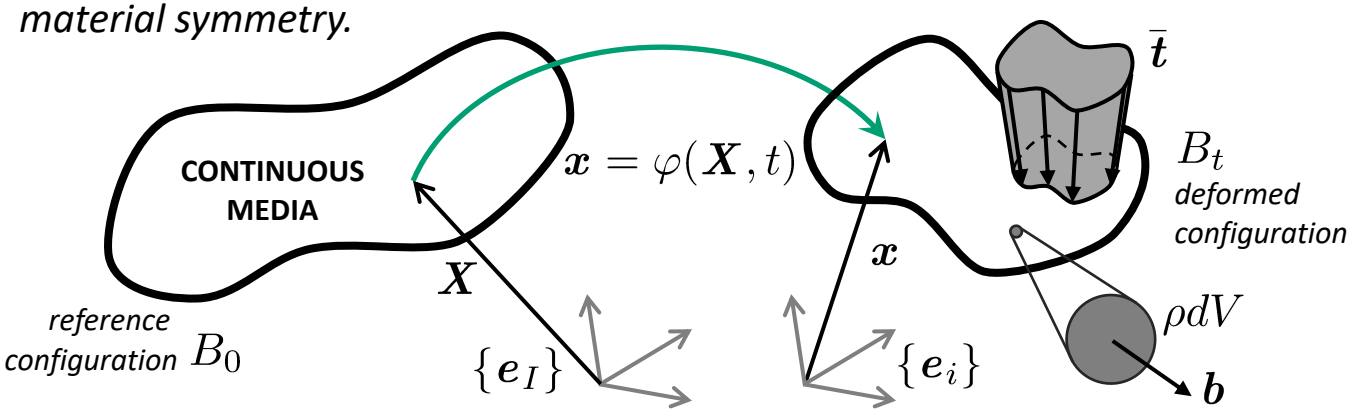
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)},\ \boldsymbol{q} = \mathbf{0}, \rho_0 u(\boldsymbol{F}, s),\ \dot{T} \neq 0\ \ (\text{with } \dot{s} = 0)$$ isentropic processes

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)},\ \boldsymbol{q} = \mathbf{0}, W(\boldsymbol{F}, T),\ \dot{s} \neq 0\ \ (\text{with } \dot{T} = 0)$$ isothermal processes

- Can these constitutive relations be selected arbitrarily? NO!

  They must follow the following fundamental principles:

  Principle of determinism, principle of local action, second law of thermodynamics restrictions (Clausius-Duhem inequality), principle of material frame indifference, material symmetry.



Constitutive relations

Coleman-Noll procedure – Isothermal processes

- *Isothermal* processes, where the motion and deformation occur at such a low temporal rate that the temperature is uniform and constant.

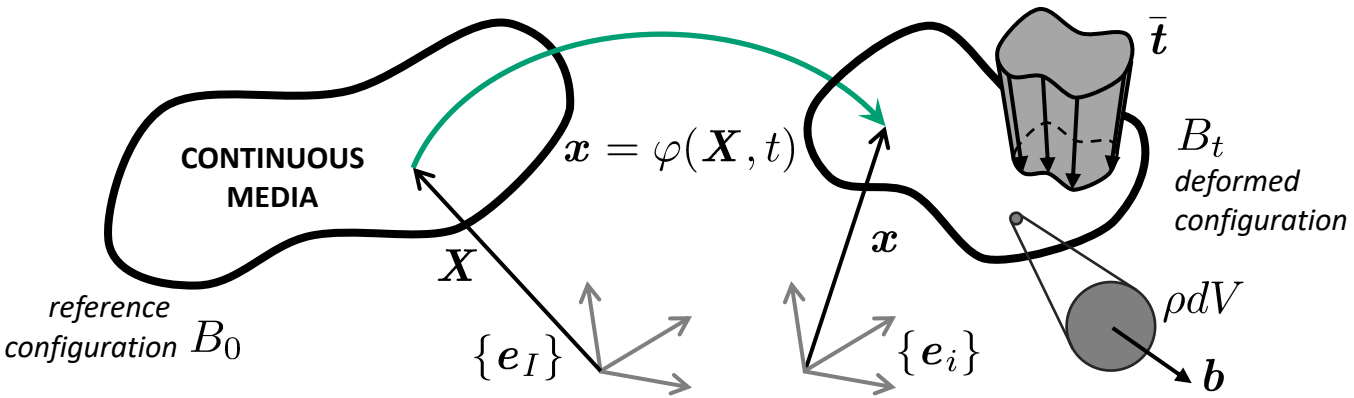
… based on the Helmholtz free energy

$$\dot{T} = 0 \qquad \nabla T = 0 \qquad \boldsymbol{q} = -k\nabla T = 0 \qquad W = \overline{W}\left(\boldsymbol{F}, T\right)$$

$$s = \overline{s}\left(\boldsymbol{F}, T\right) \equiv -\frac{1}{\rho_0}\frac{\partial W}{\partial T} \qquad \dot{s} \neq 0$$

$$\boldsymbol{\sigma}^{(e)} \equiv \frac{1}{J}\frac{\partial W}{\partial \boldsymbol{F}}\boldsymbol{F}^T \qquad \bar{\boldsymbol{\sigma}}^{(v)}\left(\boldsymbol{F}, T, \boldsymbol{d}\right)$$

If $\boldsymbol{\sigma}^{(v)} = \mathbf{0}$ then the process is reversible, and the material is called **hyperelastic**



Hyperelastic solids - Isotropic

Coleman-Noll procedure + Frame indifference + Isotropy

$$W = W\left(I_1, I_2, I_3\right)$$

strain energy density function
(*a function of the principal invariants of the right/left Cauchy-Green tensor*)

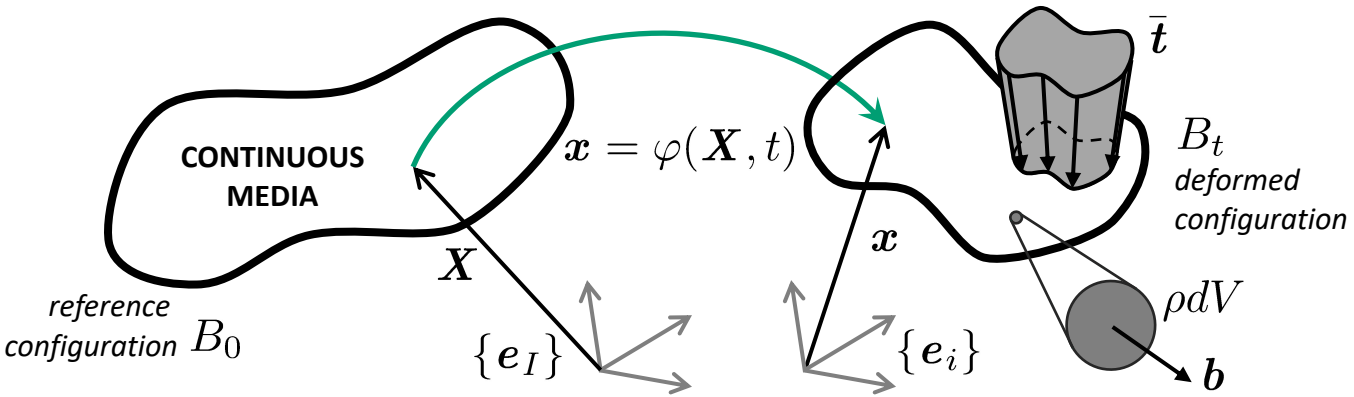
$$I_1 = \mathrm{tr}(\boldsymbol{C}) = \mathrm{tr}(\boldsymbol{B})$$
$$I_2 = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)] = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{B})^2 - \mathrm{tr}(\boldsymbol{B}^2)]$$
$$I_3 = \det(\boldsymbol{C}) = \det(\boldsymbol{B}) = J^2$$

DIY

$$\boldsymbol{\sigma}^{(e)} = \frac{2}{J}\boldsymbol{F}\frac{\partial \widetilde{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}\boldsymbol{F}^T$$

$$\boldsymbol{S}^{(e)} = 2\frac{\partial \widetilde{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}$$

elastic part of the stress tensor
(*the viscous part is zero and thus, the process is reversible*)



Hyperelastic solids - Isotropic

Coleman-Noll procedure + Frame indifference + Isotropy

$$W = W\left(I_1, I_2, I_3\right)$$

$$\boldsymbol{S} = 2\left[(W_{,I_1} + I_1 W_{,I_2})\boldsymbol{I} - W_{,I_2}\boldsymbol{C} + I_3 W_{,I_3}\boldsymbol{C}^{-1}\right]$$

$$\boldsymbol{\sigma} = \frac{2}{I_3^{1/2}}\left[I_3 W_{,I_3}\boldsymbol{I} + (W_{,I_1} + I_1 W_{,I_2})\boldsymbol{B} - W_{,I_2}\boldsymbol{B}^2\right]$$

$$\boldsymbol{S} = 2\sum_{i=1}^{3} \frac{\partial W(I_1,I_2,I_3)}{\partial I_i}\frac{\partial I_i}{\partial \boldsymbol{C}}$$

$$\frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I}$$

$$\frac{\partial I_2}{\partial \boldsymbol{C}} = I_1\boldsymbol{I} - \boldsymbol{C}$$

$$\frac{\partial I_3}{\partial \boldsymbol{C}} = I_3\boldsymbol{C}^{-1}$$

$$W = W\left(\lambda_1, \lambda_2, \lambda_3\right)$$

The right and left stretch tensors have the same principal stretches (eigenvalues).

Principal directions (eigenvectors).

$$\boldsymbol{\Lambda}_\alpha^{\boldsymbol{C}} = \boldsymbol{\Lambda}_\alpha^{\boldsymbol{U}} = \boldsymbol{N}_\alpha \quad \alpha = \{1,2,3\} \quad \|\boldsymbol{N}_\alpha\| = 1$$

$$\boldsymbol{U}\boldsymbol{N}_\alpha = \lambda_\alpha \boldsymbol{N}_\alpha \quad \alpha = \{1,2,3\}$$

$$\boldsymbol{C}\boldsymbol{N}_\alpha = \lambda_\alpha^2 \boldsymbol{N}_\alpha \quad \alpha = \{1,2,3\}$$

$$\frac{\partial \lambda_\alpha}{\partial \boldsymbol{C}} = \frac{1}{2\lambda_\alpha}\boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \quad \alpha = \{1,2,3\}$$

Hyperelastic solids - Isotropic (*and anisotropic*)

Strain energy density function

Isotropic hyperelastic solids.

Blatz-Ko materials (hyperelastic solid) $W(I_2, I_3) = c_1 \left[\frac{I_2}{I_3} + 2\sqrt{I_3} - 5 \right]$

Neo-Hookean materials (incompressible) $W(I_1, I_2) = c_1(I_1 - 3) \qquad I_3 = 1$

Moony-Rivlin materials (incompressible) $W(I_1, I_2) = c_1(I_1 - 3) + c_2(I_2 - 3)$

Ogden model (compressible/incompressible) $W(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p} \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3 \right)$

Note: incompressibility is an internal constraint (or kinematic constraint) of the material

Question: how is the pressure determined? The undetermined part of the pressure is introduced as a Lagrange multiplier and it is determined from boundary conditions

$$\boldsymbol{\sigma} = \frac{2}{J} \boldsymbol{F} \frac{\partial \widetilde{W}(\boldsymbol{C}, T)}{\partial \boldsymbol{C}} \boldsymbol{F}^T - c_0 \boldsymbol{I} \qquad \text{with} \quad I_3 = 1 \quad \text{or} \quad \lambda_3 = 1/\lambda_1 \lambda_2$$

Anisotropic elastic solids. Saint Venant materials.

$$\widetilde{W}(\boldsymbol{E}) = \frac{1}{2}(\boldsymbol{C} : \boldsymbol{E}) : \boldsymbol{E}$$

$$\boldsymbol{S} = \frac{\partial \check{W}(\boldsymbol{E})}{\partial \boldsymbol{E}} = \mathbb{C} : \boldsymbol{E} \qquad C_{IJKL} = C_{JIKL} = C_{IJLK} = C_{KLIJ}$$

Materials with memory

Internal variable, evolution equation, small strains

- The memory or path-dependency of a material can be represented through an array of **internal variables** (scalars and second-order tensors)

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}, T, \boldsymbol{\epsilon}^t) \equiv \begin{cases} \boldsymbol{\sigma}(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) \\ \dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) \quad \text{evolution equations} \end{cases}$$

  The presence of additional variables in the constitutive laws requires additional constitutive equations, namely **evolution equations**. The hypothesis is that the rate of evolution of the internal variables is also determined from the local state.

Examples:

- ***viscoelasticity**, i.e., time- and rate-dependent reversible behavior (the stress–strain relation depends on the loading rate); internal variables, e.g., the inelastic strain.*
- ***plasticity**, i.e., history-dependent irreversible behavior (the stress–strain relation depends on the loading history); internal variables, e.g., plastic strain, accumulated plastic strain.*
- ***viscoplasticity**, i.e., history- and time-dependent irreversible behavior.*
- ***damage**, i.e., irreversible degradation of the elastic stiffness with loading; internal variable is a damage parameter, e.g., a scalar measure.*

Materials with memory: Viscoelastic solids

Example of viscoelastic solid: Standard solid model

From equilibrium and compatibility:

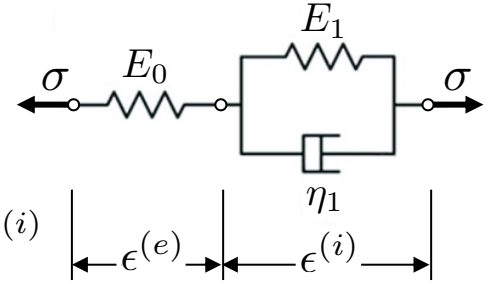
$$\dot{\epsilon}^{(i)} = \frac{1}{\eta_1}\sigma - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

$$\dot{\epsilon}^{(i)} = \frac{E_0}{\eta_1}\left(\epsilon - \epsilon^{(i)}\right) - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

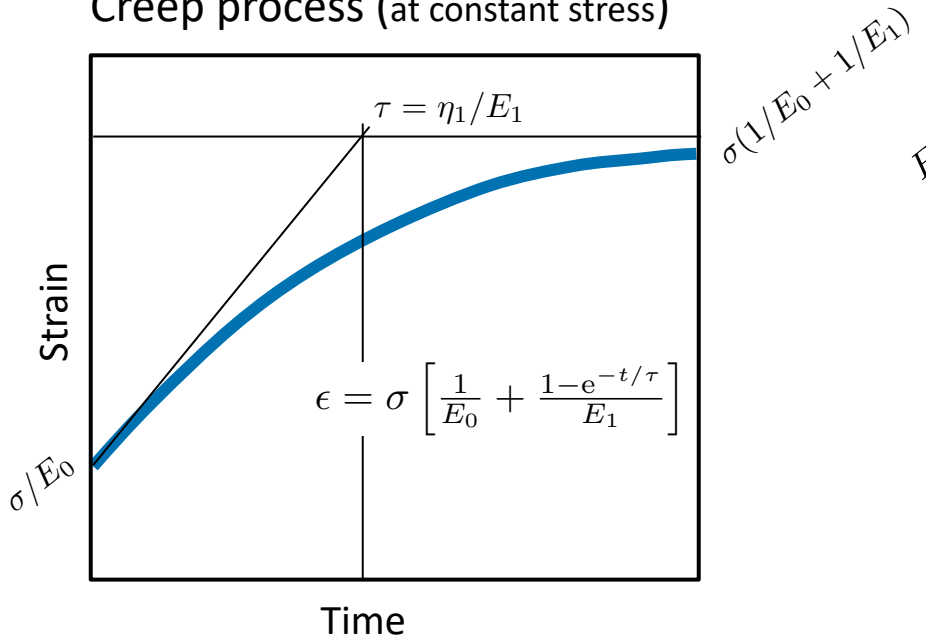
$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$

$$\sigma = E_0\epsilon^{(e)}$$

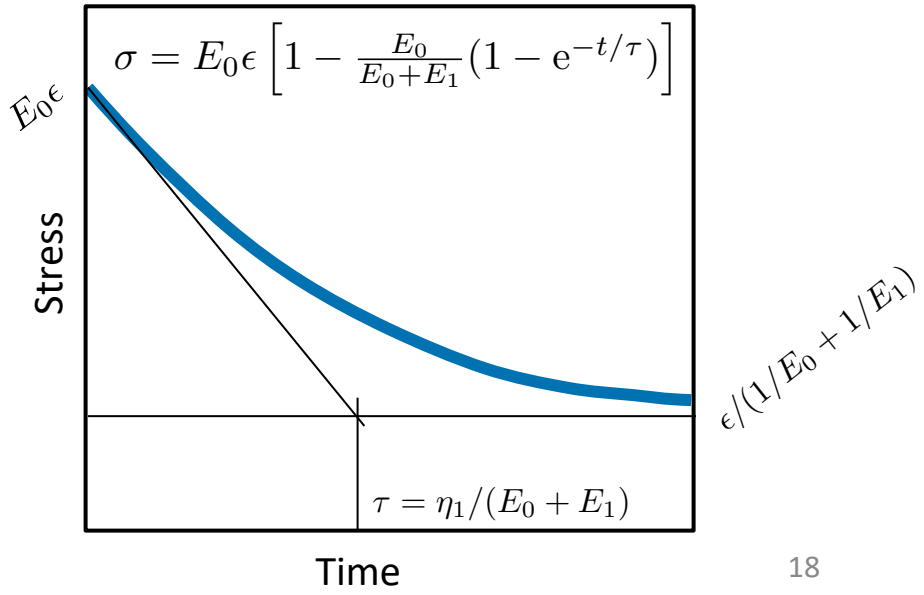
$$\sigma = E_1\epsilon^{(i)} + \eta_1\dot{\epsilon}^{(i)}$$



Creep process (at constant stress)



Relaxation process (at constant strain)



Materials with memory: Viscoelastic solids

Internal variable, evolution equation, small strains

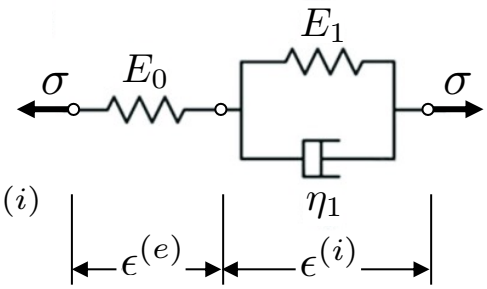
- The memory or path-dependency of a material can be represented through an array of **internal variables** (scalars and second-order tensors)

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}, T, \boldsymbol{\epsilon}^t) \equiv \begin{cases} \boldsymbol{\sigma}(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) \\ \dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) \quad \text{evolution equations} \end{cases}$$

The presence of additional variables in the constitutive laws requires additional constitutive equations, namely **evolution equations**. The hypothesis is that the rate of evolution of the internal variables is also determined from the local state.

- Example: standard solid model

$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0 \epsilon^{(e)}$$
$$\sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$



internal variable

$$\xi_1 = \epsilon^{(i)}$$

evolution equation (kinetic equation)

$$\dot{\epsilon}^{(i)} = g_1(\sigma, \xi_1) = \frac{1}{\eta_1}\sigma - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

$$\dot{\epsilon}^{(i)} = g_1(\epsilon, \xi_1) = \frac{E_0}{\eta_1}(\epsilon - \epsilon^{(i)}) - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

(*an evolution equation in terms of strain is not always possible*)

Materials with memory

Internal variables and thermodynamics

Using conservation of energy, the Clausius-Duhem inequality can be written as

$$\rho T \dot{s}^{\text{int}} = -\rho \left[\frac{1}{\rho_0} \frac{\partial W}{\partial T} + s \right] \dot{T} + \left[\boldsymbol{\sigma} - \frac{\rho}{\rho_0} \frac{\partial W}{\partial \boldsymbol{\epsilon}} \right] : \dot{\boldsymbol{\epsilon}} - \frac{1}{T} \boldsymbol{q} \cdot \nabla T \geq 0$$

Coleman and Noll made the argument that this inequality must be satisfied for <u>every</u> admissible process.

Let's now consider path-dependent behavior characterized by a set of internal variables and corresponding evolution equations, that is

$$W = \overline{W}(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) \qquad \dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma}, T, \boldsymbol{\xi})$$

- Isentropic/Adiabatic processes $\boldsymbol{q} = \boldsymbol{0}, W(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}),\ \dot{s} \neq 0 \ \ (\text{with } \dot{T} = 0)$

Assuming $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)}(\boldsymbol{\sigma}, T) + \boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi})$, then there exists $W(\boldsymbol{\epsilon}, T, \boldsymbol{\xi})$ if and only if

$$W(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}), T) + W^{(i)}(\boldsymbol{\xi}, T)$$

Materials with memory: Viscoelastic solids

Internal variables and thermodynamics

- Example: multiaxial behavior and time integration

$$W(\boldsymbol{\epsilon}, T, \boldsymbol{\xi}) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}), T) + W^{(i)}(\boldsymbol{\xi}, T)$$

Internal variable $\boldsymbol{\epsilon}^{(i)}(\boldsymbol{\xi}) = \boldsymbol{\xi}$

$$W^{(i)}(\boldsymbol{\xi}, T) = \tfrac{1}{2} E_1 \boldsymbol{\epsilon}^{(i)} : \boldsymbol{\epsilon}^{(i)}$$

Evolution equation $\dot{\boldsymbol{\epsilon}}^{(i)} = \frac{\boldsymbol{\sigma} - E_1 \boldsymbol{\epsilon}^{(i)}}{\eta_1}$

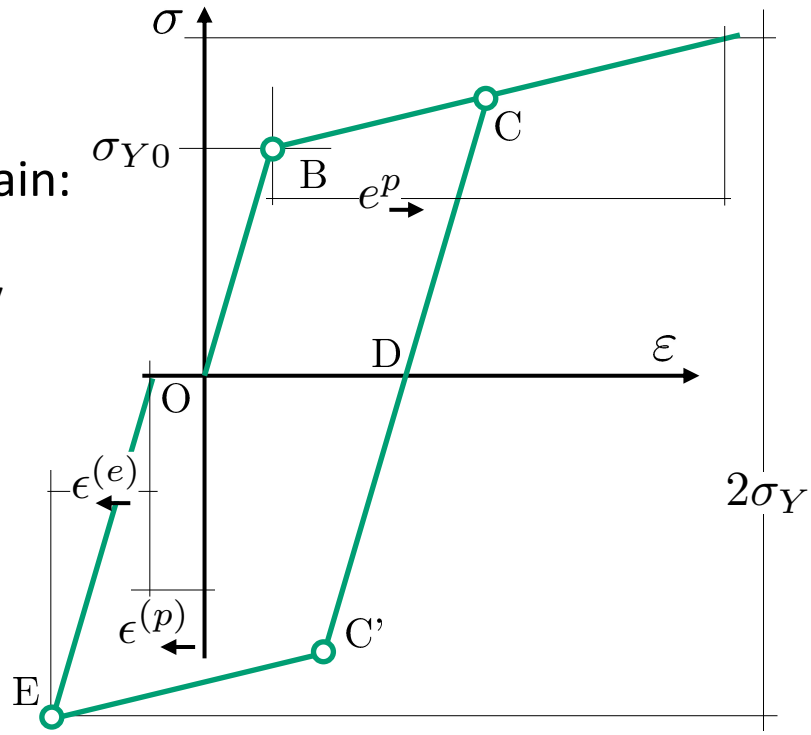
Time integration $t_n \rightarrow t_{n+1} = t_n + \Delta t$

- Given $\left\{ \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_n, \boldsymbol{\epsilon}_n^{(i)} \right\}$
- Update internal variable $\boldsymbol{\epsilon}_{n+1}^{(i)} = \frac{1}{\eta_1 / \Delta t} \boldsymbol{\sigma}_n + \boldsymbol{\epsilon}_n^{(i)} \left[1 - \frac{E_1}{\eta_1 / \Delta t} \right]$
- Update stress ..... $\boldsymbol{\sigma}_{n+1} = \frac{\rho_{n+1}}{\rho_0} \frac{\partial W_{n+1}^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}}$

Materials with path-dependency: Plastic solids

Phenomenological observations

- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function
  $$\phi(\sigma, \sigma_Y) \leq 0$$
- Additive decomposition of (small) strain:
  + elastic part is related to the stress
  + plastic part depends on material history
  $$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$$
- Metals: plastic flow is incompressible
  (any change in volume is elastic)
- Bauschinger effect $|\sigma_{Y|r}| \leq |\sigma_{Y|f}|$
  Strain hardening:
  + isotropic hardening (e^p)
  + kinematic hardening
- Metals: Bridgman (c. 1940)
  Yield function is independent of hydrostatic pressure



Isotropic plastic solids

Ideal plastic solids – Small strains

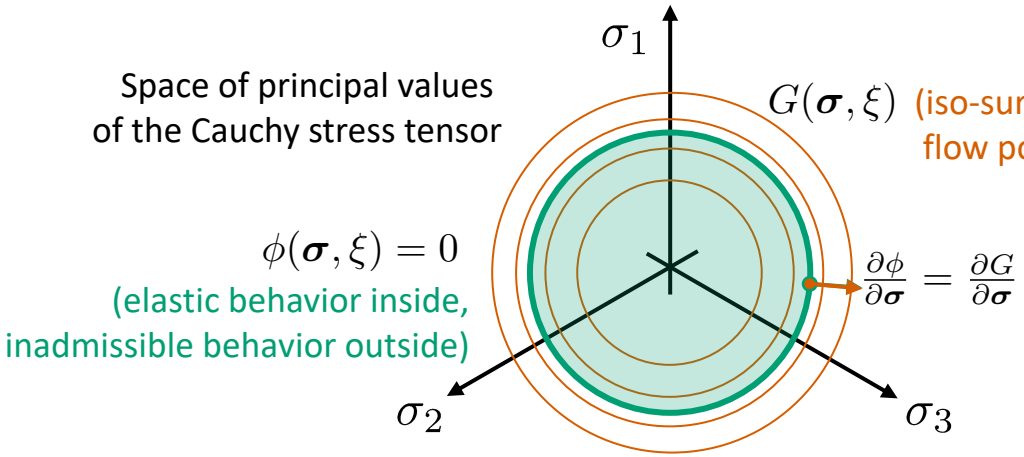
Constitutive law defined by yield function, flow potential and hardening rule

- Plasticity manifests when a stress threshold is reach (yield stress)
  Yield function $\phi(\boldsymbol{\sigma}, \xi) \leq 0$
  Internal variable (*for isotropic hardening*) $\xi = \int_0^t ||\dot{\boldsymbol{\epsilon}}^{(p)}|| \mathrm{d}t$
- Additive decomposition of (small) strain: $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(p)}$
- Consistency condition $\dot{\phi} = 0 \rightarrow \text{solve for } \dot{\lambda}$
  (*determine* $\dot{\lambda}$ *under plastic loading;* $\dot{\lambda} = 0$*, otherwise*)
- Strain energy density: $W(\boldsymbol{\epsilon}, \xi) = W^{(e)}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{(p)}(\xi)) + W^{(p)}(\xi)$
- Evolution equations:
  - Flow rule (flow potential) $\dot{\boldsymbol{\epsilon}}^{(p)} = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma}, \xi) = \dot{\lambda}\frac{\partial G}{\partial \boldsymbol{\sigma}}$
  - Hardening rule $\dot{\xi} = \dot{\lambda} h(\boldsymbol{\sigma}, \xi)$
- Cauchy stress tensor: $\boldsymbol{\sigma} = \frac{\rho}{\rho_0}\frac{\partial W^{(e)}}{\partial \boldsymbol{\epsilon}^{(e)}} := \boldsymbol{c}^e : \boldsymbol{\epsilon}^{(e)}$

Isotropic plastic solids

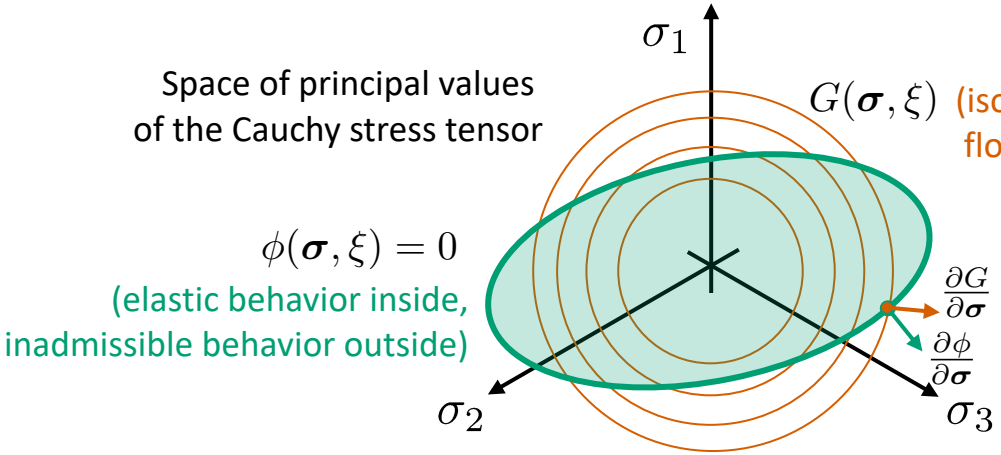
Ideal plastic solids – Small strains

Constitutive law defined by yield function, flow potential and hardening rule



$$G(\boldsymbol{\sigma},\xi)=\phi(\boldsymbol{\sigma},\xi)$$

associated flow rule



$$G(\boldsymbol{\sigma},\xi)\neq\phi(\boldsymbol{\sigma},\xi)$$

non-associated flow rule

Isotropic plastic solids

Ideal plastic solids – Small strains

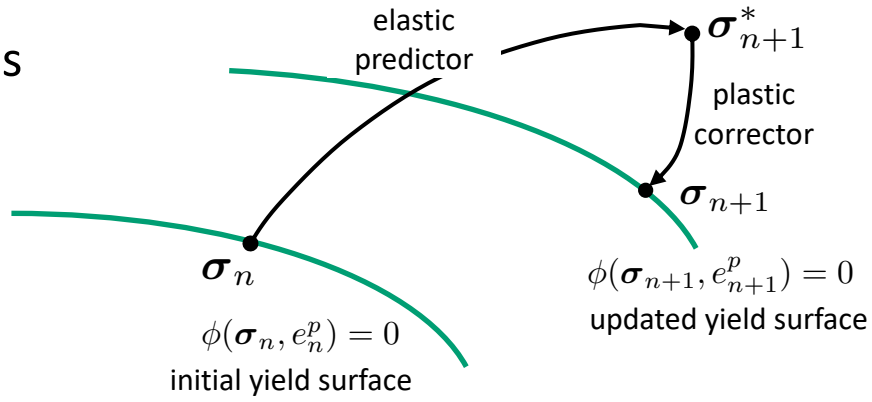
Constitutive law defined by yield function, flow potential and hardening rule

- J_2 or von Mises plasticity
- Drucker-Prager plasticity (associated flow rule)
- Drucker-Prager plasticity (non-associated flow rule)
- Tresca Criterion (or Maximum-shear-stress)
- Mohr-Coulomb plasticity

Q: Which are the chief characteristics of each plastic model?

For example: $\phi(J_2, e^p)$ $\phi(I_1, J_2)$

Q: How is the consistency condition enforced in a time-integration scheme?



Review session for the exam

Study hard!