Spring, 2022

ME 597 – Solid Mechanics II

Lecture 20
Structural elements: beams, plates, shells

PURDUE UNIVERSITY | Mechanical Engineering
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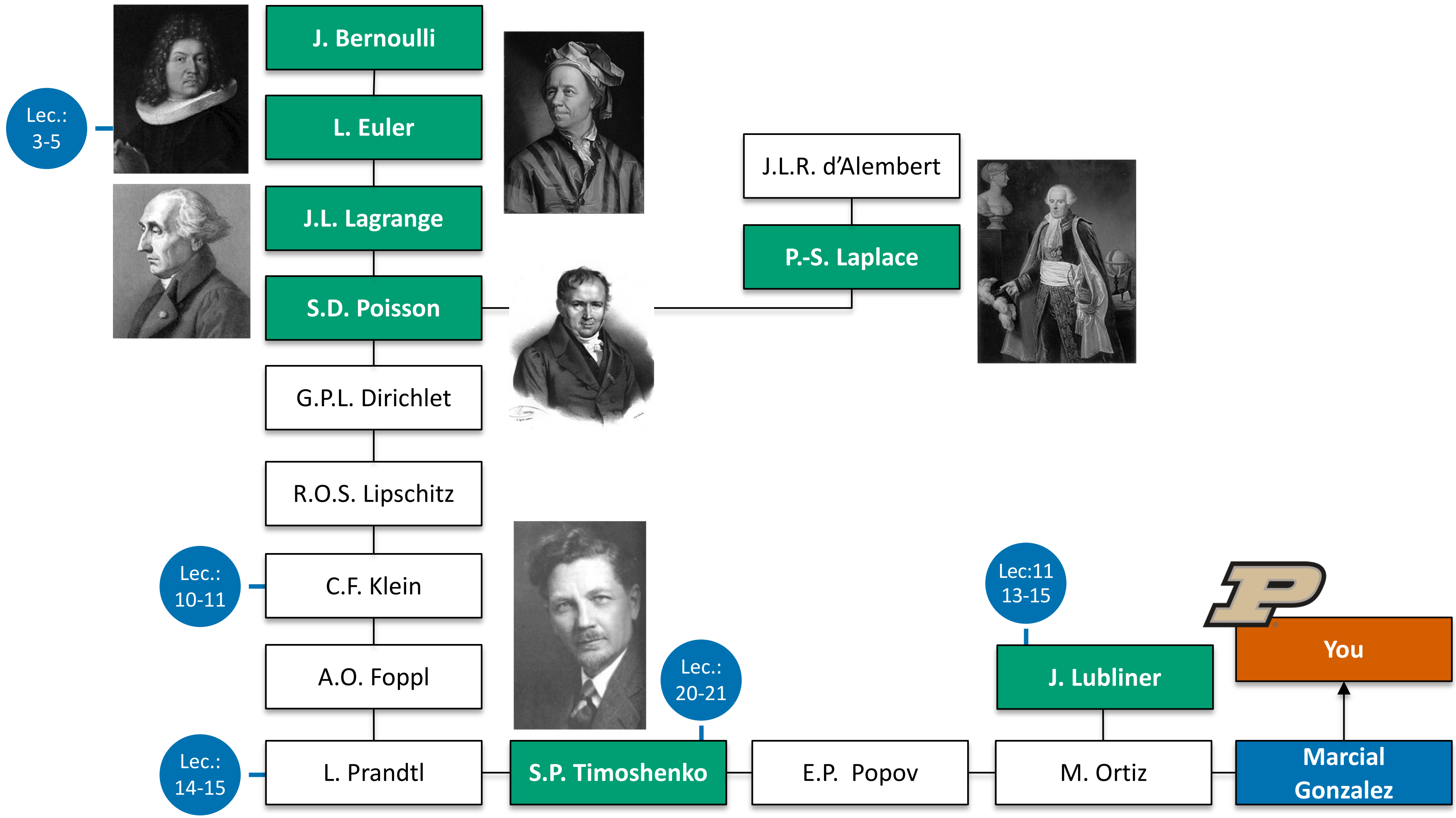
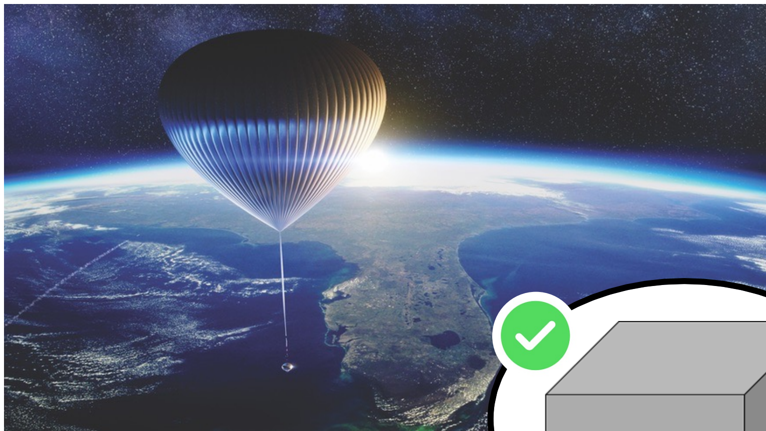
General information

Know your ~~history~~ genealogy

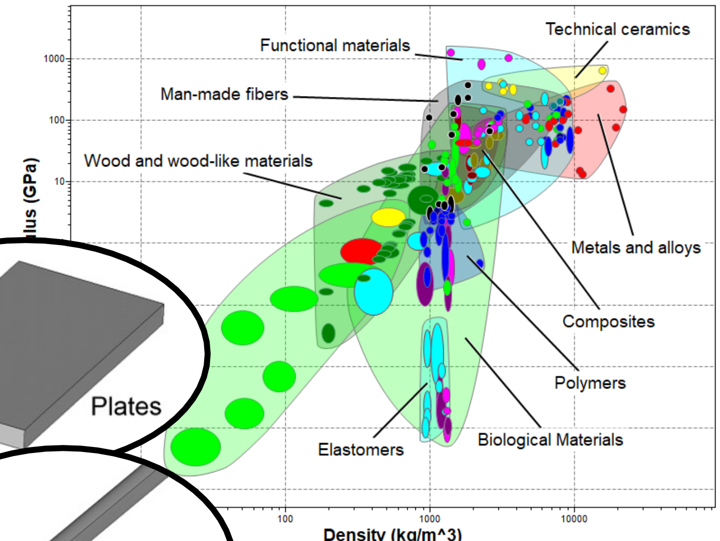


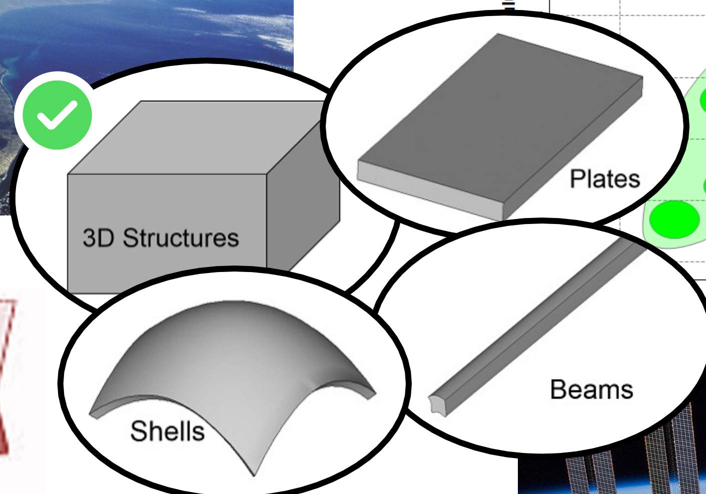
Structural elements: beams, plates, shells

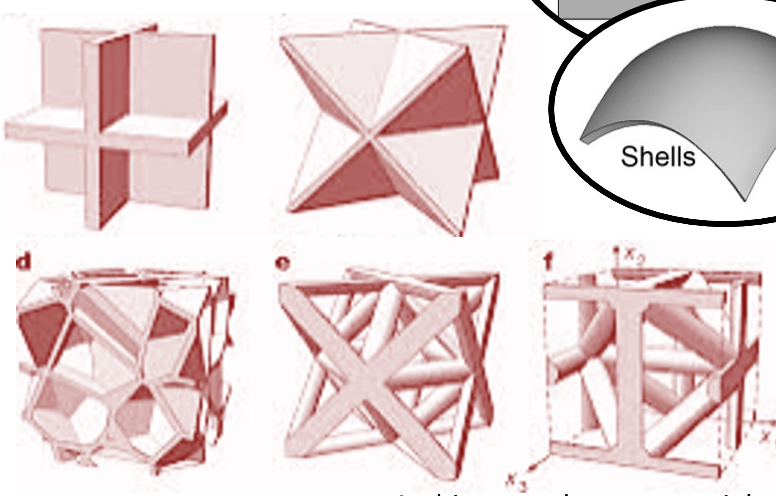
Formulation of **structural elements (beams, plates, shells)** as the analytical upscaling of continuum solids under kinematic assumptions.



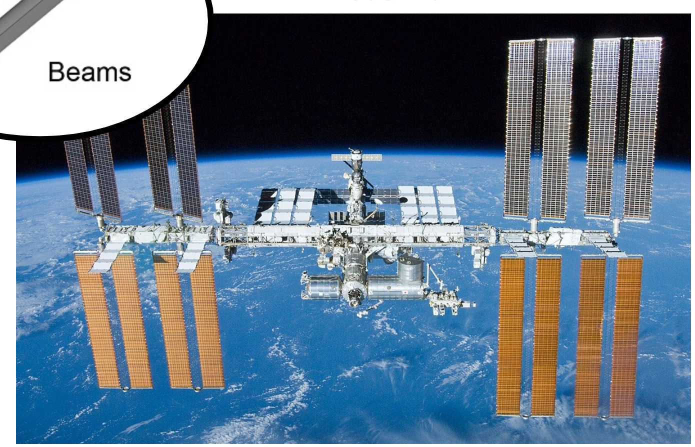
Stratospheric ballooning system







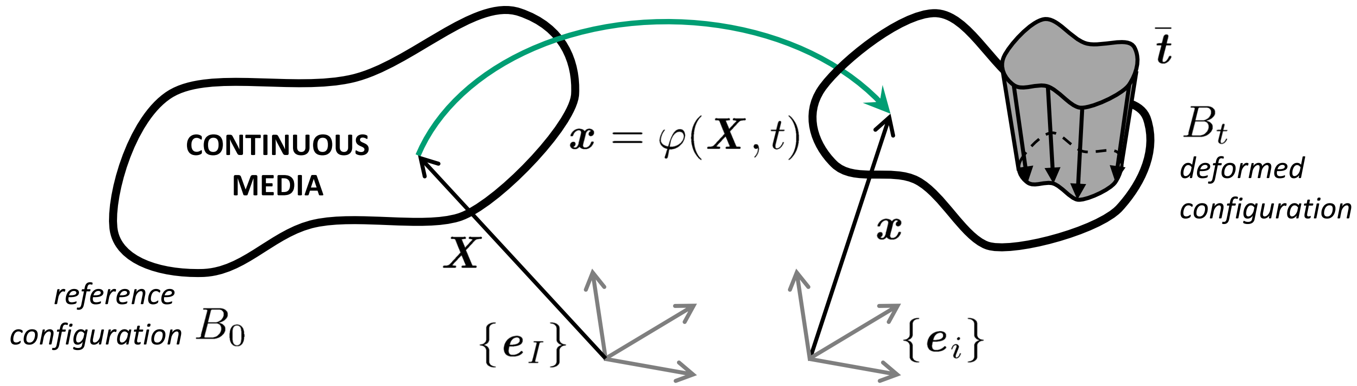
Architectured metamaterials



International space station

Structural elements: beams, plates, shells

Formulation of **structural elements (beams, plates, shells)** as the analytical upscaling of continuum solids under kinematic assumptions.



Linearized kinematic – Small trains

- The linearization is evaluated in the undeformed configuration (i.e., $\boldsymbol{X} \rightarrow \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$ and $\boldsymbol{F} = \boldsymbol{I}, \nabla_0 \boldsymbol{u} = \nabla \boldsymbol{u}$):

$$\langle \nabla_\varphi \boldsymbol{E}; \boldsymbol{u} \rangle = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] = \boldsymbol{\epsilon}$$

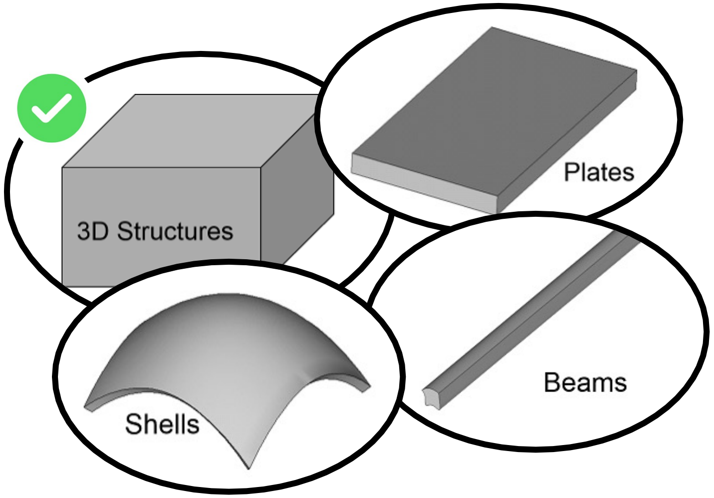
small-strain tensor
(*employed in elasticity theory*)

- Simplified geometry
- Simplifying kinematic assumptions (i.e., simplified displacement field $\boldsymbol{u} = (u_1, u_2, u_3)$)

Structural elements: beams, plates, shells

Simplified geometry

- Many everyday engineering applications utilize structural members such as rods, beams, cables, plates, and shells.
- Structural members can be idealized as one-dimensional (rods, beams, and cables) and two-dimensional (plates and shells) members.
- Assumption: two dimensions (in the case of beams) or one dimension (in the case of plates and shells) are significantly smaller than the other dimensions.



- Stress-strain behavior is upscaled to a relationship between internal resultants and kinematic variables (of the mid-plane).

Structural elements: beams, plates, shells

Timoshenko beams

- Kinematic assumptions, with $\bar{w} \ll h$

$$u_1(x,z) = \bar{u}(x) + z\theta_2(x)$$
$$u_2 = 0$$
$$u_3(x) = \bar{w}(x)$$

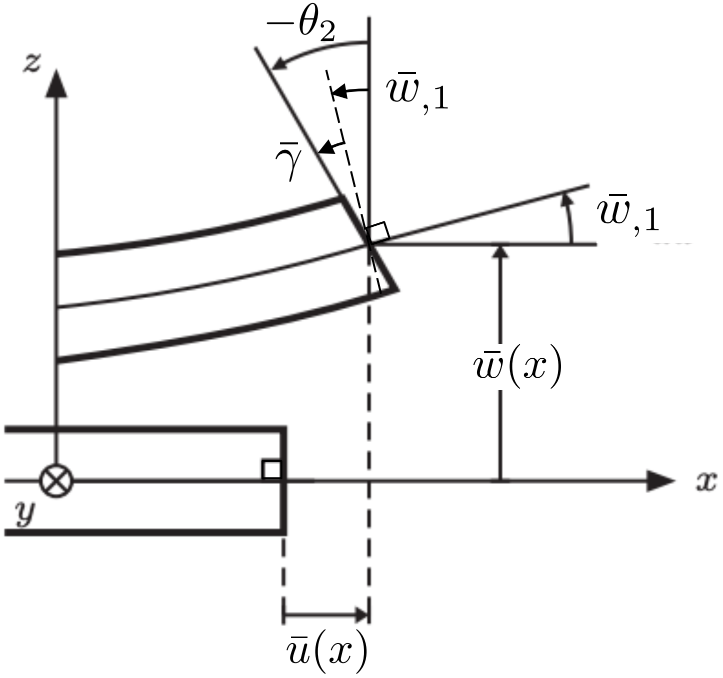
- Small-strain tensor, with $(\bar{w}_{,1})^2 \ll 1$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\epsilon_{ij} = \begin{bmatrix} \bar{u}_{,1} + z\theta_{2,1} & 0 & (\theta_2 + \bar{w}_{,1})/2 \\ 0 & 0 & 0 \\ (\theta_2 + \bar{w}_{,1})/2 & 0 & 0 \end{bmatrix}$$

$$\epsilon_{ij} = \begin{bmatrix} \bar{\epsilon}_{11} + z\bar{\kappa}_{11} & 0 & \bar{\gamma}_{13}/2 \\ 0 & 0 & 0 \\ \bar{\gamma}_{13}/2 & 0 & 0 \end{bmatrix}$$

Simplified geometry: $b \ll L \;,\; h \ll L$



kinematic variables

$\bar{\epsilon}_{11}$ $\bar{\kappa}_{11}$ $\bar{\gamma}_{13}$

extensional, bending, and shear components of beam strain

Structural elements: beams, plates, shells

Aside: Euler-Bernoulli beams

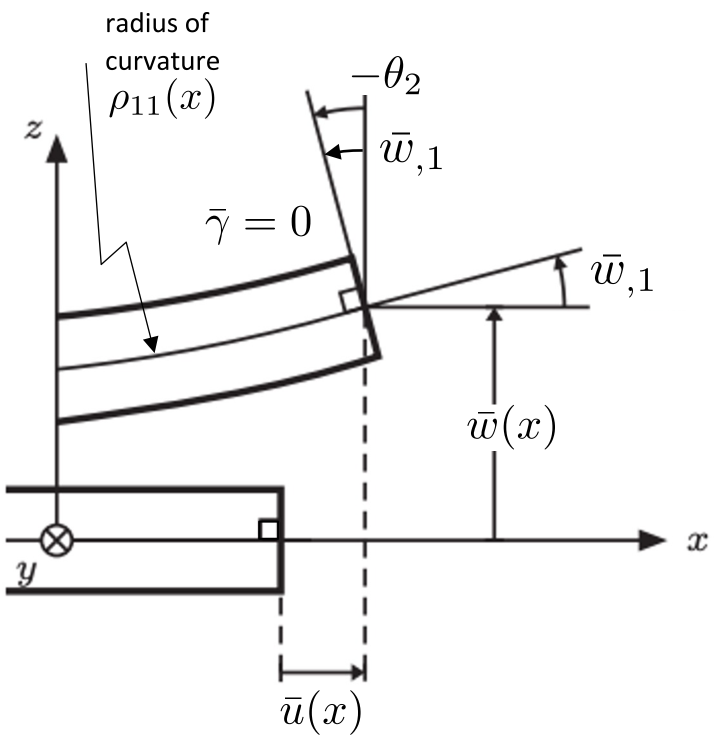
- Displacement field, with $\theta_2 = -\bar{w}_{,1}$

$$u_1(x,z) = \bar{u}(x) - z\bar{w}_{,1}(x)$$
$$u_2 = 0$$
$$u_3(x) = \bar{w}(x)$$

- Small-strain tensor, with $(\bar{w}_{,1})^2 \ll 1$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\epsilon_{ij} = \begin{bmatrix} \bar{u}_{,1} - z\bar{w}_{,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Simplified geometry: $b \ll L$, $h \ll L$
$\bar{w} \ll h$

$$\bar{\epsilon}_{11} = 0 \qquad \bar{w}_{,11} = \frac{1}{\rho_{11}} = -\kappa_{11} \implies \epsilon_{11} = -\frac{z}{\rho_{11}}$$

Euler-Bernoulli kinematic assumption

Structural elements: beams, plates, shells

Timoshenko beams

- Small-strain tensor

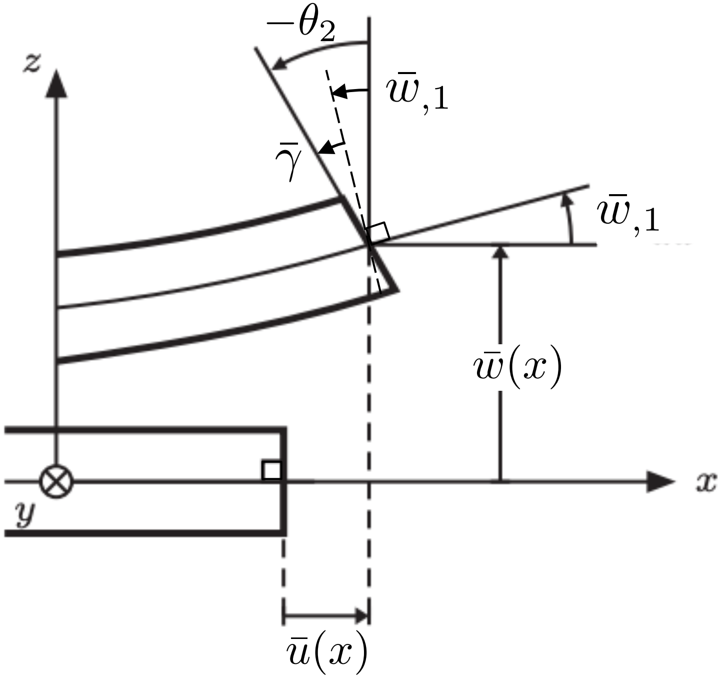
$$\epsilon_{ij} = \begin{bmatrix} \bar{\epsilon}_{11} + z\bar{\kappa}_{11} & 0 & \bar{\gamma}_{13}/2 \\ 0 & 0 & 0 \\ \bar{\gamma}_{13}/2 & 0 & 0 \end{bmatrix}$$

- Linear elasticity (generalized Hooke's law) assuming $\sigma_{22} \ll \sigma_{11}$ and $\sigma_{33} \ll \sigma_{11}$ Since lateral surfaces are traction free, stresses in 22 and 33 directions must be zero on free surfaces.

$$\sigma_{ij} = \begin{bmatrix} E(\bar{\epsilon}_{11} + z\bar{\kappa}_{11}) & 0 & \frac{E}{2(1+\nu)}\bar{\gamma}_{13} \\ 0 & 0 & 0 \\ \frac{E}{2(1+\nu)}\bar{\gamma}_{13} & 0 & 0 \end{bmatrix}$$

- Reconciled if the displacement field has higher order terms.

Simplified geometry: $b \ll L \ , \ h \ll L$



solution of a mixed boundary condition problem

$$u_1(x,z) = \bar{u}(x) - z\bar{w}_{,1}(x) + \text{h.o.t}$$
$$u_2 = 0 + \text{h.o.t}$$
$$u_3(x) = \bar{w}(x) + \text{h.o.t}$$

Structural elements: beams, plates, shells

Timoshenko beams

- Small-strain tensor

$$\epsilon_{ij} = \begin{bmatrix} \bar{\epsilon}_{11} + z\bar{\kappa}_{11} & 0 & \bar{\gamma}_{13}/2 \\ 0 & 0 & 0 \\ \bar{\gamma}_{13}/2 & 0 & 0 \end{bmatrix} \qquad \sigma_{ij} = \begin{bmatrix} E(\bar{\epsilon}_{11} + z\bar{\kappa}_{11}) & 0 & \frac{E}{2(1+\nu)}\bar{\gamma}_{13} \\ 0 & 0 & 0 \\ \frac{E}{2(1+\nu)}\bar{\gamma}_{13} & 0 & 0 \end{bmatrix}$$

- Linear elasticity (generalized Hooke's law) assuming $\sigma_{22} \ll \sigma_{11}$ and $\sigma_{33} \ll \sigma_{11}$
Since lateral surfaces are traction free, stresses in 22 and 33 directions must be zero on free surfaces.

incompatible with strain field

Reissner's theory overcomes this issue

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}$$

Structural elements: beams, plates, shells

Timoshenko beams

- Linear elasticity (generalized Hooke's law) assuming $\sigma_{22} \ll \sigma_{11}$ and $\sigma_{33} \ll \sigma_{11}$ Since lateral surfaces are traction free, stresses in 22 and 33 directions must be zero on free surfaces.

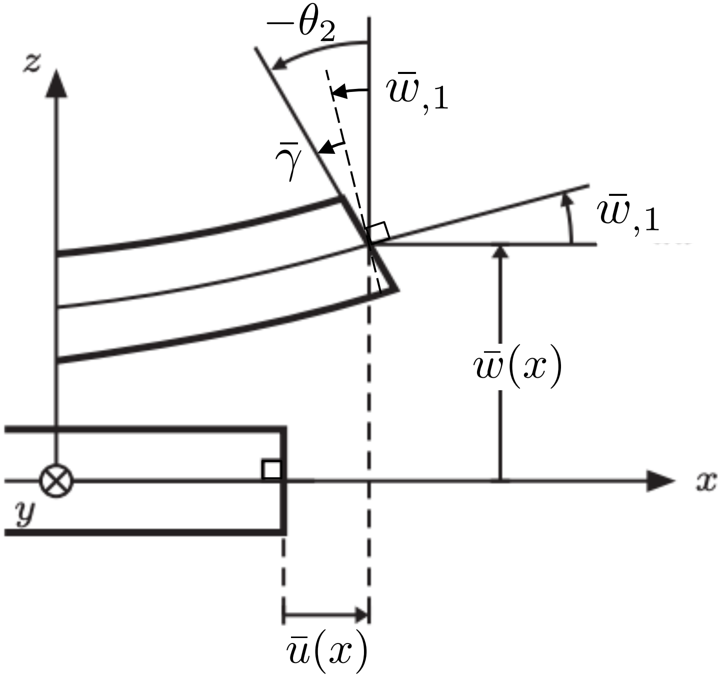
$$\sigma_{ij} = \begin{bmatrix} E(\bar{\epsilon}_{11} + z\bar{\kappa}_{11}) & 0 & \frac{E}{2(1+\nu)}\bar{\gamma}_{13} \\ 0 & 0 & 0 \\ \frac{E}{2(1+\nu)}\bar{\gamma}_{13} & 0 & 0 \end{bmatrix}$$

- Internal resultants

$$N_1 = \int_A \sigma_{11} \mathrm{d}A \qquad M_2 = \int_A z\sigma_{11} \mathrm{d}A$$

$$V = K_\mathrm{s} \int_A \sigma_{13} \mathrm{d}A$$

Simplified geometry: $b \ll L$, $h \ll L$



beam geometry
height (h), width (b)
cross-sectional area (A)
moment of inertia (I)
Shear correction factor (K_s)

Structural elements: beams, plates, shells

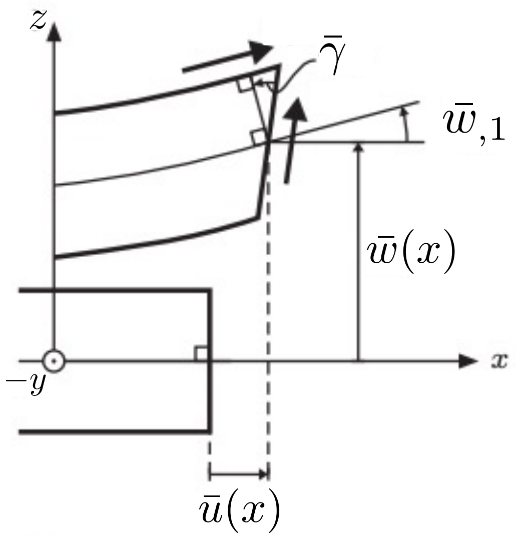
Timoshenko beams

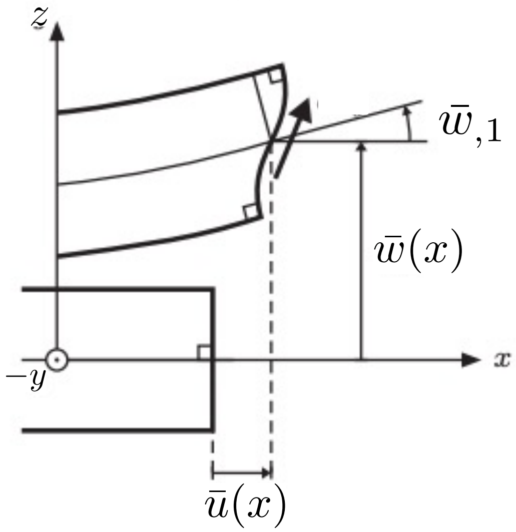
- Internal resultants

$$N_1 = b \int_{-h/2}^{h/2} E(\bar{\epsilon}_{11} + z\bar{\kappa}_{11})\mathrm{d}z = EA\bar{\epsilon}_{11}$$

$$M_2 = b \int_{-h/2}^{h/2} E(z\bar{\epsilon}_{11} + z^2\bar{\kappa}_{11})\mathrm{d}z = EI_{22}\bar{\kappa}_{11}$$

$$V = K_\mathrm{s} b \int_{-h/2}^{h/2} \frac{E}{2(1+\nu)}\bar{\gamma}_{13}\mathrm{d}z = \frac{K_\mathrm{s} AE}{2(1+\nu)}\bar{\gamma}_{13}$$





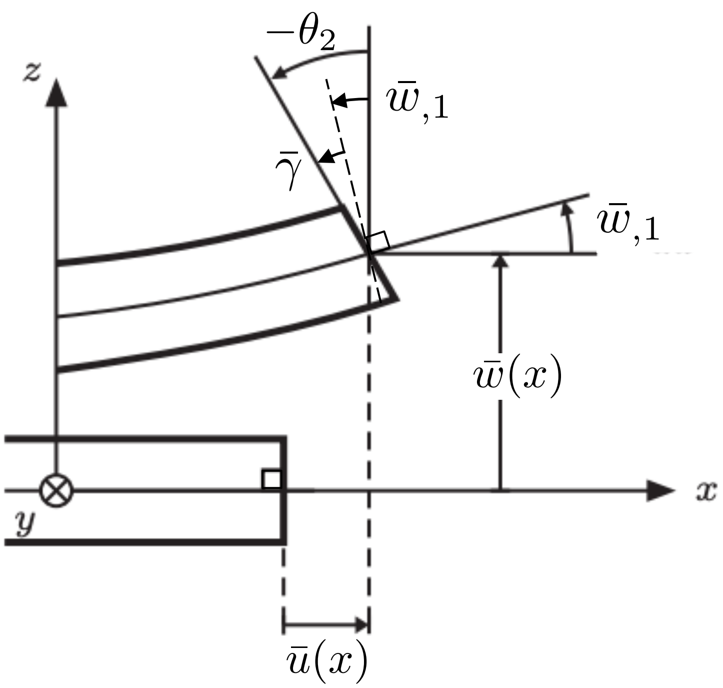
Note: The shear correction factor K_s is required in the formulation due to the absence of shear stress and strain at the top and bottom boundaries of the beam. K_s is computed such that resultant shear force V creates the same strain energy as does the true transverse stresses predicted by the three-dimensional elasticity theory.

Structural elements: beams, plates, shells

Timoshenko beams

- Constitutive relation
  (relationship between internal resultants and kinematic variables)

$$\begin{Bmatrix} N_1 \\ M_2 \\ V \end{Bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{22} & 0 \\ 0 & 0 & \frac{K_s EA}{2(1+\nu)} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_{11} \\ \bar{\kappa}_{11} \\ \bar{\gamma}_{13} \end{Bmatrix}$$



Euler-Bernoulli beams

- Constitutive relation

$$\begin{Bmatrix} N_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} EA & 0 \\ 0 & EI_{22} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_{11} \\ \bar{\kappa}_{11} \end{Bmatrix}$$

kinematic variables

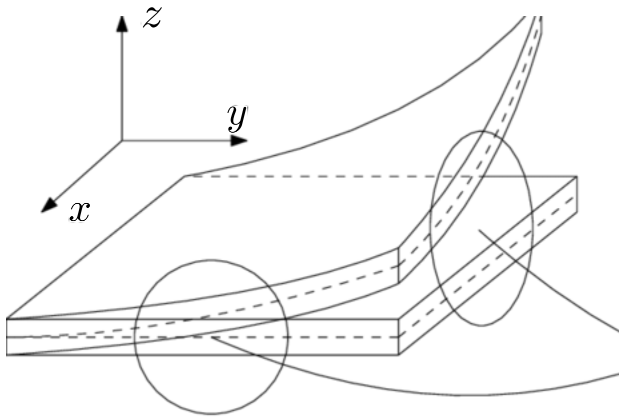
$\bar{\epsilon}_{11}(x)$ $\bar{\kappa}_{11}(x)$ $\bar{\gamma}_{13}(x)$

extensional, bending, and shear components of beam strain

Structural elements: beams, plates, shells

Mindlin's plates

Simplified geometry: the plate is initially flat



- Kinematic assumptions, with $\bar{w} \ll h$

$$u_1(x,y,z) = \bar{u}(x,y) + z\theta_1(x,y)$$
$$u_2(x,y,z) = \bar{v}(x,y) + z\theta_2(x,y)$$
$$u_3(x,y,z) = \bar{w}(x,y)$$

- Small-strain tensor, with $(\bar{w}_{,1})^2 \ll 1$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\epsilon_{ij} = \begin{bmatrix} \bar{\epsilon}_{11} + z\bar{\kappa}_{11} & \bar{\gamma}_{12}/2 + z\bar{\kappa}_{12} & \bar{\gamma}_{13}/2 \\ \bar{\gamma}_{12}/2 + z\bar{\kappa}_{12} & \bar{\epsilon}_{22} + z\bar{\kappa}_{22} & \bar{\gamma}_{23}/2 \\ \bar{\gamma}_{13}/2 & \bar{\gamma}_{23}/2 & 0 \end{bmatrix}$$

kinematic variables

$\bar{\epsilon}_{11} \quad \bar{\epsilon}_{22} \quad \bar{\kappa}_{11} \quad \bar{\kappa}_{22}$

extensional strains, bending strains

kinematic variables (cont.)

$\bar{\gamma}_{12} \quad \bar{\kappa}_{12} \quad \bar{\gamma}_{13} \quad \bar{\gamma}_{23}$

contribution of the mid-plane displacement to in-plane shear, contribution of normal rotations to in-plane shear, and shear components

Structural elements: beams, plates, shells

Mindlin's plates

- Small-strain tensor

$$\epsilon_{ij} = \begin{bmatrix} \bar{\epsilon}_{11} + z\bar{\kappa}_{11} & \bar{\gamma}_{12}/2 + z\bar{\kappa}_{12} & \bar{\gamma}_{13}/2 \\ \bar{\gamma}_{12}/2 + z\bar{\kappa}_{12} & \bar{\epsilon}_{22} + z\bar{\kappa}_{22} & \bar{\gamma}_{23}/2 \\ \bar{\gamma}_{13}/2 & \bar{\gamma}_{23}/2 & 0 \end{bmatrix}$$

- Linear elasticity (generalized Hooke's law) assuming $\sigma_{33} \ll \sigma_{22}$ and $\sigma_{33} \ll \sigma_{11}$ Since top/bottom surfaces are traction free, the stress in 33 direction must be zero on free surfaces.

solution of a mixed boundary condition problem

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}$$

Structural elements: beams, plates, shells

Kirchhoff plates

- Small-strain tensor, with $\theta_2 = -\bar{w}_{,1}$, $\theta_1 = -\bar{w}_{,2}$

$$\epsilon_{ij} = \begin{bmatrix} \bar{\epsilon}_{11} + z\bar{\kappa}_{11} & \bar{\gamma}_{12}/2 + z\bar{\kappa}_{12} & \cancel{\bar{\gamma}_{13}/2} \\ \bar{\gamma}_{12}/2 + z\bar{\kappa}_{12} & \bar{\epsilon}_{22} + z\bar{\kappa}_{22} & \cancel{\bar{\gamma}_{23}/2} \\ \cancel{\bar{\gamma}_{13}/2} & \cancel{\bar{\gamma}_{23}/2} & 0 \end{bmatrix}$$

- Linear elasticity (generalized Hooke's law) assuming $\sigma_{33} \ll \sigma_{22}$ and $\sigma_{33} \ll \sigma_{11}$ Since top/bottom surfaces are traction free, the stress in 33 direction must be zero on free surfaces.

solution of a mixed boundary condition problem

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \cancel{\sigma_{33}} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \cancel{\gamma_{12}} \\ \cancel{\gamma_{23}} \\ \gamma_{13} \end{Bmatrix}$$

Structural elements: beams, plates, shells

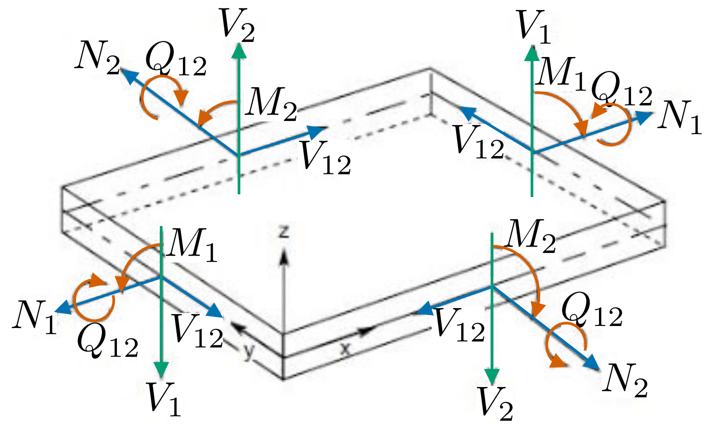
Mindlin's plates (next lecture …)

- Internal resultants

$$N_1 = b\int_{-h/2}^{h/2} E(\bar{\epsilon}_{11} + z\bar{\kappa}_{11})\mathrm{d}z = EA\bar{\epsilon}_{11}$$

$$M_2 = b\int_{-h/2}^{h/2} E(z\bar{\epsilon}_{11} + z^2\bar{\kappa}_{11})\mathrm{d}z = EI_{22}\bar{\kappa}_{11}$$

$$V = K_\mathrm{s} b\int_{-h/2}^{h/2} \frac{E}{2(1+\nu)}\bar{\gamma}_{13}\mathrm{d}z = \frac{K_\mathrm{s}AE}{2(1+\nu)}\bar{\gamma}_{13}$$



… and more!!!

- Constitutive relation
  (relationship between internal resultants and kinematic variables)

kinematic variables

$\bar{\epsilon}_{11}$ $\bar{\epsilon}_{22}$ $\bar{\kappa}_{11}$ $\bar{\kappa}_{22}$

extensional strains, bending strains

kinematic variables (cont.)

$\bar{\gamma}_{12}$ $\bar{\kappa}_{12}$ $\bar{\gamma}_{13}$ $\bar{\gamma}_{23}$

contribution of the mid-plane displacement to in-plane shear, contribution of normal rotations to in-plane shear, and shear components

Structural elements: beams, plates, shells

Any questions?