Heteroscedasticity

Econ 671

Purdue University

Is the OLS estimator still consistent? Let's see ...

$$\begin{aligned}\hat{\beta} &= \beta + (X'X)^{-1}X'\epsilon \\ &= \beta + \left(\frac{X'X}{n}\right)^{-1}\frac{X'\epsilon}{n}.\end{aligned}$$

Therefore,

$$plim(\hat{\beta}) = \beta + plim\left[(X'X/n)^{-1}\right] plim\left(X'\epsilon/n\right).$$

From our previous lecture, we continue to assume that

$$plim\left[(X'X/n)^{-1}\right] = Q^{-1},$$

where Q^{-1} has finite elements.

Is the OLS estimator still consistent? Let us now consider the second term. Again, we can write:

$$n^{-1}X'\epsilon = \frac{1}{n}\sum_i x_i'\epsilon_i.$$

In our previous proofs, we employed a law of large numbers based on *iid* data together with mean independence to argue:

$$\frac{1}{n}\sum_i x_i'\epsilon_i \xrightarrow{p} E(x_i'\epsilon_i) = 0.$$

In this case, however, *the data are independent, but NOT identically distributed.*

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We can, however, use a different LLN for independent observations. For example, Chebychev's LLN can be applied. This states:

Theorem

Let x_i be a series of independent, but not necessarily identically distributed random variables with $E(x_i) = \mu_i$ and $Var(x_i) = \sigma_i^2 < \infty$. If

$$\frac{1}{n^2}\sum_{i=1}^{\infty}\sigma_i^2 \rightarrow 0$$

then

$$\overline{X}_n - \frac{1}{n}\sum_{i=1}^{n}\mu_i \xrightarrow{p} 0$$

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In our regression application, provided the sequence of partial sums

$$\sum_i \sigma_i^2 x_i' x_i$$

doesn't grow too fast, we can use the theorem to show

$$n^{-1} X' \epsilon \to 0$$

so that the OLS estimator remains consistent.