Economics 671
Solutions: Problem Set #1

(1) First, let's expand the traditional formula. Note that

$$\sum_i (x_i - \overline{x})(y_i - \overline{y}) = \sum_i x_i y_i - n\overline{xy} = \sum_i x_i y_i - \frac{1}{n}(\sum_i x_i)(\sum_i y_i).$$

Likewise, for the denominator:

$$\sum_i (x_i - \overline{x})^2 = \sum_i x_i^2 - n\overline{x}^2 = \sum_i x_i^2 - \frac{1}{n}\left(\sum_i x_i\right)^2.$$

Multiplying each of these by n, it follows that the traditional formula for the OLS slope estimator is equivalent to:

$$\hat{\beta}_2 = \frac{n\sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n\sum_i x_i^2 - (\sum_i x_i)^2}.$$

Now, direct application of our $(X'X)^{-1}X'y$ formula gives:

$$\hat{\beta} = \begin{bmatrix} n & \sum_i x_i \\ \sum_i x_i & \sum_i x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i y_i \\ \sum_i x_i y_i \end{bmatrix}.$$

Inverting the 2×2 matrix and picking of the second element of $\hat{\beta}$ shows that

$$\hat{\beta}_2 = \frac{n\sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n\sum_i x_i^2 - (\sum_i x_i)^2},$$

which is the same as the above.

(2a) To accomplish this, first define A as a $k \times k$ matrix that permutes the columns of X. The matrix A will take the form that each row of A will contain only one 1, and all other entries will be zero. Similarly, each column of A will contain only one 1, and all the other elements will be zero.

Just to provide a specific example, suppose that X is an $n \times 3$ matrix. And, let A be defined as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

If we let

$$Z = XA,$$

it follows that Z is simply an $n \times 3$ matrix that puts the third column of X as the first column of Z, the first column of X as the second column of Z, and the second column of X as the third column of Z. So, A is constructed to simply permute the column order of X in this particular way. This generalizes beyond the three-variable case and A can be constructed to accomplish column rearranging of any desired sort. Finally, note that, however constructed,

$$AA' = A'A = I_k$$

so that $A^{-1} = A'$. Now, let us consider the OLS estimator of a regression of y on Z and call this estimator $\hat{\theta}$:

$$\begin{aligned} \hat{\theta} &= (Z'Z)^{-1}Z'y \\ &= \left([XA]'[XA]\right)^{-1}[XA]'y \\ &= \left(A'(X'X)A\right)^{-1}A'X'y \\ &= A^{-1}(X'X)^{-1}(A')^{-1}A'X'y \\ &= A^{-1}(X'X)^{-1}X'y \\ &= A'\hat{\beta} \end{aligned}$$

This also implies

$$A\hat{\theta} = \hat{\beta},$$

where $\hat{\beta}$ is the OLS estimator of a regression of X on y. Thus, the OLS estimator produced when rearranging the columns of X is simply a rearrangement of the initial OLS estimates.

(2b) No, this does not violate the rank condition. r and x will be correalted typically, but this construction does not imply that any one of the coumns of the X matrix can be expressed as a linear combination of the other columns of X.

(2c) This follows similarly to $(2a)$ with the construction

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where it follows that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, if we stack our first regression into a $n \times 4$ matrix X and the second regression into an $n \times 4$ matrix Z, and apply the same result as was described in (2a), we have

$$\hat{\pi} = A^{-1}\hat{\beta}$$

or

$$\begin{bmatrix} \hat{\pi}_0 \\ \hat{\pi}_1 \\ \hat{\pi}_2 \\ \hat{\pi}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 - \hat{\beta}_1 \\ \hat{\beta}_3 \end{bmatrix}.$$

This makes sense since, upon subbing $w_i = r_i - x_i$, we get

$$\begin{aligned} y_i &= \beta_0 + \beta_1(r_i - x_i) + \beta_2 x_i + \beta_3 z_i + u_i \\ &= \beta_0 + \beta_1 r_i + (\beta_2 - \beta_1)x_i + \beta_3 z_i + u_i \end{aligned}$$

This substituion gives the same relationship between the coefficients β and π as suggested among the estimates $\hat{\pi}$ and $\hat{\beta}$ above.