Economics 671
Solutions: Problem Set #3

An accompanying MATLAB program is provided on the website that answers most of the questions posed in the problem set. It remains for us to analytically derive the sampling distribution for arbitrary n.

(a) First note,

$$\begin{aligned}
E(\exp(tx)) &= \int_0^\infty \lambda^{-1}\exp(-\lambda^{-1}x)\exp(tx)dx \\
&= \lambda^{-1}\int_0^\infty \exp(-(\lambda^{-1}-t)x)dx \\
&= \lambda^{-1}(\lambda^{-1}-t)^{-1}\int_0^\infty (\lambda^{-1}-t)\exp(-(\lambda^{-1}-t)x)dx \\
&= \lambda^{-1}(\lambda^{-1}-t)^{-1} \\
&= (1-\lambda t)^{-1}, t < \lambda^{-1}
\end{aligned}$$

where the integrand in the third line is recognized as the integral of an $Exponential[(\lambda^{-1}-t)^{-1}]$ density, which equals one provided $t < \lambda^{-1}$.

(b) Note that

$$\begin{aligned}
E(\exp(tY)) &= E[\exp(t\sum_i x_i)] \\
&= \prod_{i=1}^n E(\exp(tx_i)) \\
&= \prod_{i=1}^n (1-\lambda t)^{-1} \\
&= (1-\lambda t)^{-n}
\end{aligned}$$

where the second line follows since the x_i are independent, and the third line follows since they are assumed to follow an $Exponential(\lambda)$ distribution, and given the result in part (a).

(c) As for the sampling distribution of our estimator, note

$$\hat{\mu}_n = \overline{y}_n = \mu + \overline{\epsilon}_n = (\mu-\lambda) + \frac{1}{n}\sum_i x_i,$$

where $x_i \sim Exponential(\lambda)$ and $\sum_i x_i \sim Gamma(n,\lambda)$. The sampling distribution of $\hat{\mu_n}$ is thus established by a change of variable.

Specifically, we note

$$\hat{\mu}_n = (\mu-\lambda) + \frac{1}{n}Y$$

so that

$$Y = n[\hat{\mu}_n - (\mu - \lambda)].$$

The density of $\hat{\mu}_n$ can be otained using the change of variable technique, noting that the Jacobian of this transformation, $|\partial Y/\partial \hat{\mu}_n|$ is simply n. Thus,

$$\begin{aligned} p(\hat{\mu}_n) &= n[\Gamma(n)\lambda^n]^{-1}[n(\hat{\mu}_n - (\mu-\lambda)]^{n-1}\exp(-n[\hat{\mu}_n - (\mu-\lambda)]/\lambda), \quad \hat{\mu}_n > \mu - \lambda \\ &= [\Gamma(n)\lambda^n]^{-1}n^n[(\hat{\mu}_n - (\mu-\lambda)]^{n-1}\exp(-n[\hat{\mu}_n - (\mu-\lambda)]/\lambda), \quad \hat{\mu}_n > \mu - \lambda. \end{aligned}$$

(d) To obtain the distribution of our statistic, which we will call $W = \sqrt{n}(\hat{\mu}_n - \mu)$, we can use our change of variable method one more time. Specifically, we know that

$$\hat{\mu}_n = Wn^{-1/2} + \mu,$$

yelding a Jacobian equal to $n^{-1/2}$. We thus obtain

$$p(W) = [\Gamma(n)\lambda^n]^{-1}n^{n-1/2}[Wn^{-1/2} + \lambda]^{n-1}\exp(-n[Wn^{-1/2} + \lambda)]/\lambda), \quad W > -\lambda\sqrt{n}.$$

MATLAB code that does the plotting and calculations is posted in the course webpage. Note that plots of the density functions are not directly comparable, as the exact sampling distribution of the estimator is discrete, whereas the limiting normal density is continuous. So, the two *density functions* would have the same general shape, although different scales. To make these comparableone could simply scale the discrete distribution up by a factor of $\sqrt{n}$ (the Jacobian of the transformation for the continuous case). Finally, to side-step this issue altogether (and directly address the issue of convergence in distribution), we plot the two CDF's instead.

Results can be changed by changing the number of observations (*nobs*) in my code, as well as *iter*, as the sample size changes in the different experiments.