Economics 690
Problem Set: Using the Inverse Transform and Importance Sampling

Consider the triangular density:

$$p(x) = 1 - |x|, \quad -1 \leq x \leq 1.$$

The purpose of this problem set is to illustrate how the methods of composition and inversion can be used to simulate draws from this triangular density.

We begin by noting the identity:

$$p(x) = p(x|0 < x \leq 1)\Pr(0 < x \leq 1) + p(x| - 1 \leq x \leq 0)\Pr(-1 \leq x \leq 0). \tag{1}$$

The density function $p(x|0 < x \leq 1)$, for example, is simply a truncated version of the marginal $p(x)$, and is obtained by noting

$$p(x|0 < x \leq 1) = \frac{p(x)}{\Pr(0 < x \leq 1)} I(0 < x \leq 1).$$

(a) Derive expressions for the two truncated densities in (1) as well as the probabilities appearing in that equation.

The *method of composition* notes that

$$p(x) = \int_Z p(x|z)p(z),$$

and thus a draw from $p(x)$ can be obtained by first drawing a z^* from the marginal $p(z)$ and then drawing x from $p(x|z = z^*)$.

We can apply this technique to our particular problem given the representation of $p(x)$ in (1). Specifically, let z be a binary variable denoting the event that $0 < x \leq 1$. That is, $z = 1$ if this event is true, while $z = 0$ if $-1 \leq x \leq 0$.

With this notation, we can write an equivalent version of (1):

$$p(x) = p(x|z = 1)\Pr(z = 1) + p(x|z = 0)\Pr(z = 0). \tag{2}$$

In this form, we can see the potential application of the method of composition, as the last equation on the previous page is simply a discrete version of the $p(x) = \int_Z p(x|z)p(z)$ formula. Thus, we can obtain a draw from $p(x)$ as follows:

1. Draw z from its binary distribution, [which is straight-forward given your answer to part (a)].
2. Depending on the realized value of z, draw from either $p(x|z = 1)$ or $p(x|z = 0)$. Note that these correspond to the truncated densities of x given in part (a).

(b) The first step of the above process is rather simple. The second requires a little more thought. Using the *method of inversion*, describe a technique for drawing from $p(x|z = 1)$ and $p(x|z = 0)$. [This, of course, requires you to calculate the cdfs associated with each of these truncated densities, and then solving the implied equation. If you do these correctly, each equation will be a quadratic, and only one solution of the quadratic polynomial will make any sense.]

(c) Using your answers to (a) and (b), write a MATLAB program to obtain 25,000 draws from the triangular density. [If you write your code efficiently, this should be really fast]. If you have done everything correctly, a histogram of your draws should resemble, well, a triangle!

(2) Reproduce the importance sampling estimates presented in your lecture notes. That is, consider the exercise of using importance sampling to calculate the first two moments of a standard normal random variable using Laplace(0,2) and Laplace(3,1) distributions as importance functions.

Your "final answer" should be a replication of the two sampling distribution graphs presented in your lecture notes (which contain sampling distributions associated with the first two moments as well as RNE).