Economics 690
Problem Set: Prior-Posterior Analysis and Conjugacy

**(1)** Using MATLAB, reproduce the graphs shown in the conjugate Bernoulli analysis example of your lecture notes. (This application made use of the Purdue football data).

Produce graphs for the following cases (note that your graphs should resemble those in your lecture notes, though in some cases they may be a little different:

- A uniform prior for $\theta$ is employed. (Plot the prior and posterior). *Note:* "beta" in MATLAB will calculate the Beta function and the "hold on" command in MATLAB will allow you to plot several curves on the same graph.
- An "optimistic" prior for $\theta$ is employed, i.e., $\theta \sim B(16, 4)$. (Plot the prior and posterior).
- Generated data experiments with sample sizes equal to $N = 25, 100$ and 1,000 under the "optimistic" prior. Note that the actual data are no longer used, but instead, you must generate data sets of size $N$ where each observation is drawn with "success" probability equal to .25. Note: The "rand" command in MATLAB will generate a number which is drawn uniformly over the unit interval. The realized value of this process can be used to generate the desired $y$ variables.

**(2)** As discussed in the course lecture notes, consider again the Bernoulli sampling model:

$$p(y_t|\theta) = \begin{cases} \theta & \text{if } y_t = 1 \\ 1-\theta & \text{if } y_t = 0 \end{cases} = \theta^{y_t}(1-\theta)^{1-y_t}$$

with associated likelihood function:

$$L(\theta) = \theta^m (1-\theta)^{T-m},$$

$m = \sum_t y_t$. Instead of using the beta prior, as discussed in the notes, suppose instead that you employ a *truncated* Beta prior of the form:

$$p_B(\theta|\underline{\alpha}, \underline{\delta}) \propto \theta^{\underline{\alpha}-1}(1-\theta)^{\underline{\delta}-1} I\left(a < \theta < b\right),$$

where $I$ is an indicator function and $(a, b)$ with $0 < a < b < 1$ denotes the support of the density. (That is, the indicator function assigns zero mass outside this region).

Derive the posterior density for $\theta$ under this prior. Is the truncated Beta prior a conjugate prior for the Bernoulli sampling model?

**(3)** Consider the exponential sampling model, as presented in your class notes, with likelihood function:

$$L(\theta) = \prod_{t=1}^{T} f_{EXP}(y_t|\theta) = \prod_{t=1}^{T} \theta \exp(-\theta y_t) = \theta^T \exp(-T\overline{y}\theta).$$

In your class notes, we showed that the Gamma prior is a conjugate prior for this model. Suppose instead that you specify a *lognormal prior* for $\theta$ of the form:

$$p(\theta|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\theta} \exp\left(-\frac{1}{2\sigma^2}(\ln\theta - \mu)^2\right), \quad \theta > 0.$$

**(a)** Derive the posterior density for $\theta$. Is the lognormal prior a conjugate prior for this model?

**(b)** Suppose that $T = 25$, $\overline{y} = 1$, $\sigma^2 = .5$ and $\mu = 1$. Using MATLAB, provide a program that plots the *prior and posterior* densities for $\theta$. Here is how such a program might proceed:

- Make a grid of $\theta$ values over which the densities (both prior and posterior) will be plotted. [The "linspace" command is useful for this. A reasonable range for this example is the interval $(0, 4)$.]
- Plot the prior density over this interval.
- Note that the prior times likelihood only gives the posterior up to proportionality. (That is, the density you found in (a) will not integrate to unity). Calculate the (unnormalized) posterior density that you found in (a) over this grid of values.
- You need to make the posterior density proper (i.e., make it integrate to unity). To do this, you can numerically approximate the integral under the (unnormalized) posterior using the "trapz" command. [For example, "trapz(thetagrid,density)" in MATLAB will approximate the area under the curve "density" evaluated at discrete support points "thetagrid"]. Once you calculate this integral, you can normalize the posterior by multiplying the (unnormalized) density by the reciprocal of this integral.
- Plot the normalized (proper) posterior together with the lognormal prior for $\theta$.