Lesson 32, §16.3 The Fundamental Theorem for Line Integrals

Fundamental Theorem of Calculus

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Theorem Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(\mathbf{b})) - f(\mathbf{r}(a))$$

Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Theorem Suppose $mathbfF$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field[1] on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem If $\mathbf{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is a conservative vector field, where p and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Theorem Let $\mathbf{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open **simply-connected region** D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.
Note: A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D. Intuitively speaking, a imply-connected region contains no hole and con't consist of two separate pieces.

[1] Lesson 30-31