On Nash-Cournot Games with Price Caps

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Abstract

In this paper, we study an N-person Nash-Cournot game with a price cap. Under certain mild conditions, we show that this game can be formulated as a complementarity problem. Based on this formulation, we study various properties of the game, including equilibrium existence, computability, and a description of the equilibrium set when there are multiple equilibria.

Keywords: Generalized Nash equilibrium, Nash-Cournot, linear complementarity problem

1. Introduction

Generalized Nash equilibrium problems (GNEP) describe a class of games in which each player’s strategy space is a set parameterized by other players’ actions. While first appeared in Rosen \cite{Rosen65} in 1965, only recently have such problems gained increasing attention, due to their appearance in various application areas, including electricity markets with system operators, emissions trading, and games with shared resource constraints, to name a few. GNEPs in general are very hard problems \cite{Facchinei2007}, which lack both theory foundation and efficient computational methods. A prevailing approach to deal with GNEPs is described in Harker \cite{Harker1991}, which states that under general convexity assumptions, a generalized Nash equilibrium can be obtained through solving a variational inequality (VI). Though this approach greatly simplifies the computation of an equilibrium of GNEPs, the relationship between a (convex) GNEP and a VI is not one-to-one correspondence. More specifically, while a solution to the VI derived from a GNEP is an equilibrium to the original problem, the reverse is not true; that is, not all GNEP’s equilibria satisfy a VI. Consequently, theories on the uniqueness of VI solutions are
not applicable to show uniqueness of equilibria to GNEPs. In this work, we consider a specific class of games, Nash-Cournot games with price caps, that can be converted to GNEPs, and provide more definite answers to the existence, uniqueness and computability of an equilibrium of such games than the VI approach can provide.

Nash-Cournot games, originated from [4], are widely-used economic models to describe firms’ competition on the amount of outputs (i.e., quantities) they produce. Such games represent one form of market power abuse in the sense that firms strategically manipulate their production output of a certain goods to raise its price above the perfectly competitive level. One regulatory tool to curb this type of market power abuse is to set a market price cap on certain goods. Examples of price-cap regulation abound in various industries, including wholesale electricity markets [5, 6], telecommunication [7], airports [8], privatized public utilities [9], and emissions trading markets [10].

While Nash-Cournot games are well-studied both theoretically and computationally, Nash-Cournot games with price caps are not, mainly due to the non-smoothness of the price function caused by a fixed price cap. The games we consider are directly inspired by the work in Hobbs and Pang [5], where they present an instance of games with price caps arising from a deregulated wholesale electricity market. The games are further extended in Pang and Sun [11] to include Nash games with piecewise quadratic cost functions. Though existence of equilibria of such games are shown in [11], neither of the paper discusses the uniqueness of equilibria. While uniqueness of equilibria cannot be expected for the general setting of such games, we nonetheless can characterize the entire equilibrium set. In addition, we present a simple reformulation of the games to linear complementarity problems (LCPs) and show the convergence of Lemke’s algorithm to solve the resulting LCPs. Our major goal of this paper is to provide a solid theoretical and computational foundation of Nash-Cournot games with price caps, which would allow sophisticated analysis of various policy issues involving price-cap regulation that are otherwise difficult to perform. Such policy issues include optimal price-cap regulation [12], second-best equilibrium [13], investments under price caps [14, 15] and safety valves on cap-and-trade markets of emissions permits [16].

The rest of the paper is organized as follows. In Section 2, we provide the mathematical model of Nash-Cournot games with price caps. Section 3 presents the existence of equilibria. The complementarity problem reformulation is presented in Section 4. Computability and uniqueness of equilibria are discussed in Section 5. We then conclude the paper in Section 6 and discuss potential application areas.
2. Nash-Cournot Games with Price Caps

Consider a set of $F$ firms indexed by $f$, each with a capacity $K_f$. Each firm $f$ aims to maximize its profit by choosing a production level $q_f$, under the assumption that its action would not affect other firms’ actions. Firm $f$’s optimization problem is as follows.

$$
\max_{q_f} \pi_f(q_f; Q_{-f}) \equiv P(Q_{-f} + q_f)q_f - c_f q_f \\
\text{s.t. } q_f \in X_f \equiv \{q_f \in \mathbb{R} : 0 \leq q_f \leq K_f\},
$$

where $Q_{-f}$ is the total production of all rival firms, i.e.,

$$
Q_{-f} = \sum_{f \neq \nu = 1}^{F} q_{\nu},
$$

$c_f$ is the marginal production cost of firm $f$, and $P(\tau)$ is the price function of the market. Throughout this study, we consider a linear price function (i.e., linear inverse demand function) with a fixed price cap. The explicit form of the price function is as follows.

$$
P(\tau) = \begin{cases} 
P & \text{if } \tau \leq Q \\
\alpha - \beta \tau & \text{if } \tau \geq Q,
\end{cases}
$$

where $P$ is the price cap. $\alpha > 0$ and $\beta > 0$ represent the linear price function’s intersection at the quantity-axis and its slope, respectively. The quantity corresponding to the price cap based on the linear price function, denoted as $Q$, can then be expressed explicitly as $Q = \frac{\alpha - P}{\beta}$. An illustration of $P(\tau)$ is shown in Figure 1.

Throughout the paper, we make the following assumptions.

(A) The price cap is greater than the marginal production cost of every firm, i.e., $P > c_f$, for each $f = 1, \cdots, F$.

(B) The sum of the capacities of all the firms is greater than or equal to $Q$, i.e.,

$$
\sum_{f=1}^{F} K_f \geq Q.
$$
Remark 1. Assumptions (A) and (B) are reasonable. For (A), if it is not valid for a certain firm, then under the rationality of profit maximization, the firm will produce zero quantity in an equilibrium, and hence there is no need to consider this firm in the game. If (B) is not satisfied, then the game is the same as a non-price-capped game.

An equilibrium of the Nash-Cournot game with a price cap is a vector of the firms’ productions \( q^* = (q_f^*)_{f=1}^F \), such that no firm has the incentive to deviate from its production level given that other firms stick to their equilibrium production level. Mathematically, \( q^* \) satisfies: for each \( f = 1, \ldots, F \),

\[
q_f^* \in \text{argmax } P(Q_{-f}^* + q_f)q_f - c fq_f \\
\text{s.t. } 0 \leq q_f \leq K_f.
\]

3. Equilibrium Existence

To study this Nash-Cournot game, we start with examining individual firm’s optimization problem (1). We first show that the payoff function is a concave function for each \( f \).

Lemma 1. Assume that (A) holds. With a given \( Q_{-f} \geq 0 \), the payoff function \( \pi_f(q_f; Q_{-f}) \) with the price function given in (2) is concave with respect to \( q_f \), for each \( f = 1, \ldots, F \).
Proof. We prove this by considering the different ranges of $Q_{-f}$. It is easy to show concavity of $\pi_f(q_f; Q_{-f})$ when $Q_{-f} \geq \overline{Q}$ or $Q_{-f} \leq \overline{Q} - K_f$. We focus on the case when $\overline{Q} - K_f < Q_{-f} < \overline{Q}$. Let $q_f^1$ and $q_f^2$ be any two distinct points from firm $f$’s feasible region $X_f$, and $\tilde{q}_f = \lambda q_f^1 + (1 - \lambda)q_f^2$ with $\lambda \in [0, 1]$. If

\[
Q_{-f} + q_f^i > \overline{Q} \quad \text{or} \quad Q_{-f} + q_f^i < \overline{Q}
\]

for $i = 1, 2$, it is again easy to show that

\[
\pi_f(\tilde{q}_f; Q_{-f}) \geq \lambda \pi_f(q_f^1; Q_{-f}) + (1 - \lambda)\pi_f(q_f^2; Q_{-f}).
\]

Now without loss of generality, assume that

\[
Q^1 \equiv Q_{-f} + q_f^1 < \overline{Q} < Q_{-f} + q_f^2 \equiv Q^2.
\]

Then $P(Q^1) = \overline{P} > P(Q^2)$. If $Q_{-f} + \tilde{q}_f < \overline{Q}$, we have that

\[
\pi_f(\tilde{q}_f; Q_{-f}) = (\overline{P} - c_f)\tilde{q}_f = \lambda(\overline{P} - c_f)q_f^1 + (1 - \lambda)(\overline{P} - c_f)q_f^2
\]

\[
> \lambda\pi_f(q_f^1; Q_{-f}) + (1 - \lambda)\pi_f(q_f^2; Q_{-f}),
\]

where the last inequality is directly implied by the fact that $P(Q^1) = \overline{P} > P(Q^2)$. If $Q_{-f} + \tilde{q}_f > \overline{Q}$, let $\tilde{P}(Q)$ denote the uncapped price function; i.e., $\tilde{P}(Q) = \alpha - \beta Q$ for $Q \geq 0$. Then

\[
\pi_f(\tilde{q}_f; Q_{-f}) = \tilde{P}(Q_{-f} + \tilde{q}_f) - c_f\tilde{q}_f
\]

\[
\geq \lambda[\tilde{P}(Q^1) - c_f]q_f^1 + (1 - \lambda)[\tilde{P}(Q^2) - c_f]q_f^2
\]

\[
> \lambda\pi_f(q_f^1; Q_{-f}) + (1 - \lambda)\pi_f(q_f^2; Q_{-f}),
\]

where the first inequality uses the concavity of $\pi_f(q_f; Q_{-f})$ for all $q_f > \overline{Q} - Q_{-f}$, and the second inequality comes from the fact that $\tilde{P}(Q^1) > P(Q^1) = \overline{P}$. Hence, we have shown that $\pi_f(q_f; Q_{-f})$ is (strictly) concave when $\overline{Q} - K_f < Q_{-f} < \overline{Q}$, which completes the concavity proof of $\pi_f(q_f; Q_{-f})$ for $Q_{-f} \geq 0$. 

With concavity (and continuity) of firms’ payoff functions, together with the nonempty, compact and convex set of each firm’s feasible actions, the existence of a (pure-strategy) equilibrium to the Nash-Cournot game with a price cap follows directly from well-established results in economics (see Theorem 2.2 in [17], for example).

Theorem 2. Under Assumption (A), an equilibrium $q^*$ satisfying (3) exists. 

A consequence of Lemma 1 is the continuity of the best response functions, which are optimal solution mappings, denoted by $\phi(Q_f)$, and are defined as follows.

$$
\phi_f(Q_f) \equiv \arg\max_{0 \leq q_f \leq K_f} \{P(Q_f + q_f)q_f - c_f q_f\}, \ f = 1, \ldots, F.
$$

(4)

Best response functions are important in understanding a game and play an important role in evolutionary game theory. Hence, the results shown here can be useful to other studies involving games with price caps.

**Proposition 3.** Under Assumption (A) and (B), $\phi_f(Q_f)$ is continuous with respect to $Q_f$ for all $Q_f \geq 0$ for each $f = 1, \ldots, F$.

**Proof.** Note that the best response function $\phi_f(Q_f)$ may not be a single-valued function. It can been shown however for the game defined in (3), it is single-valued. This follows from the fact that with $\beta > 0$, $\pi_f(q_f; Q_f)$ is strictly concave with respect to $q_f$ when $Q_f > Q - K_f$. When $Q_f < Q - K_f$, it is not difficulty to see that the unique optimal solution to firm $f$ is $q_f^* = K_f$. With concavity of firms’ payoff function and convexity of firms’ feasible action regions, the continuity of the optimal solution mapping with respect to a parameter follows directly from the result in Robinson and Day [18].

4. Complementarity Problem Formulation

In this section we show that the Nash-Cournot game with a price cap (3) can be formulated into a linear complementarity problem (LCP), which will facilitate both theoretical analysis and computation of the equilibria. We first show a result needed to establish the LCP formulation.

**Lemma 4.** Assume that (A) and (B) hold. Let $q^*$ be an equilibrium satisfying (3). It follows that

$$
\sum_{f=1}^{F} q_f^* \geq Q.
$$

(5)

**Proof.** Assume for the sake of contradiction that there is an equilibrium solution $q^*$ such that

$$
\sum_{f=1}^{F} q_f^* < Q.
$$
Since
\[ \sum_{f=1}^{F} K_f \geq \overline{Q}, \]
there must exist a firm \( \nu \) such that \( q^*_\nu < K_\nu \). Let \( \epsilon > 0 \) be a positive number satisfying
\[ \epsilon < \min\{\overline{Q} - \sum_{f=1}^{F} q^*_f, K_\nu - q^*_\nu\}. \]

It is clear that
\begin{align*}
P(Q^*_\nu + q^*_\nu + \epsilon)(q^*_\nu + \epsilon) - c_\nu \cdot (q^*_\nu + \epsilon) \\
= \overline{P} \cdot (q^*_\nu + \epsilon) - c_\nu \cdot (q^*_\nu + \epsilon) \\
> \overline{P}q^*_\nu - c_\nu q^*_\nu \\
= P(Q^*_\nu + q^*_\nu)q^*_\nu - c_\nu q^*_\nu. \end{align*}

This contradicts the equilibrium condition (3).

With Lemma 4, we can formulate the Nash-Cournot game as a linear complementarity problem. Notice that, from Lemma 4 we know that
\[ \overline{Q} \leq Q^*_{-f} + q^*_f \leq Q^*_{-f} + K_f, \]
for each firm \( f \). Therefore, in an equilibrium solution, each individual firms optimization problem is shown below:
\begin{align*}
\max & \quad -\beta q^2_f + (\alpha - \beta Q^*_{-f} - c_f)q_f \\
\text{s.t.} & \quad q_f \leq K_f \\
& \quad q_f \geq \overline{Q} - Q^*_{-f} \\
& \quad q_f \geq 0. \end{align*}

Notice that we have converted the original Nash-Cournot game into a generalized Nash game where the rivals’ variables not only appear in the objective function but also appear in the constraints.

Now rewrite (6) and introduce multipliers for the constraints, we have:
\begin{align*}
\min & \quad \beta q^2_f - (\alpha - \beta Q^*_{-f} - c_f)q_f \\
\text{s.t.} & \quad q_f \leq K_f, \quad \cdots \cdots \lambda_f \\
& \quad -q_f \leq Q^*_{-f} - \overline{Q}, \quad \cdots \cdots \mu_f \\
& \quad -q_f \leq 0, \quad \cdots \cdots \xi_f \end{align*}

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It is clear that (7) is a quadratic problem with strongly convex objective function. Therefore, \( q_f^* \) is the optimal solution of (7) if and only if there exist multipliers \( \lambda_f \), \( \mu_f \), and \( \xi_f \) such that

\[
2\beta q_f^* - (\alpha - \beta Q^*_f - c_f) + \lambda_f - \mu_f - \xi_f = 0
\]
\[
0 \leq \lambda_f \perp K_f - q_f^* \geq 0
\]
\[
0 \leq \mu_f \perp Q^*_f - Q + q_f^* \geq 0
\]
\[
0 \leq \xi_f \perp q_f^* \geq 0.
\]  

Solving \( \xi_f \) from the first equation in (8) and substituting into the last complementarity condition, we obtain the following equivalent linear complementarity problem.

\[
0 \leq \lambda_f \perp K_f - q_f^* \geq 0
\]
\[
0 \leq \mu_f \perp Q^*_f - Q + q_f^* \geq 0
\]
\[
0 \leq q_f^* \perp 2\beta q_f^* - \alpha + \beta Q^*_f + c_f + \lambda_f - \mu_f \geq 0.
\]  

Stacking up the optimality conditions for the individual firms, we have that \( q^* \) is an equilibrium if and only if there exist vectors \( \lambda = (\lambda_f)_{f=1}^F \) and \( \mu = (\mu_f)_{f=1}^F \) such that

\[
0 \leq q^* \perp \beta q^* - \alpha 1 + \beta Eq^* + c + \lambda - \mu \geq 0
\]
\[
0 \leq \lambda \perp K - q^* \geq 0
\]
\[
0 \leq \mu \perp Eq^* - Q 1 \geq 0,
\]  

where \( K = (K_f)_{f=1}^F \), \( 1 \) is the all one vector of size \( F \), \( c = (c_f)_{f=1}^F \), matrix \( E \) is an all one matrix of size \( F \times F \), and matrix \( I \) is the identity matrix of size \( F \times F \).

**Theorem 5.** Assume conditions (A) and (B) hold. A vector \( q^* \) is an equilibrium of the Nash-Cournot game with price cap if and only if there exists \( \lambda \), and \( \mu \) such that \( (q^*, \lambda, \mu) \) is a solution of (10).

5. Solution Analysis

In the last section, we have derived an LCP formulation of the game. This LCP can be written as the standard form as follows:

\[
0 \leq x \perp Mx + u \geq 0,
\]  

\[8\]
where

\[ x \equiv \begin{pmatrix} q \\ \lambda \\ \mu \end{pmatrix}, \quad u \equiv \begin{pmatrix} \alpha 1 + c \\ K \\ -Q1 \end{pmatrix}, \quad \text{and} \quad M \equiv \begin{bmatrix} \beta(I + E) & I & -I \\ -I & 0 & 0 \\ I & 0 & 0 \end{bmatrix}. \]

Notice that

\[ M = \begin{bmatrix} \beta(I + E) & I & -I \\ -I & 0 & 0 \\ I & 0 & 0 \end{bmatrix}_M + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E - I & 0 & 0 \end{bmatrix}_M, \]

with the first matrix \( M_1 \) being positive semidefinite and the second matrix \( M_2 \) being nonnegative (entry-wise). Hence \( M \) is copositive. However, due to the fact that even all the firms have a common constraint \( \sum_{f=1}^F q_f \geq \bar{Q} \), the multiplier of this constraint for different firms might be different, we are not able to show the successful termination of Lemke’s method [19]. Therefore, similar to [19], we study a restricted multiplier formulation of the game.

5.1. Restricted multiplier formulation

In the restricted multiplier formulation, we require the multipliers of the common constraint to be the same. Therefore, we have the following LCP formulation where the vector of multipliers \( \mu \) in (10) has been replaced by a scalar \( \kappa \).

\[
\begin{align*}
0 \leq q & \perp \beta Iq - \alpha 1 + \beta Eq + c + \lambda - \kappa 1 \geq 0 \\
0 \leq \lambda & \perp K - q \geq 0 \\
0 \leq \kappa & \perp 1^Tq - \bar{Q} \geq 0
\end{align*}
\]  \hspace{1cm} (12)

By letting

\[ y \equiv \begin{pmatrix} q \\ \lambda \\ \kappa \end{pmatrix}, \quad w \equiv \begin{pmatrix} \alpha 1 + c \\ K \\ -Q \end{pmatrix}, \quad \text{and} \quad N \equiv \begin{bmatrix} \beta(I + E) & I & -1 \\ -I & 0 & 0 \\ 1^T & 0 & 0 \end{bmatrix}, \]

we can rewrite (12) as the following standard form:

\[ 0 \leq y \perp Ny + w \geq 0. \]
Recall that a matrix $M$ is semimonotone if for every vector $x \succeq 0$, there exists a component $x_i > 0$ such that $(Mx)_i \geq 0$ [20]. Applying the following lemma, we can establish the successful termination of the Lemke’s method.

**Lemma 6.** [21, Lemma 2] Let $M$ be a semimonotone matrix. If $d > 0$ is such that for every $\tau > 0$, SOL$(q + \tau d, M)$ is bounded, then the LCP$(q, M)$ has a solution that can be computed by Lemke’s algorithm with $d$ as the covering vector.

**Theorem 7.** Assume (A) and (B) hold. The LCP (12) has a solution which can be computed by Lemke’s algorithm with 1 as the covering vector.

**Proof.** We first show that the matrix $N$ is semimonotone. In fact, notice that

$$N + N^T = \begin{bmatrix} 2\beta(I + E) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Since $I + E$ is the sum of a positive definite matrix and a positive semi-definite matrix, it is positive definite. This shows that $N + N^T$ is positive semidefinite. It follows that $N$ is positive semidefinite and hence semimonotone. Now assume that there exists $\tau > 0$ such that SOL$(w + \tau 1, N)$ contains a sequence of solutions $(q_k, \lambda_k, \kappa_k)_{k=1}^\infty$ such that

$$0 \leq q_k \perp \beta I q_k - \alpha 1 + \beta E q_k + c + \lambda_k - \kappa_k 1 + \tau 1 \geq 0$$

$$0 \leq \lambda_k \perp K - q_k + \tau 1 \geq 0$$

$$0 \leq \kappa_k \perp 1^T q_k - \bar{Q} + \tau \geq 0. \quad (13)$$

It is clear that $q_k$ must be bounded since $q_k \leq K + \tau 1$, i.e., $q_f^k \leq K_f + \tau$ for each $f = 1, \cdots, F$. We claim that $\kappa_k$ must be bounded. Suppose not, then we can assume without loss of generality that $\kappa^k > 0$ for all $k$ sufficiently large, and hence by the third complementarity condition in (13)

$$\sum_{f=1}^F q_f^k = \bar{Q} - \tau \quad (14)$$

for all $k$ sufficiently large. From the first complementarity condition we have

$$\lambda_f^k \geq \kappa - \tau + \alpha - c_f - \beta \sum_{f=1}^F q_f^k - \beta q_f^k,$$
for all $k$ and all $f$. Since $\kappa^k$ is unbounded and $q_f^k$ is bounded for all $f$, we must have that $\{\lambda_f^k\}_{k=1}^\infty$ is unbounded for all $f$. Therefore, without loss of generality we may assume that $\lambda_f^k > 0$ for all $f$ and all $k$ sufficiently large. This, by the second complementarity condition in (13), implies that

$$q_f^k = \tau + K_f$$

for all $f$ and all $k$ sufficiently large. Thus

$$\sum_{f=1}^F q_f^k = F\tau + \sum_{f=1}^F K_f$$

for all $f$ and all $k$ sufficiently large. Compared with (14) we have

$$F\tau + \sum_{f=1}^F K_f = Q - \tau,$$

which in turn implies that

$$\sum_{f=1}^F K_f < Q.$$  

This contradicts Assumption (B). Next we show that $\lambda^k$ is also bounded. Assume that there exists an $f$ such that $\lambda_f^k$ is unbounded. Then we can assume without loss of generality that $\lambda_f^k > 0$ for all $k$ sufficiently large, and thus by the second complementarity condition in (13) we have

$$q_f^k = K_f + \tau > 0$$

for all $k$ sufficiently large. On the other hand, since $q^k$ and $\kappa^k$ are both bounded, we must have

$$\beta q_f^k + \sum_{f=1}^F q_f^k - \alpha + c_f + \lambda_f^k - \kappa^k + \tau > 0$$

for all $k$ sufficiently large. This implies that $q_f^k = 0$ for all $k$ sufficiently large due to the first complementarity condition in (13). We obtain a contradiction. Therefore, the sequence $(q^k, \lambda^k, \kappa^k)$ must be bounded. The theorem now follows from Lemma 6 readily.

We next study the uniqueness of equilibria. We first look at the restricted multiplier formulation (12). Since the matrix $N$ is a positive semi-definite matrix
(not symmetric), therefore for any two solutions $y^1 = (q^1, \lambda^1, \kappa^1)^T$ and $y^2 = (q^2, \lambda^2, \kappa^2)^T$ of the LCP $(w, N)$, we must have:

$$(N + N^T)(y^1 - y^2) = 0.$$ 

Therefore, we have

$$2(\beta I + E)(q^1 - q^2) = 0,$$

which implies that $q^1 = q^2$ since $\beta I + E$ is positive definite. It follows that even though the restricted multiplier formulation (12) might have multiple solutions, the equilibrium production $q^*$ is unique. This fact is stated in the following proposition.

**Proposition 8.** Given any two solutions $(q^1, \lambda^1, \kappa^1)$ and $(q^2, \lambda^2, \kappa^2)$ of (12), it holds that $q^1 = q^2$. 

5.2. Properties of the equilibrium set

We now move on to the original formulation (10). A natural question to ask is whether the $q$-uniqueness or the $Q$-uniqueness holds for (10) also. Given a solution $x = (q, \lambda, \mu)$ of (10), notice that if $\sum_{f=1}^{F} q_f = Q$, then $\mu = 0$ and hence the solution is also a solution of the restricted multiplier formulation. Therefore, the $q$ part must be unique. We now focus on the case where $\sum_{f=1}^{F} q_f = Q$. We prove the following technical lemma.

**Lemma 9.** Assume that conditions (A) and (B) hold. Let $q$ be an equilibrium of the Nash Cournot game with a price cap. Suppose it holds that $\sum_{f=1}^{F} q_f = Q$, then $q_f > 0$ for all $f = 1, \cdots, F$.

**Proof.** Let $q$ be an equilibrium. Suppose for the sake of contradiction that $q_f = 0$ for some $f$. Since $q_f = 0 < K_f$, by the second complementarity condition in (10) we have $\lambda_f = 0$. Notice that by the first complementarity condition in (10) we have

$$0 \leq \beta q_f - \alpha + \beta \sum_{f=1}^{F} q_f + c_f + \lambda_f - \mu_f$$

$$\leq -\alpha + \beta Q + c_f = c_f - \bar{P} < 0.$$ 

(15)

This is a contradiction. Therefore this lemma holds readily. 

With Lemma 9, we can prove the following $Q$-uniqueness result.
Theorem 10. Let \( q^1 \) and \( q^2 \) be two equilibria of the Nash-Cournot game with a price cap. It holds that
\[
\sum_{f=1}^{F} q^1_f = \sum_{f=1}^{F} q^2_f.
\]

Proof. Let \( q^1 \) and \( q^2 \) be two equilibria of the Nash-Cournot game with a price cap. It is clear that there exist vectors \( \lambda^1, \lambda^2, \mu^1 \) and \( \mu^2 \) such that \((q^1, \lambda^1, \mu^1)\) and \((q^2, \lambda^2, \mu^2)\) both satisfy complementarity conditions \((10)\). Assume that this theorem does not hold. Then we may assume without loss of generality that \( Q^1 \equiv \sum_{f=1}^{F} q^1_f > \sum_{f=1}^{F} q^2_f \equiv Q^2 \). We claim that \( Q^2 = Q \). Suppose not, then \( Q^1 > Q^2 > Q \). By the third complementarity condition in \((10)\) we derive that \( \mu^1 = \mu^2 = 0 \). Therefore \((q^1, \lambda, 0)\) and \((q^2, \lambda^2, 0)\) must also satisfy the restricted multiplier formulation \((12)\). By Proposition 8, we have \( q^1 = q^2 \). This is a contradiction. Thus we must have \( Q^2 = Q \).

By Lemma 9 we have \( q^2_f > 0 \) for all \( f = 1, \ldots, F \). Since
\[
\sum_{f=1}^{F} q^1_f > \sum_{f=1}^{F} q^2_f,
\]
there must exist an \( f \) such that \( q^1_f > q^2_f > 0 \). By the first complementarity condition in \((10)\) we have
\[
\beta q^1_f - \alpha + \beta \sum_{f=1}^{F} q^1_f + c_f + \lambda^1_f - \mu^1_f = \beta q^2_f - \alpha + \beta \sum_{f=1}^{F} q^2_f + c_f + \lambda^2_f - \mu^2_f = 0,
\]
which implies that
\[
\beta (q^1_f - q^2_f) + \beta (Q^1 - Q^2) + (\lambda^1_f - \lambda^2_f) - (\mu^1_f - \mu^2_f) = 0.
\]
Notice that \( q^1_f \leq K_f \), and thus \( q^2_f < K_f \). It follows from the second complementarity condition in \((10)\) that \( \lambda^2_f = 0 \leq \lambda^1_f \). Notice also that \( \mu^1_f = 0 \leq \mu^2_f \) by the third complementarity condition in \((10)\). Therefore
\[
\beta (q^1_f - q^2_f) + \beta (Q^1 - Q^2) + (\lambda^1_f - \lambda^2_f) - (\mu^1_f - \mu^2_f) \geq \beta (q^1_f - q^2_f) + \beta (Q^1 - Q^2) > 0.
\]
This is a contradiction which concludes the proof.

Next we present a result which provides the explicit form of the equilibrium set, given that an equilibrium \( q^* \) is known. In fact, when \( Q^* \equiv \sum_{f=1}^{F} q^*_f > Q \), we know that \( q^* \) is the only solution. In the case when \( Q^* = Q \), we have the following theorem.

**Theorem 11.** Assume conditions (A) and (B) hold. Let \( q^* \) be an equilibrium of the Nash-Cournot game, and define \( Q^* \equiv \sum_{f=1}^{F} q^*_f \). Assume that \( Q^* = Q \), then the set of equilibria, denoted by \( \Omega \), is given as follows.

\[
\Omega = \left\{ q \in \mathbb{R}^F_{+} \left| \begin{array}{l l}
q_f = K_f, & \forall f \in \gamma \\
\frac{\alpha - \beta Q - c_f}{\beta} \leq q_f \leq K_f, & \forall f \in \{1, \ldots, F\} \setminus \gamma \\
\sum_{f=1}^{F} q_f = Q
\end{array} \right. \right\}, \quad (16)
\]

where \( \gamma \triangleq \{ f \mid \beta K_f + \beta Q - \alpha + c_f \leq 0 \} \).

**Proof.** We first take an arbitrary equilibrium \( q' \) of the game, we will show that \( q' \in \Omega \). It is clear that there exist vectors \( \lambda' \) and \( \mu' \) such that \( (q', \lambda', \mu') \) satisfies (10). By Theorem 10 and the assumption \( Q^* = Q \), it is clear that

\[
\sum_{f=1}^{F} q'_f = Q.
\]

We claim that \( q'_f = K_f \) for all \( f \in \gamma \). Suppose not, then there exists an \( f \in \gamma \) such that \( q'_f < K_f \). By the second complementarity condition in (10), we have \( \lambda'_f = 0 \). By the first complementarity condition in (10), we have

\[
\beta q'_f + \beta Q - \alpha + c_f - \mu'_f \geq 0.
\]

However, by the definition of \( \gamma \), it holds that

\[
\beta q'_f + \beta Q - \alpha + c_f - \mu'_f < \beta K_f + \beta Q - \alpha + c_f \leq 0.
\]
This is a contradiction. Now it suffices to show that \( q'_f \geq \frac{\alpha - \beta Q - c_f}{\beta} \) for all \( f \in \{1, \cdots, F\} \setminus \gamma \). Suppose not, then we must have an \( f \in \{1, \cdots, F\} \setminus \gamma \) such that
\[
q'_f < \frac{\alpha - \beta Q - c_f}{\beta} < K_f,
\]
where the second inequality is due to the definition of \( \gamma \). Therefore \( \lambda'_f = 0 \). Now we have
\[
\beta q'_f + \beta Q - \alpha + c_f - \mu'_f \leq \beta q'_f + \beta Q - \alpha + c_f < 0.
\]
This contradicts the first complementarity condition in (10). Therefore, we have shown the desired result.

Remark 2. Theorem 11 provides us a description of the equilibrium set when a single equilibrium is known. Notice that when no solutions are known, the theorem is not applicable. This is similar to the monotone LCPs and affine variational inequalities (AVIs), when a solution is available, a full description of the solution set is available, see [20, Theorem 3.1.7] and [22, Lemma 2.4.12] for more details. Theorem 11 also implies a procedure to obtain the whole equilibria set instead of a single equilibrium. More specifically, one can apply Lemke’s method on the restricted multiplier formulation (12) and obtain an equilibrium \( q^* \). If \( Q^* > Q \) then the set of equilibria is a singleton \( \Omega = \{q^*\} \). If \( Q^* = Q \), the set of equilibria \( \Omega \) is given by equation (16).

Corollary 12. Let \( \Omega \) be the set of equilibria of the Nash-Cournot game with a price cap. It holds that \( \Omega \) is a compact convex set. Moreover, \( \Omega \) is a polyhedron. \( \Box \)
6. Conclusion

In this paper, we present a reformulation of Nash-Cournot games with price caps to LCPs. The approach is in line with that in [11], while the reformulation is simpler than that in [19] as it does require auxiliary variables. Our major contributions are two-fold. First, we show the uniqueness of a Nash equilibrium under the restrictive multiplier case, and the applicability of Lemke’s algorithm to find such an equilibrium. Second, we provide a definite answer to the uniqueness of Nash equilibrium for the general price-cap games and characterize the set of equilibria.

With the obtained results, further analyses involving price-cap regulation can be built upon the models in this paper. For example, in welfare economics literature, the “theory of the second best” studies optimal market structures when one or more assumptions leading to social-welfare maximizing results do not hold (i.e., there exist one or more forms market failure). Given that market failure is caused by imperfect competition (à la a Cournot game), then price-cap regulation may (or may not) lead to a second-best equilibrium. Such a problem can be numerically analyzed by setting up a bilevel program with the regulator’s welfare maximizing problem as the upper level, in which the regulator chooses a price cap; while the lower level problem is the Nash-Cournot game with price caps, which can be represented by an LCP. The resulting problem is then a Mathematical Problem with Equilibrium Constraint (MPEC) [23]. The properties of the LCP obtained in this paper can then be used to design efficient algorithms to solve the MPEC. Such analyses will be pursued in our future research.

References


