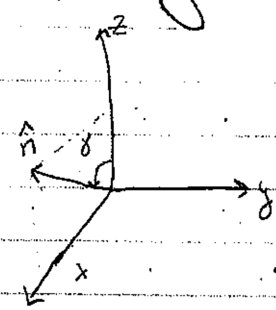
HW#2
Phys 660
9/22/09

1.12 1) A spin $\frac{1}{2}$ system is known to be in an eigenstate of $S\cdot\hat{n}$ w/ eigenvalue $\hbar/2$, ($\hat{n}$; unit vector in xz plane, w/ angle γ from the pos z axis)

a) if S_x is measured what is the probability of getting $\hbar/2$

$\phi = 0$

$$|\hat{n},+\rangle = \cos\tfrac{\gamma}{2}\cancel{e^{-i\phi/2}}|+\rangle + \sin\tfrac{\gamma}{2}\cancel{e^{+i\phi/2}}|-\rangle$$



to measure S_x

$$\langle S_x;+|\hat{n},+\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\left(\cos\tfrac{\gamma}{2}|+\rangle + \sin\tfrac{\gamma}{2}|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos\tfrac{\gamma}{2}\langle +|+\rangle + \cancel{\sin\tfrac{\gamma}{2}\langle +|-\rangle}^{\,0} + \cancel{\cos\tfrac{\gamma}{2}\langle -|+\rangle}^{\,0} + \sin\tfrac{\gamma}{2}\langle -|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos\tfrac{\gamma}{2} + \sin\tfrac{\gamma}{2}\right)$$

the propability of obtaining this value is

$$\left|\langle S,+|\hat{n},+\rangle\right|^2$$

$$\left|\frac{1}{\sqrt{2}}\left(\cos\tfrac{\gamma}{2} + \sin\tfrac{\gamma}{2}\right)\right|^2$$

$$\frac{1}{2}\left(\underbrace{\cos^2\tfrac{\gamma}{2} + \sin^2\tfrac{\gamma}{2}}_{1} + 2\cos\tfrac{\gamma}{2}\sin\tfrac{\gamma}{2}\right)$$

$$\frac{1}{2}\left(1 + 2\cos\tfrac{\gamma}{2}\sin\tfrac{\gamma}{2}\right)$$

$\boxed{\frac{1}{2}(1+\sin\gamma)}$ the probability of measuring $+\hbar/2$ for S_x

(The solutions are provided by Laura Boon, Katie Davis
Ting-fung Chung, Shuo Liu.
Thank them!)

b) Evaluate the dispersion in S_x, that is

$$\langle (S_x - \langle S_x \rangle)^2 \rangle$$

$$\langle S_x^2 - S_x \langle S_x \rangle - \langle S_x \rangle S_x + \langle S_x \rangle^2 \rangle$$

$$\langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_x \Rightarrow \langle \hat{n}_j;+| S_x |\hat{n}_j;+\rangle \qquad S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)$$

$$\left(\cos\tfrac{\gamma}{2}\langle +| + \sin\tfrac{\gamma}{2}\langle -|\right)\left(\frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)\right)\left(\cos\tfrac{\gamma}{2}|+\rangle + \sin\tfrac{\gamma}{2}|-\rangle\right)$$

$$\cos\tfrac{\gamma}{2}\tfrac{\hbar}{2}\langle +|+\rangle\langle -| + \cancel{\cos\tfrac{\gamma}{2}\tfrac{\hbar}{2}\langle +|-\rangle}\langle +| + \tfrac{\hbar}{2}\cancel{\sin\tfrac{\gamma}{2}\langle -|+\rangle}\langle -| + \tfrac{\hbar}{2}\sin\tfrac{\gamma}{2}\langle -|-\rangle\langle +|\Big)\left(\cos\tfrac{\gamma}{2}|+\rangle + \sin\tfrac{\gamma}{2}|-\rangle\right)$$

$$\frac{\hbar}{2}\left[\left(\cos\tfrac{\gamma}{2}\langle -| + \sin\tfrac{\gamma}{2}\langle +|\right)\left(\cos\tfrac{\gamma}{2}|+\rangle + \sin\tfrac{\gamma}{2}|-\rangle\right)\right]$$

$$\frac{\hbar}{2}\left(\cancel{\cos^2\tfrac{\gamma}{2}\langle -|+\rangle} + \cos\tfrac{\gamma}{2}\sin\tfrac{\gamma}{2}\langle -|-\rangle + \sin\tfrac{\gamma}{2}\cos\tfrac{\gamma}{2}\langle +|+\rangle + \cancel{\sin^2\tfrac{\gamma}{2}\langle +|-\rangle}\right)$$

$$\langle S_x \rangle = \frac{\hbar}{2} 2\cos\tfrac{\gamma}{2}\sin\tfrac{\gamma}{2} \Rightarrow \langle S_x \rangle^2 = \frac{\hbar^2}{4} 4\sin^2\tfrac{\gamma}{2}\cos^2\tfrac{\gamma}{2} = \frac{\hbar^2}{4}\sin\gamma$$

$$\langle S_x^2 \rangle = \langle \hat{n}_j+| S_x^2 |n_j+\rangle \qquad S_x^2 = \frac{\hbar^2}{4}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)^2$$

$$\frac{\hbar^2}{4}\left(|+\rangle\langle +| + |-\rangle\langle -|\right)$$

$$\left(\cos\tfrac{\gamma}{2}\langle +| + \sin\tfrac{\gamma}{2}\langle -|\right)\frac{\hbar^2}{4}\left(|+\rangle\langle +| + |-\rangle\langle -|\right)\left(\cos\tfrac{\gamma}{2}|+\rangle + \sin\tfrac{\gamma}{2}|-\rangle\right)$$

$$\frac{\hbar^2}{4}\Big[\cos\tfrac{\gamma}{2}\langle +|+\rangle\langle +| + \cancel{\cos\tfrac{\gamma}{2}\langle +|-\rangle}\langle -| + \cancel{\sin\tfrac{\gamma}{2}\langle -|+\rangle}\langle +| + \sin\tfrac{\gamma}{2}\langle -|-\rangle\langle -|\Big)\left(\cos\tfrac{\gamma}{2}|+\rangle + \sin\tfrac{\gamma}{2}|-\rangle\right)$$

$$\frac{\hbar^2}{4}\left[\cos\tfrac{\gamma}{2}\langle +| + \sin\tfrac{\gamma}{2}\langle -|\right)\left(\cos\tfrac{\gamma}{2}|+\rangle + \sin\tfrac{\gamma}{2}|-\rangle\right)$$

$$\frac{\hbar^2}{4}\left(\cos^2\tfrac{\gamma}{2}\langle +|+\rangle + \cancel{\sin\tfrac{\gamma}{2}\cos\tfrac{\gamma}{2}\langle -|+\rangle} + \cancel{\cos\tfrac{\gamma}{2}\sin\tfrac{\gamma}{2}\langle +|-\rangle} + \sin^2\tfrac{\gamma}{2}\langle -|-\rangle\right)$$

$$\frac{\hbar^2}{4}\left(\cancel{\cos^2\tfrac{\gamma}{2} + \sin^2\tfrac{\gamma}{2}}\right)^{1}$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4}$$

$$\langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$\frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2\gamma = \boxed{\frac{\hbar^2}{4}\left(1 - \sin^2\gamma\right)}$$

✓

$\gamma = 0 \Rightarrow \frac{\hbar^2}{4}$

$\gamma = \frac{\pi}{2} \Rightarrow 0$

$\gamma = \pi \Rightarrow \frac{\hbar^2}{4}$

13.

Measurement a :

$$M_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Measurement b :

$$M_b = \begin{pmatrix} \cos \beta/2 \\ \sin \beta/2 \end{pmatrix} (\cos \tfrac{\beta}{2} \quad \sin \beta/2) = \begin{pmatrix} \cos^2 \beta/2 & \frac{1}{2}\sin\beta \\ \frac{1}{2}\sin\beta & \sin^2 \beta/2 \end{pmatrix}.$$

Measurement c :

$$M_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Total measurement $= M_c M_b M_a = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}\sin\beta & 0 \end{pmatrix}$

$\Rightarrow$ Probability $= \frac{1}{4}\sin^2\beta$

Maximaze : $\beta = \pi/2 \Rightarrow P = 1/4$

#18) (c) $\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4} e^{\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right]}$ (1)

Prove $\langle x'|\Delta x|\alpha\rangle = z\langle x'|\Delta p|\alpha\rangle$ where z is imaginary.

We know $\Delta x = x - \langle x\rangle$ & $\Delta p = p - \langle p\rangle$

$$\langle x'|\Delta x|\alpha\rangle = \langle x'|x - \langle x\rangle|\alpha\rangle$$
$$= \langle x'|x|\alpha\rangle - \langle x'|\langle x\rangle|\alpha\rangle$$
$$= \int \langle x'|x''\rangle x''\langle x''|\alpha\rangle dx'' - \int \langle x'|x''\rangle\langle x\rangle\langle x''|\alpha\rangle dx''$$

note that $\langle x'|x''\rangle = \delta(x'-x'') \Rightarrow$

$$\langle x'|\Delta x|\alpha\rangle = \delta(x'-x'')\left[\int (x''\langle x''|\alpha\rangle - \langle x\rangle\langle x''|\alpha\rangle)\,dx''\right]$$

$$\langle x'|\Delta p|\alpha\rangle = \langle x'|p - \langle p\rangle|\alpha\rangle \quad \text{where } p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$
$$= \langle x'|p|\alpha\rangle - \langle x'|\langle p\rangle|\alpha\rangle$$
$$= \int \langle x'|x''\rangle \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x''|\alpha\rangle - \langle x'|x''\rangle\langle p\rangle\langle x''|\alpha\rangle dx''$$

note that $\langle x'|x''\rangle = \delta(x'-x'')$ again $\Rightarrow$

$$= \delta(x'-x'')\left[\int \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x''|\alpha\rangle - \langle p\rangle\langle x''|\alpha\rangle dx''\right]$$

$$\frac{\partial}{\partial x}\langle x''|\alpha\rangle = \left(\frac{i\langle p\rangle}{\hbar} - \frac{2(x''-\langle x\rangle)}{4d^2}\right)\langle x''|\alpha\rangle \text{ by (1)}$$

So, $\langle x'|\Delta p|\alpha\rangle = \delta(x'-x'')\left[\int \cancel{\langle p\rangle\langle x''|\alpha\rangle} - \frac{\hbar}{2id^2}(x''-\langle x\rangle)\langle x''|\alpha\rangle \cancel{- \langle p\rangle\langle x''|\alpha\rangle} dx''\right]$

$$\langle x'|\Delta p|\alpha\rangle = -\frac{\hbar}{2id^2}\,\delta(x'-x'')\int x''\langle x''|\alpha\rangle - \langle x\rangle\langle x''-\alpha\rangle\,dx''$$

$$= \frac{-\hbar}{2id^2}\,\langle x''|\Delta x|\alpha\rangle$$

$$\Rightarrow \langle x''|\Delta x|\alpha\rangle = \underbrace{\frac{-2id^2}{\hbar}}_{\text{z the imaginary \#}}\langle x'|\Delta p|\alpha\rangle .$$

✓

#32) (a) verify (1.7.39a) & (1.7.39b) for the expectation value of p & p^2 from the Gaussian wave packet!

$$\langle x'|\alpha\rangle = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right] e^{\left[ikx' - x'^2/2d^2\right]}$$

Show : $\langle p\rangle = \hbar k$

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x'} \Rightarrow$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \langle\alpha|x'\rangle \frac{\hbar}{i}\frac{\partial}{\partial x'}\langle x'|\alpha\rangle dx'$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{\pi^{1/2} d}\right]\left(\frac{\hbar}{i}\right) e^{\left[-ikx' - x'^2/2d^2\right]} \underbrace{\frac{\partial}{\partial x}\left(e^{\left[ikx' - x'^2/2d^2\right]}\right)}_{(A)} dx'$$

$$(A) = (ik - x'/d^2)\, e^{\left[ikx' - x'^2/2d^2\right]} \Rightarrow$$

$$\langle p\rangle = \int_{-\infty}^{\infty}\left[\frac{\hbar}{\sqrt{\pi}\, d}\right]\frac{1}{i}\left(ik - \frac{x'}{d^2}\right) e^{-x'^2/d^2} dx'$$

$$= \int_{-\infty}^{\infty}\left[\frac{\hbar}{\sqrt{\pi}\, d}\right]\left(k + \frac{ix'}{d^2}\right) e^{-x'^2/d^2} dx'$$

= 0 b/c odd function from $-\infty \to \infty$

$$= \left[\frac{\hbar}{\sqrt{\pi}\, d}\right]\left[k\int_{-\infty}^{\infty} e^{-x'^2/d^2} dx' + \underbrace{\frac{i}{d^2}\int_{-\infty}^{\infty} x' e^{-x'^2/d^2} dx'}_{=0}\right]$$

$$\int_{-\infty}^{\infty} e^{-x'^2/d^2} dx' = 2\int_0^{\infty} e^{-x^2/d^2} dx$$

$$= \boxed{\frac{\hbar k}{\sqrt{\pi}\, \cancel{d}}}\sqrt{\pi}\left(\frac{\cancel{d}}{\cancel{2}}\right)(\cancel{2}) = \hbar k$$

$\therefore \langle p\rangle = \hbar k$ ∎

(b) evaluate the expectation value of p & p^2 using the momentum-space wave function (1.7.42)

$$\langle p'|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right]}$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \langle\alpha|p'\rangle\, p' \langle p'|\alpha\rangle\, dp'$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{\hbar\sqrt{\pi}}\right) p'\, e^{\left[-(p'-\hbar k)^2 d^2/\hbar^2\right]}\, dp'$$ let $p'-\hbar k = u$, $dp' = du$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{\hbar\sqrt{\pi}}\right)(u+\hbar k)\, e^{-u^2 d^2/\hbar^2}\, du$$

$$= \frac{d}{\hbar\sqrt{\pi}}\left[\underbrace{\int_{-\infty}^{\infty} u\, e^{-u^2 d^2/\hbar^2}\, du}_{\text{integral = zero b/c odd b/n } -\infty \text{ \& } \infty} + \hbar k \int_{-\infty}^{\infty} e^{-u^2 d^2/\hbar^2}\, du\right]$$

$$= \frac{\cancel{d}}{\cancel{\hbar}\cancel{\sqrt{\pi}}}\, \hbar k\, \cancel{\sqrt{\pi}} \left(\frac{\cancel{\hbar}}{\cancel{2}\cancel{d}}\right)(\cancel{2}) = \hbar k$$ which is the same value we obtained for $\langle p\rangle$ using the gaussian wave function & $\hat{p}$.

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \langle\alpha|p'\rangle\, p'^2 \langle p'|\alpha\rangle\, dp'$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{\hbar\sqrt{\pi}}\right) p'^2\, e^{\left[-(p'-\hbar k)^2 d^2/\hbar^2\right]}\, dp'$$ let $p'-\hbar k = u$, $dp' = du$

$$\langle p^2 \rangle = -\hbar^2\left[-k^2 - \frac{1}{d^2} + \frac{1}{2d^2}\right]$$

$$= -\hbar^2\left[-k^2 - \frac{1}{2d^2}\right] = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

✓

$$\langle p^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

$$\hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x'^2}$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \langle \alpha | x'\rangle \left(-\hbar^2 \frac{\partial^2}{\partial x'^2}\right) \langle x' | \alpha\rangle \, dx'$$

$$= \int_{-\infty}^{\infty} \left[\frac{-\hbar^2}{\sqrt{\pi}\, d}\right] e^{[-ikx' - x'^2/2d^2]} \underbrace{\frac{\partial^2}{\partial x'^2}\left(e^{[ikx' - x'^2/2d^2]}\right)}_{(A)} dx'$$

$$(A) = \frac{\partial}{\partial x'}\left((ik - x'/d^2)\, e^{[ikx' - x'^2/2d^2]}\right)$$

$$= ik(ik - x'/d^2)\, e^{[ikx' - x'^2/2d^2]} - x'/d^2 (ik - x'/d^2)\, e^{[ikx' - x'^2/2d^2]} - \frac{1}{d^2} e^{[ikx' - x'^2/2d^2]}$$

$$\Rightarrow \langle p^2\rangle = \int_{-\infty}^{\infty} \left[\frac{-\hbar^2}{\sqrt{\pi}\, d}\right] e^{-x'^2/d^2} \left(-k^2 \underbrace{- ikx'/d^2 - ikx'/d^2}_{\text{those are odd integrals b/n } -\infty \text{ \& } \infty \text{ and thus equal zero}} + x'^2/d^4 - \frac{1}{d^2}\right) dx'$$

$$\Rightarrow \langle p^2\rangle = -\frac{\hbar^2}{\sqrt{\pi}\, d}\left[\int_{-\infty}^{\infty} \left(-k^2 - \frac{1}{d^2}\right) e^{-x'^2/d^2} dx' + \int_{-\infty}^{\infty} \frac{x'^2}{d^4} e^{-x'^2/d^2} dx'\right]$$

$$= \left(\frac{-\hbar^2}{\cancel{\sqrt{\pi}}\, \cancel{d}}\right)\left[\left(-k^2 - \frac{1}{d^2}\right)\cancel{\sqrt{\pi}}\left(\frac{\cancel{d}}{\cancel{2}}\right) + \frac{1}{d^4}\cancel{\sqrt{\pi}}(2)\left(\frac{d}{2}\right)^{\cancel{3}\,2}\right](2)$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty}\left(\frac{d}{\hbar\sqrt{\pi}}\right)(u+\hbar k)^2 e^{-u^2d^2/\hbar^2}\,du$$

$$= \int_{-\infty}^{\infty}\left(\frac{d}{\hbar\sqrt{\pi}}\right)(u^2+\hbar^2k^2+\underbrace{2u\hbar k}_{\substack{\text{odd}\\ \text{integral}\\ -\infty\to\infty \Rightarrow = \text{zero}}}) e^{-u^2d^2/\hbar^2}\,du$$

$$=\left(\frac{d}{\hbar\sqrt{\pi}}\right)\left[\int_{-\infty}^{\infty} u^2 e^{-u^2d^2/\hbar^2}\,du + \hbar^2k^2\int_{-\infty}^{\infty} e^{-u^2d^2/\hbar^2}\,du\right]$$

$$=\left(\frac{d}{\hbar\sqrt{\pi}}\right)\left[\sqrt{\pi}(2)\left(\frac{\hbar}{2d}\right)^{\cancel{3}\,2}(2) + \hbar^2k^2\sqrt{\pi}\left(\frac{\hbar}{2d}\right)(2)\right]$$

$$= \cancel{2}\left(\frac{\hbar^2}{\cancel{4}_{2}d^2}\right)+\hbar^2k^2 = \frac{\hbar^2}{2d^2}+\hbar^2k^2 \quad \checkmark$$

again, this is the same value we obtained for $\langle p^2\rangle$ using the gaussian wave packet & $\hat{p}^2$.

1.30 Solution

$\mathcal{T}(\vec{l}) = \exp(-i\frac{\vec{p}\cdot\vec{l}}{\hbar})$ Thus $\mathcal{T}(\vec{l})|\vec{x}\rangle = |\vec{x}+\vec{l}\rangle$.

1. $[\hat{x}_i, \mathcal{T}(\vec{l})]|\vec{x}\rangle = (\hat{x}_i \mathcal{T}(\vec{l}) - \mathcal{T}(\vec{l})\hat{x}_i)|\vec{x}\rangle$

$$= x_i|\vec{x}+\vec{l}\rangle - \mathcal{T}(\vec{l})\, x_i|\vec{x}\rangle$$

$$= (x_i+l_i)|\vec{x}+\vec{l}\rangle - x_i|\vec{x}+\vec{l}\rangle$$

$$= l_i|\vec{x}+\vec{l}\rangle$$

Therefore $[x_i, \mathcal{T}(\vec{l})] = l_i$

2. Under translation.

$$|\psi\rangle \to \mathcal{T}(\vec{l})|\psi\rangle.$$

and $\langle\psi|\hat{\vec{x}}|\psi\rangle \to \langle\psi|\mathcal{T}^\dagger(\vec{l})\,\hat{\vec{x}}\,\mathcal{T}(\vec{l})|\psi\rangle$

In $|\vec{x}\rangle$ representation

$$\langle\psi|\mathcal{T}^\dagger(\vec{l})\,\hat{\vec{x}}\,\mathcal{T}(\vec{l})|\psi\rangle = \iint \langle\psi|\mathcal{T}^\dagger|\vec{x}\rangle\langle\vec{x}|\hat{\vec{x}}|\vec{x}'\rangle\langle\vec{x}'|\mathcal{T}|\psi\rangle\, d\vec{x}\,d\vec{x}'$$

$$= \iint \psi^*(\vec{x}-\vec{l})\,\vec{x}\,\delta(\vec{x}-\vec{x}')\,\psi(\vec{x}'-\vec{l})\,d\vec{x}\,d\vec{x}'$$

$$= \int_V \psi^*(\vec{x}-\vec{l})\,\vec{x}\,\psi(\vec{x}-\vec{l})\,d\vec{x}$$

$$= \int_V \psi^*(\vec{r})(\vec{r}+\vec{l})\psi(\vec{r})\,d\vec{r} = \int_V \psi^*(\vec{x})(\vec{x}+\vec{l})\psi(\vec{x})\,d\vec{x}$$

$$= \langle\vec{x}+\vec{l}\rangle$$

Therefore, under $\mathcal{T}$, $\langle\vec{x}\rangle \to \langle\vec{x}+\vec{l}\rangle$

32. continue

(b) $\langle p^2\rangle = \langle \psi_{p'}|p^2|\psi_{p'}\rangle \quad ; \quad \psi_{p'} = \langle p'|\alpha\rangle$

$$= \frac{d}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} p'^2 \exp\left(\frac{-(p'-\hbar k)^2 d^2}{\hbar^2}\right) dp'$$

let $n = p'-\hbar k$, $dn = dp'$

$$\Rightarrow \frac{d}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} (n+\hbar k)^2 \exp\left(-\frac{n^2 d^2}{\hbar^2}\right) dn$$

$$= \frac{d}{\hbar\sqrt{\pi}}\int_{-\infty}^{\infty} (n^2+2\hbar k n+\hbar^2 k^2) \exp\left(-\frac{n^2}{(\frac{\hbar}{d})^2}\right) dn$$

$$= \frac{d}{\hbar\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{d}\right)^3 + \hbar^2 k^2 \sqrt{\pi}\frac{\hbar}{d}\right] \qquad \because \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{a^2}\right) dx = \frac{\sqrt{\pi}}{2}a^3$$

$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

// ✓.

33. (a) (i) Prove $\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

$$\langle p'|x|\alpha\rangle = \int_{-\infty}^{\infty}\langle p'|x|p''\rangle\langle p''|\alpha\rangle dp'' \quad \text{——(1)} \qquad \int_{-\infty}^{\infty} dp''|p''\rangle\langle p''| = 1$$

$$\langle p'|x|p''\rangle = \int_{-\infty}^{\infty}\langle p'|x|x'\rangle\langle x'|p''\rangle dx' \qquad \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'| = 1$$

$$= \int_{-\infty}^{\infty} x'\langle p'|x'\rangle\langle x'|p''\rangle dx'$$

As $\langle x'|p''\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ip''x'}{\hbar}}$ and $\langle p'|x'\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{ip'x'}{\hbar}}$

$$\Rightarrow \langle p'|x|p''\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} x' e^{\frac{ip''x'}{\hbar}} e^{-\frac{ip'x'}{\hbar}} dx'$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} x' \exp\left(-\frac{ix'}{\hbar}(p'-p'')\right) dx' \quad \text{——(2)}$$

$$\langle p'|p''\rangle = 2\pi\hbar\,\delta(p'-p'')$$

$$= \int_{-\infty}^{\infty}\langle p'|x'\rangle\langle x'|p''\rangle dx'$$

$$= \int_{-\infty}^{\infty}\exp\left(-\frac{ix'}{\hbar}(p'-p'')\right) dx'$$

$$\frac{\partial}{\partial p'}\delta(p'-p'') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\exp\left(-\frac{ix'}{\hbar}(p'-p'')\right)\left(-\frac{ix'}{\hbar}\right) dx' \qquad \left(\frac{\partial}{\partial p'}\langle p'|p''\rangle = 2\pi\hbar\frac{\partial}{\partial p'}\delta(p'-p'')\right)$$

$$i\hbar\frac{\partial}{\partial p'}\delta(p'-p'') = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} x' \exp\left(-\frac{ix'}{\hbar}(p'-p'')\right) dx' \quad \text{——(3)}$$

Sub. (3) into (2), we get

$$\langle p'|x|p''\rangle = i\hbar \frac{\partial}{\partial p'} \delta(p'-p'')$$

$$\Rightarrow \langle p'|x|\alpha\rangle = \int_{-\infty}^{\infty} i\hbar \frac{\partial}{\partial p'} \delta(p'-p'') \langle p''|\alpha\rangle \, dp''$$

$$= i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \quad //$$

✓

(a) (ii) Prove $\langle\beta|x|\alpha\rangle = \int dp' \phi_\beta^*(p') \, i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p')$; $\phi_\alpha(p') = \langle p'|\alpha\rangle$
$\phi_\beta(p') = \langle p'|\beta\rangle$

$$\langle\beta|x|\alpha\rangle = \int_{-\infty}^{\infty} \langle\beta|p'\rangle\langle p'|x|\alpha\rangle \, dp'$$

$$= \int_{-\infty}^{\infty} \langle\beta|p'\rangle \, i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \, dp' \qquad \left\{ \begin{array}{l} \text{by part (a)(i)} \\ \langle p'|x|\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \end{array} \right.$$

$$= \int_{-\infty}^{\infty} \langle p'|\beta\rangle^* \, i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle \, dp'$$

$$= \int_{-\infty}^{\infty} dp' \, \phi_\beta^*(p') \, i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p') \quad //$$

(b) $\exp\left(\frac{i\hat{x}\Xi}{\hbar}\right)$ where Ξ is a no. with dimension of momentum.

This ~~function~~ operator has a structure which is similar to space translation op., $U_a = \exp\left(\frac{-ia\hat{p}}{\hbar}\right)$. But position and momentum elements are switched.

$$\hat{p}|U_\Xi|p'\rangle = (U_\Xi \hat{p} + [\hat{p}, U_\Xi])|p'\rangle \qquad \because [\hat{p}, U_\Xi] = \hat{p}U_\Xi - U_\Xi\hat{p}$$

$$\hat{p}U_\Xi = U_\Xi\hat{p} + [\hat{p}, U_\Xi]$$

Also $[\hat{p}, U_\Xi] = -i\hbar \frac{\partial}{\partial x} \exp\left(\frac{i x \Xi}{\hbar}\right)$

$$= -i\hbar \exp\left(\frac{i\hat{x}\Xi}{\hbar}\right)\cdot\left(\frac{i\Xi}{\hbar}\right) = (\Xi U_\Xi)$$

✓

$$\Rightarrow \hat{p}|U_\Xi|p' = (U_\Xi\hat{p} + \Xi U_\Xi)|p'\rangle$$

$$= U_\Xi p|p'\rangle + \Xi U_\Xi|p'\rangle$$

$$= \cancel{U_\Xi p} \, (p' + \Xi) U_\Xi|p'\rangle \quad //$$

$\therefore U_\Xi = \exp\frac{i\hat{x}\Xi}{\hbar}$ is a momentum translation op.