Renormalization

Practical calculations

i) Define a regularization through a parameter ($\Lambda \to \infty$, $\varepsilon \to 0$, etc)

ii) Define a way to subtract counter terms in the Lagrangian.

$\Rightarrow$ At each order we get finite answers.
equivalent to redefine the parameters of the Lagrangian

theoretical β.

•) Prove that such procedures gives finite results at every order in pert. theory.

•) Prove that it preserves the symmetries of cntbr...

Counter terms should have the same form & symmetries of the original Lagrangian.

Physical

\- Renormalization group (scale dependence of coupling constant) e.g.

\- Large momentum behaviour of Green functions.

Regularization + subtraction

Regularization

•) **Lattice** : good for numerics after Wickrotation + lattice pert. theory.

Reduces F.T. to usual QM. system.

⇒ Used to define F.T. non-perturbatively.

– Clear physics

– Breaks Poincare symmetry, ~~my~~ maybe supersymmetry, problems w/ fermions. Continuum limit tricky

•) Pauli-Villars

$$\frac{i}{p^2-m^2+i\varepsilon} \to \frac{i}{p^2-m^2+i\varepsilon} - \frac{i}{p^2-\Lambda^2+i\varepsilon} =$$

$$= i\,\frac{-\Lambda^2 \cancel{+i\varepsilon} + m^2 \cancel{-i\varepsilon}}{(p^2-m^2+i\varepsilon)(p^2-\Lambda^2+i\varepsilon)} \overset{p\to\infty}{\sim} \frac{1}{p^4}$$

improved propagator (can add more terms if necessary).

Very good at preserving symmetries.

– cumbersome for computations.

– Somewhat unclear physics

-) Dimensional reg.

analytic continuation in dimension (as for Γ function).

-) very good for computations (very little extra complication)
specially w/ MS ($\overline{MS}$) scheme

-) preserves gauge symmetry, flavor symmetry.

-) not so good w/ fermions. γ-matrices ok but problems with supersymmetry (# of fermions states different than bosons).

↳ can be fixed with DR (dimensional reduction).
One takes a higher dim. theory and dimensionally reduce to 4-ε dim. (# of fields depends on ε).

-) unclear physical meaning.

-) modifies propagator at short (ok) and large distances (?)

$$\text{e.g.}\quad \Delta(x-y) = \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip(x-y)}}{p^2+i\varepsilon} \sim (x-y)^{2-d} = \frac{1}{(x-y)^{\cancel{2}-\varepsilon}}$$

$2-4+\varepsilon$

$-2+\varepsilon$

modified at $x\to y$ or $(x-y)\to\infty$.

good for IR divergences
but one has to be careful.

→) Momentum subtraction.

Expand Integrand of diagram in powers of external momenta. Subtract the divergent part (first terms) to leave an integrand that gives a finite integral.

Subtraction prescription.

→) MS → subtract pole in dim. reg.
→) $\overline{MS}$ as MS but w/ γ and 4π.
} less physical

→) Define coupling constants as Green's function at fix momenta. More physical

e.g. $-i\lambda = \Gamma^{(4)}(s = t = 4m^2/3)$

self energy $\Sigma(p^2) = \Sigma(m_0^2) + (p^2 - m_0^2)\,\Sigma'(m_0^2) + \tilde{\Sigma}(p^2)$

$\cancel{(i\Delta(p))^{-1} = p^2 - m_0^2}$ Define mass as position of pole of $\Delta(p^2)$ and residue to be 1.

$i\Delta_0(p) = \frac{i}{\quad\quad} \to i Z_\phi^{-1} \Delta_0(p) \qquad \tilde{\Sigma}(m^2) = 0$

$$i\Delta_R(p) = \frac{i}{p^2 - m^2 + \underbrace{\Sigma(p^2)}_{\text{loop. corr.}} \underbrace{- \Sigma_1 p^2 - \Sigma_0 m^2}_{\text{added counterterms.}} + i\varepsilon}$$

$i\Delta_R(p^2 = m^2) \rightarrow$ pole

$$\Sigma(m^2) - \Sigma_1 m^2 - \Sigma_0 m^2 = 0$$

$$\Sigma_0 + \Sigma_1 = \frac{\Sigma(m^2)}{m^2}$$

residue.

$$i\Delta_R(p^2 \to m^2) = \frac{i}{(p^2 - m^2) + \cancel{\Sigma(m^2)} + \Sigma'(m^2)(p^2 - m^2) - \Sigma_1(p^2 - m^2) - \cancel{- \Sigma_1 m^2 - \Sigma_0 m^2}}$$

$$\Sigma'(m^2) - \Sigma_1 = 0$$

$$\Sigma_1 = \Sigma'(m^2)$$

$$\Sigma_0 = \frac{1}{m^2}\Sigma(m^2) - \Sigma'(m^2)$$

$$\Sigma'(m^2) = \left.\frac{\partial}{\partial p^2}\Sigma(p^2)\right|_{p^2 = m^2}$$

Can be used with dim. reg. also

Naive estimate of degree of divergence.

All ~~loop momenta~~ internal lines. $k_\ell \to \lambda k_\ell$

$$I_G \sim \lambda^{\omega} \quad ; \quad \lambda \to \infty$$

$\omega \geq 0$ divergent

$\omega < 0$ superficially convergent.

 vertex λ^{δ_v} ← # of derivatives.

- - - λ^{-2}

→ λ^{-1}

$\int d^4 q$ λ^4

L: loops

I_B: boson lines

I_f: fermion lines.

V: vertices.

$$\omega(G) = 4L - 2I_B - I_f + \sum_v \delta_v = 4 + 2I_B + 3I_f + \sum_v (\delta_v - 4)$$

$$I_B + I_f - V + 1 = L$$

$$\omega(G) - 4 = 2I_B + 3I_f + \sum_v (\delta_v - 4)$$

internal

f_v : # of fermions

b_v : # of bosons.

$$I_f = \frac{1}{2}\sum_v f_v \qquad I_b = \frac{1}{2}\sum_v b_v$$

$$\omega(G) = 4 + \sum_v b_v + \frac{3}{2}\sum_v f_v + \sum_v (\delta_v - 4)$$

$$\omega(G) - 4 = \sum_v \big(\underbrace{b_v + \tfrac{3}{2} f_v + \delta_v}_{\hat{\omega}_v} - 4 \big)$$

$$\omega(G) - 4 = \sum_v (\hat{\omega}_v - 4)$$

exter B   extern f   external deriv.

$$= \sum_v (\omega_v - 4) - E_B - \frac{3}{2} E_f - \delta$$

coupling constant

$$\omega_v + [g_v] = 4$$

$$\omega(G) = \left(4 - E_B - \frac{3}{2} E_f - \delta\right) + \sum_v [g_v]$$

$\boxed{[g_v] \geq 0}$ if $[g_v] < 0$ then adding vertices make the diag. (more) divergent.

Assume $\sum_v [g_v] \geq 0$

also $\underbrace{E_B + \frac{3}{2} E_f + \delta > 4}_{\text{finite}}$ $\Rightarrow$ $\underline{\omega(G) < 0}$

Lag. should have $\omega(g) \leq 4$ for each term

$d(\omega_G \leq 4)$ are divergent $\Rightarrow$

Consider ϕ^4 theory and Pauli-Villars.

$$\omega(G) = 4L - 4I_B$$

$$I_B - V + 1 = L$$

$$\omega(G) = 4(1-V)$$

$$I_B = \frac{4V - E_B}{2} = 2V - \frac{1}{2}E_B$$

$$\omega(G) = 4(1-V) \qquad V=1 \to 0 \qquad V=2 -$$

only divergent diagram. ⇒ use separate regularization or normal-order

Pauli-Villars makes ϕ^4 theory finite at all orders.

<u>Convergence theorem</u> (1 PI Green's functions)

$i\varepsilon \to$ convenient to go to Euclidean space.

$\mathcal{F}$: family of all 1-PI connected subdiagrams of G

Feynman diag.

$G \in \mathcal{F}$

if $\omega(g) < 0 \ \forall g \in \mathcal{F}$ then G is absolutely convergent in the Euclidean region.

<u>Corollary</u>:

if G has no superficially divergent diagram

$$(\omega(g) < 0 \quad \forall g \in \mathcal{F},\ g \neq G)$$

but $\omega(G) \geqslant 0$ (G itself is superficially divergent)

then the divergent part is a polynomial of degree $\leqslant \omega(G)$ in the external momenta P and masses.

•) the $(\omega(G)+1)$-th derivatives have degree of homogeneity (-1) and then are superficially convergent

$$\frac{\partial^{\omega+1}}{\partial p^{\omega+1}} I_G(P) \quad \text{are finite}$$

 $\omega_v - 4 = 0$

e.g. ϕ^4 theory.

$$\omega(G) = \sum_v (\omega_v - 4) - E_B - \delta + 4$$

$$\omega(G) = -E_B - \delta + 4$$

<u>divergent</u>:

$E_B = 4 \quad \delta = 0 \rightarrow \phi^4$ no derivatives.

$E_B = 2 \rightarrow \delta = 0 \rightarrow m^2 \phi^2$

$\qquad\quad \searrow \delta = 2 \rightarrow (\partial\phi)^2$

renormalizable

In general

$$E_B + \frac{3}{2} E_f + \delta \leq 4 \quad \text{divergent}$$

$\Rightarrow [g_{\hat{V}}] \geq 0$ are divergent ← renormalizable if we include all $[g] \geq 0$ terms...

In that case the all counterterms can be absorbed in a redefinition of the parameters of the Lagrangian.

E.g.

$$\mathcal{L} + \delta\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

$$+ \frac{1}{2}(Z-1)(\partial_\mu\varphi)^2 - \frac{1}{2}(Zm_0^2 - m^2)\varphi^2 - \frac{1}{4!}(Z^2\lambda_0 - \lambda)\varphi^4$$

$$= \frac{1}{2}Z(\partial_\mu\varphi)^2 - \frac{1}{2}Zm_0^2\varphi^2 - \frac{1}{4!}Z^2\lambda_0\varphi^4$$

$$= \frac{1}{2}(\partial_\mu\varphi_0)^2 - \frac{1}{2}m_0^2\varphi_0^2 - \frac{1}{4!}\lambda_0\varphi_0^4$$

$$\boxed{\varphi_0 = \sqrt{Z}\,\varphi}$$

$$G_R^{(n)}(p_1 \ldots p_n, m, \lambda) = Z^{-n/2}\, G_{reg}^{(n)}(p_1 \ldots p_n, m_0, \lambda_0, \Lambda)$$

$$\Gamma_R^{(n)}(p_1 \ldots p_n, m, \lambda) = \underbrace{Z^{n/2}\, \Gamma_{reg}^{(n)}(p_1 \ldots p_n, m_0, \lambda_0, \Lambda)}_{\Lambda \to \infty}$$

<u>Dim. reg.</u> Bare Lagrangian is defined in dimension $d = 4 - \varepsilon$ and everything is finite but diverges as $\varepsilon \to 0$.

The subtraction procedure requires a mass scale

eg. define coupling constants at a fixed scale μ.

e.g.

$$\Gamma_R^{(4)}\left(s=t=u=\frac{4\mu^2}{m}\right) = -\lambda$$

off-shell

$$\Gamma_R^{(2)}(p^2)\Big|_{p^2=\mu^2} = \mu^2 - m^2 \qquad \frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p^2)\Big|_{p^2=\mu^2} = 1.$$

in dim. reg. $\underline{MS}$ we only subtract poles. But λ & λ_0 have different dimensions. $\lambda \to$ adim. $\lambda_0 \to$

$$\lambda \sim \mu^{-\varepsilon}\lambda_0 \qquad \int d^{4-\varepsilon}x \; \partial\phi_0\partial\phi_0 + \lambda_0\phi^4$$

$-4+\varepsilon$ $\quad 1-\varepsilon/2 \quad \varepsilon \quad 4-2\varepsilon$

$$\underbrace{\Gamma_{reg}^{(n)}(p_1 \dots p_n, m_0, \lambda_0, \underset{\Lambda}{\varepsilon})}_{} = Z_\phi\left(\frac{\mu}{m}, \lambda \dots\right)^{-n/2}\Gamma_R^{(n)}(p_1 \dots p_n, m(\mu), \lambda(\mu), \mu)$$

indep. of μ.

$$\partial_\mu = 0 = Z_\phi^{-n/2}\left(-\frac{n}{2}\frac{\partial_\mu Z_\phi}{Z_\phi}\Gamma_R + \partial_\lambda\Gamma_R\frac{\partial\lambda}{\partial\mu} + \frac{\partial m}{\partial\mu}\partial_m\Gamma_R + \partial_\mu\Gamma\right)$$

$$\mu\partial_\mu\Gamma_R - n\gamma\Gamma_R + \beta\frac{\partial}{\partial\lambda}\Gamma_R + m\gamma_m\partial_m\Gamma_R = 0$$

$$\boxed{\begin{aligned}\beta(\lambda, m/\mu) &= \mu\frac{\partial\lambda}{\partial\mu}\\ \gamma_m &= \mu\,\partial_\mu \ln m\\ \gamma\left(\lambda, \frac{m}{\mu}\right) &= \frac{1}{2}\frac{\partial\ln Z_\phi}{\partial\ln\mu}\end{aligned}}$$

β, γ_m, γ are finite since they represent a change in ... prescription.

MS is an mass independent subtraction scheme.

β, γ_m, γ are indep of the mass

$$\lambda_0 = \mu^\varepsilon \left[\lambda + \sum_{r=1}^{\infty} \frac{a_r(\lambda)}{\varepsilon^r} \right]$$

$$m_0 = \mu \left[1 + \sum_r \frac{b_r(\lambda)}{\varepsilon^r} \right]$$

$$\phi_0 = \phi \left[1 + \sum_r \frac{c_r(\lambda)}{\varepsilon^r} \right]$$

Series is unique (if we impose only poles).

change $\mu \Rightarrow \mu'(1+\zeta)$ $\quad \zeta \ll 1$. $\quad \left| \begin{array}{l} \mu' = \mu - \zeta\mu \\ \frac{\delta\mu}{\mu} = -\zeta \end{array} \right.$

$$\lambda_0 = (\mu')^\varepsilon (1+\varepsilon\zeta) \left[\lambda + \sum_{r=1}^{\infty} \frac{a_r(\lambda)}{\varepsilon^r} \right]$$

$$m_0 = \mu'(1+\zeta) \left[1 + \sum_{r=1}^{\infty} \frac{b_r(\lambda)}{\varepsilon^r} \right]$$

$$\lambda_0 = (\mu')^\varepsilon \left(\lambda + \varepsilon\zeta\lambda + \sum_{r=1}^{\infty} \frac{a_r}{\varepsilon^r} + \zeta a_1 + \sum_{r=1}^{\infty} \frac{a_{r+1}(\lambda)}{\varepsilon^r} \zeta \right)$$

$$\lambda = \tilde{\lambda} - \varepsilon\zeta\tilde{\lambda}$$

$$\lambda_0 = (\mu')^\varepsilon \left(\tilde{\lambda} + \sum_{r=1}^{\infty} \frac{a_r(\tilde{\lambda}) - \partial_\lambda a_r \, \varepsilon\zeta\tilde{\lambda}}{\varepsilon^r} + \zeta a_1 - \varepsilon\zeta\tilde{\lambda}\partial a_1 \right.$$

$$\left. + \sum_{r=1}^{\infty} \frac{a_{r+1}}{\varepsilon^r} \zeta \right)$$

$$\lambda_0 = (\mu')^\varepsilon \left(\tilde{\lambda} + \zeta a_1 - \partial_\lambda a_1 \zeta \tilde{\lambda} \text{ + probs}\right)$$

$$\lambda' = \lambda + \zeta(a_1 - \partial_\lambda a_1 \lambda)$$

$$\delta\lambda = -\frac{\delta\mu}{\mu}(a_1 - \partial_\lambda a_1 \lambda)$$

$$\boxed{\beta(\lambda) = -a_1 + \lambda \partial_\lambda a_1}$$

we only need $1/\varepsilon$ pole.

Behavior in the deep Euclidean region

take space-like moment $p_i^2 < 0$, non exceptional ($\sum_i p_i \neq 0$ for subsets.)

$p_i = \sigma k_i \qquad \sigma \to \infty$

(Weinberg's theorem $\Gamma_R^{(n)} \sim \sigma^{4-n} \times$ polynomial in $\ln\sigma$ to any finite order in pert theory

$$\sim \sigma^{4-n}\underbrace{\left(a_0(\ln\sigma)^{k_0} + a_1(\ln\sigma)^{k_1}\lambda + \cdots\right.}_{\text{can resum to } \sigma^{4-n-\gamma(\lambda)}}$$

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$$\Gamma^{(n)}(p_i, m, \lambda, \mu) = \mu^{4-n}\, \bar{\Gamma}(p_i/\mu, m/\mu, \lambda)$$

$$\left(\int d^4x\, \phi^n \Gamma.\right)$$

$$\Gamma^{(n)}(\sigma p_i, m, \lambda, \mu) = \mu^{4-n}\, \bar{\Gamma}(\sigma p_i/\mu, m/\mu, \lambda)$$

$$= \left(\frac{\mu}{\sigma}\right)^{4-n} \sigma^{4-n}\, \bar{\Gamma}\left(p_i/(\mu/\sigma), \frac{(m/\sigma)}{(\mu/\sigma)}, \lambda\right)$$

$$= \cancel{\phi}\, \sigma^{4-n}\, \Gamma^{(n)}\left(p_i, \frac{m}{\sigma}, \lambda, \frac{\mu}{\sigma}\right)$$

$$\sigma \partial_\sigma \Gamma^{n}(\sigma p_i, m, \lambda, \mu) = (4-n)\, \sigma^{4-n}\, \Gamma^{(n)}(p_i, m/\sigma, \lambda, \mu/\sigma) +$$

$$\cancel{+}\, \sigma^{4-n} \frac{m}{\cancel{\sigma}\sigma}\, \partial_m \Gamma^{(n)}(p_i, m/\sigma, \lambda, \mu/\sigma)$$

$$- \sigma^{4-n} \frac{\mu}{\cancel{\sigma}}\, \partial_\mu \Gamma^{(n)}(p_i, m/\sigma, \lambda, \mu/\sigma)$$

$$= (4-n)\, \Gamma^{(n)}(\sigma p_i, m, \lambda \mu) - \frac{m}{\cancel{\sigma}}\partial_m\, \Gamma^{(n)}(\sigma p_i, m, \lambda, \mu) -$$

$$- \frac{\mu}{\cancel{\sigma}}\, \partial_\mu \Gamma^{(n)}(\sigma p_i, m, \lambda, \mu)$$

$$\sigma\, \partial_\sigma \Gamma(m/\sigma) = -\frac{m}{\cancel{\sigma}} \Gamma' = -m\partial_m$$

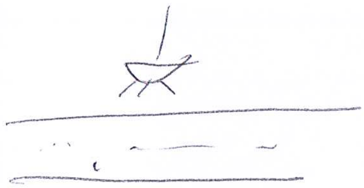
$$\partial_m \Gamma(m/\sigma) = \frac{1}{\sigma}\, \Gamma'$$

$$\sigma \partial_\sigma \Gamma(\sigma p_i) = (4-n)\,\Gamma - m\partial_m \Gamma \cancel{+} n\gamma\Gamma + \beta\, \partial_\lambda \Gamma + m\gamma_m \partial_m \Gamma$$

$$= \beta \partial \Gamma - n\gamma\Gamma + (4-n)\gamma\Gamma + m(\gamma_m - 1)\partial_m \Gamma$$

$$\sigma \partial_\sigma \Gamma - \beta \partial_\lambda \Gamma + n\gamma\Gamma + (n-4)\Gamma + m(1-\gamma_m)\partial_m \Gamma = 0$$ 12

Bacterial analogy



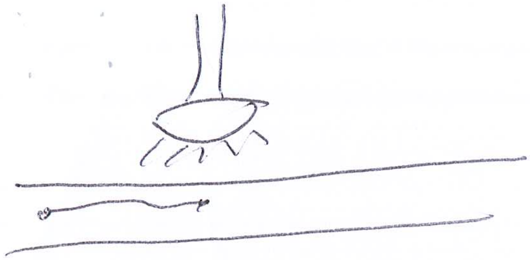
$$\frac{\partial\rho}{\partial t} + v\frac{\partial \rho}{\partial x} = L(x)\rho$$

$$\frac{dx'(x,t)}{dt} = v(x') \qquad x'(x,0) = x$$

$$\rho(x,t) = f(x'(x,-t))\, e^{\int_{-t}^{0} dt'\, L(x'(x,t'))}$$

$$\partial_t \rho = f'\, v(x')\, e^{\int} + f\, e^{\int} \left(-L(x'(x,-t))\right)$$

$$\partial_x \rho = f' \frac{\partial x'}{\partial x} e^{\int} + f\, e^{\int} \int_{-t}^{0} dt' \frac{\partial L}{\partial x'} \frac{\partial x'}{\partial x}$$



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$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = L(x)\rho$$

$$\boxed{\frac{d x'(x,t)}{dt} = v(x'(x,t))}$$

$$x'(x,t) = \int_0^t v(x'(x,t))\, dt$$

$$\frac{dx'}{dt} = v(x') \qquad \int \frac{dx'}{v(x')} = dt$$

$$t = \int_x^{x'(x,t)} \frac{dy}{v(y)} \longrightarrow \underline{\underline{x'(t)}}$$

$$\rho(x,t) = f(x'(x,-t))\, e^{\int_{-t}^{0} dt'\, L(x'(x,t'))}$$

$$y \longrightarrow x$$

$$0 = -\frac{1}{v(x)} + \frac{1}{v(x')} \frac{\partial x'(t)}{\partial x} \qquad \partial x'/\partial x = \frac{v(x')}{v(x)}$$

$$\partial \rho/\partial t = -f'\, v(x')\, e^{\int} + f\, e^{\int} \int_t L'\, v(x')\, L(x')$$

$$\partial \rho/\partial x = f' \frac{v(x')}{v(x)} e^{\int} + f\, e^{\int} \int_{-t} dt'\, L' \frac{v(x')}{v(x)}$$

$$\frac{\partial \rho}{\partial t} + v(x)\frac{\partial \rho}{\partial x} = f\, e^{\int^{L(x')}} \left(\cancel{L(x')} + \int_{-t}^{0} dt' \underbrace{L' v(x')}_{\partial_{t'} L} \right)$$ (14)

$$\partial_{t'} L = L' \, v(x')$$

$$L(x'(x,0)) - L(x'(x,-t))$$

$$L(x) - \cancel{L(x')}$$

$$= L(x)\,\rho. \quad \checkmark$$

<u>Solve:</u> $\frac{d\bar{\lambda}(t)}{dt} = \beta(\bar{\lambda})$, $\frac{d\bar{m}(t)}{dt} = [\gamma_m - 1]\,\bar{m}(t)$

$\bar{\lambda}(t=0) = \lambda$

$\bar{m}(t=0) = m$

$$\Gamma_R^{(n)}(\sigma p_i, \mu, \lambda, m) = \sigma^{4-n}\, e^{-n\int_0^t \gamma(\bar{\lambda}(t'))\,dt'}\, \Gamma_R^{(n)}(p_i, \bar{m}(t), \bar{\lambda}(t), \mu)$$

$\bar{m}(t) \underset{t\to\infty}{\to} 0$, then

$$\Gamma_R^{(n)}(\sigma p_i, \mu, \lambda, m) = \sigma^{4-n}\, e^{-n\int_0^t \gamma(\bar{\lambda}(t'))\,dt'}\, \Gamma_R^{(n)}(p_i, 0, \bar{\lambda}(t), \mu)$$