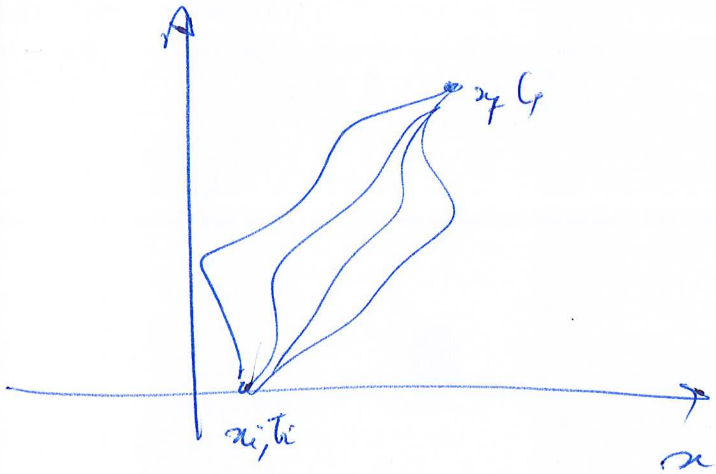
Path integrals in Q.M.



$$\langle x_f, t_f | x_i, t_i \rangle =$$

$$= \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle$$

$$= K(x_f t_f; x_i t_i)$$

$$K(x_f t_f; x_i t_i) = \int [\mathcal{D} x(t)]\, e^{iS[x(t)]}$$

$$\boxed{\begin{array}{l} \psi(x_f, t_f) = \int dx_i\, K(x_f t_f; x_i t_i)\, \psi(x_i, t_i) \\ K(x_f, t_f; x_i t_i) = \int dx\, K(x_f t_f; x t)\, K(x t; x_i t_i) \\ \qquad t_f > t > t_i \end{array}}$$

example: free particle

$$K(x_f t_f; x_i t_i) = \int \frac{dp}{2\pi} \langle x_f | e^{-\frac{ip^2}{2m}(t_f - t_i)} | p \rangle \langle p | x_i \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-\frac{ip^2}{2m}(t_f - t_i)}\, e^{ip(x_f - x_i)}$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{\frac{i}{2m}(t_f - t_i)}}\; e^{-\frac{(x_f - x_i)^2}{4\frac{i}{2m}(t_f - t_i)}} = \frac{\sqrt{m}}{\sqrt{2\pi}} \frac{e^{-\frac{i\pi}{4}}}{\sqrt{(t_f - t_i)}}\, e^{\frac{mi(x_f - x_i)^2}{2(t_f - t_i)}}$$

$$\boxed{K(x_f t_f; x_i t_i) = e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi(t_f - t_i)}}\; e^{\frac{im}{2}\frac{(x_f - x_i)^2}{(t_f - t_i)}}}$$

$t_f - t_i$ $\quad \frac{e^{-i\pi/4}}{\sqrt{\pi\sigma}}\, e^{ix^2/\sigma} \to \delta(x)$ by Fourier

$$\int_{-\infty}^{\infty} dx\, e^{ix^2/\sigma} f(x) = \int_{-\infty}^{\infty} dx\, e^{\frac{ix^2}{\sigma}} \int_{-\infty}^{\infty} dk\, e^{ikx} \tilde{f}(k)$$

$$= \sqrt{\pi\sigma} \int dk\, \tilde{f}(k)\, e^{\frac{ik^2\sigma}{4}} = \sqrt{\pi\sigma} \int dk\, \tilde{f}(k)$$

$f(0)$

h.o

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$x(t) = e^{-iHt}\, x\, e^{iHt} \qquad p(t) = e^{-iHt}\, p\, e^{iHt}$$

$$[x(t), p(t)] = i \quad ; \quad H = \frac{1}{2m} p^2(t) + \frac{1}{2} m\omega^2 x^2(t)$$

$$\partial_t x(t) = -iH\,x(t) + x(t)\,iH = -i\,[H, x(t)] =$$

$$= \frac{-i}{\cancel{2}m}\, p(-i) = -\frac{1}{\cancel{2}m}\, p(t)$$

$$\partial_t p(t) = -i\, m\omega^2\, i\, x(t) = m\omega^2 x(t)$$

$$\partial_t^2 x(t) = -\frac{1}{\cancel{2}\cancel{m}}\, \cancel{m}\omega^2 x(t) = -\omega^2 x(t)$$

$$(\partial_t^2 + \omega^2)\, x(t) = 0 \qquad x(t) = \cos\omega t\, \hat{A} + \sin\omega t\, \hat{B}$$

$$\dot{x}(t) = -\omega \sin\omega t\, \hat{A} + \omega \cos\omega t\, \hat{B} = -\frac{1}{m} p(t)$$

$$\boxed{x(t) = \hat{x}_s \cos\omega t - \frac{1}{m\omega} \sin\omega t\, \hat{p}_s}$$

$$X(t) \underbrace{e^{-iHt}|x_i\rangle}_{|\psi\rangle} = x_i \underbrace{e^{-iHt}|x_i\rangle}_{|\psi\rangle}$$

$$\langle x_f|\psi\rangle = \psi(x_f)$$

$$\cos\omega t \, x_f \psi + \frac{i}{m\omega}\sin\omega t \, \partial_{x_f}\psi = x_i \psi$$

$$\frac{\partial_{x_f}\psi}{\psi} = -i\frac{m\omega}{\sin\omega t}\left(x_i - x_f\cos\omega t\right)$$

$$\partial_{x_f}\ln\psi = -\frac{im\omega}{\sin\omega t}\left(x_i - x_f\cos\omega t\right)$$

$$\psi = e^{-\frac{im\omega}{\sin\omega t}\left(x_i x_f - \frac{1}{2}x_f^2\cos\omega t\right)} \cdot A(t, x_i)$$

Property $\langle x_f|e^{-iHt}|x_i\rangle^* = \langle x_i|e^{iHt}|x_f\rangle$

$$K(x_f, t; x_i, 0)^* = K(x_i, -t; x_f, 0)$$

$\psi^*(-t)$: same as $x_i \leftrightarrow x_f$

$$= e^{-\frac{im\omega}{\sin\omega t}\left(x_i x_f - \frac{1}{2}x_f^2\cos\omega t\right)}$$

$$\psi = A(t)\, e^{-\frac{im\omega}{\sin\omega t}\left(x_i x_f - \frac{1}{2}(x_f^2 + x_i^2)\cos\omega t\right)}$$

$$\boxed{A^*(-t) = A(t)}$$

$$\partial_t \psi = -iH\psi = -i\left(-\frac{1}{2m}\partial_x^2\psi + \frac{1}{2}m\omega^2x^2\psi\right)$$

$$\psi = Ae^{\chi} \qquad \partial_x\psi = A\partial_x\chi\, e^{\chi}$$

$$\dot A \cancel{e^{\chi}} + A\dot\chi \cancel{e^{\chi}} = \frac{i}{2m}A\chi''\cancel{e^{\chi}} + \frac{i}{2m}A\chi'^2\cancel{e^{\chi}} -$$

$$-\frac{i}{2}m\omega^2x^2A\cancel{e^{\chi}}$$

$$\frac{\dot A}{A} + \left(\frac{im\omega^2}{s^2}c\,x_ix_f - \frac{im\omega^2}{2}(x_f^2+x_i^2)\frac{1}{s^2\omega t}\right) =$$

$$= \frac{i}{2\cancel{2m}}\left(\frac{\cancel{2im}\omega}{s\omega t}c\omega t\right) + \frac{i}{2m}\left(-\frac{im\omega}{s\omega t}(x_i - x_f c\omega t)\right)^2 -$$

$$-\frac{i}{2}m\omega^2x_f^2$$

$$\frac{\dot A}{A} = -\frac{i\omega}{2s\omega t}c\omega t - \cancel{\frac{im\omega^2c}{s^2}}x_ix_f + \frac{im\omega^2}{2}(\cancel{x_i^2}+x_f^2)\frac{1}{s^2\omega t} +$$

$$+\frac{i\omega^2m}{2s^2\omega t}\left(\cancel{x_i^2}+x_f^2c^2\omega t - \boxed{\cancel{2x_ix_f}c\omega t}\right) - \frac{i}{2}m\omega^2x_f^2$$

$$= \frac{i\omega c\omega t}{s\omega t} + \cancel{\frac{im\omega^2x_f^2}{2s^2\omega t}(1-c^2\omega t)} - \cancel{\frac{i}{2}m\omega^2x_f^2}$$

$$\frac{\dot{A}}{A} = -\frac{\omega}{2}\frac{c\omega t}{s\omega t}$$

$$\partial_t \ln A = -\frac{1}{2} \ln s\omega t$$

$$A = \frac{A_0}{\sqrt{s\omega t}}$$

$$\psi = \frac{A_0}{\sqrt{s\omega t}} e^{-\frac{im\omega}{s\omega t}\left(x_i x_f - \frac{1}{2} c\omega t\,(x_i^2 + x_f^2)\right)}$$

$$t \to 0 \quad \psi \to \delta(x_i - x_f)$$

$$\psi \simeq \frac{A_0}{\sqrt{\omega t}} e^{+\frac{im\omega}{\omega t}\frac{1}{2}(x_i - x_f)^2}$$

$$\int \psi = \frac{A_0}{\sqrt{\omega \cancel{t}}} \sqrt{\frac{2\pi \cancel{t}}{-im}} = A_0 \sqrt{\frac{2\pi}{-im\omega}} = 1$$

$$A_0 = \sqrt{\frac{-im\omega}{2\pi}} = e^{-\frac{i\pi}{4}} \sqrt{\frac{m\omega}{2\pi}}$$

$$K(x_f t_f; x_i t_i) = e^{-\frac{i\pi}{4}} \sqrt{\frac{m\omega}{2\pi \sin(t_f - t_i)}}\; e^{-\frac{im\omega}{\sin(\omega(t_f - t_i))}\left(x_i x_f - \frac{1}{2}\cos(t_f - t_i)\,(x_i^2 + x_f^2)\right)}$$

Free particle.

$$\int dx_1 \cdots dx_{N-1} \; e^{+i\frac{(x_n-x_{n-1})^2}{2m\Delta t}}$$

$$\xi_1 = x_1 - x_0 \qquad \xi_2 = x_2 - x_1 \quad \cdots \quad \xi_N = x_N - x_{N-1}$$

$$\int d\xi_1 \cdots d\xi_{N-1}\, d\xi_N \; e^{+i\frac{1}{2m\Delta t}\left(\xi_1^2 + \cdots + \xi_{N-1}^2 + \xi_N^2\right)} \; \delta(\xi_1 + \cdots + \xi_N - \Delta x)$$

$$\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int d\xi_1 \cdots d\xi_N \; e^{i\lambda\xi_1 + \cdots + i\lambda\xi_N - i\lambda\Delta x} \; e^{-i\frac{1}{2m\Delta t}\xi_1^2 - \cdots - i\frac{1}{2m\Delta t}\xi_N^2}$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left(\sqrt{\frac{\pi\, 2m\Delta t}{-i}}\right)^N e^{-\frac{\lambda^2\, 2m\Delta t\, N}{4(-i)} - i\lambda\Delta x}$$

$$= (+2\pi i m\Delta t)^N \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \; e^{-\frac{i}{2} m\lambda^2 (t_f - t_i) - i\lambda\Delta x}$$

$$= \frac{(+2\pi i m\Delta t)^N}{2\pi} \sqrt{\frac{\pi}{-\frac{im}{2}(t_f - t_i)}} \; e^{+\frac{\Delta x^2}{4\frac{i}{2}m(t_f - t_i)}}$$

$$= \frac{(+2\pi i m\Delta t)^N}{2\pi} \sqrt{\frac{2\pi i}{m(t_f - t_i)}} \; e^{\frac{i\Delta x^2}{2m(t_f - t_i)}}$$

same up to N factor.

H. O.

$$\int \mathcal{D}x \; e^{i\int dt \left(\frac{m\dot{x}^2}{2} + \frac{1}{2} m\omega^2 x^2\right)}$$

$$x = x_{cl} + \delta x \qquad \delta x(0) = \delta x(T) = 0 \qquad - m\ddot{x}\,\delta x + m\omega^2 x\, \delta x$$

$$\int \mathcal{D}x \; e^{i\int dt \left(\frac{m\dot{x}_{cl}^2}{2} + 2m(\dot{x}\,\delta\dot{x}) + \frac{1}{2} m\,\delta\dot{x}^2 + \frac{1}{2} m x_{cl}^2 + \right.}$$
$$+ m\omega^2 x\,\delta x + \frac{1}{2} m\omega^2 \delta x^2$$

$$\int \mathcal{D}x \; e^{iS_{cl}} \; e^{i\int dt \left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} m\omega^2 y^2\right)}$$

$$y(0) = y(T) = 0.$$

$\underline{S_{cl}}$:

$$x = A \cos\omega t + B \sin\omega t$$

$$x = x_i \cos\omega t + B \sin\omega t$$

$$x_f = x_i\, c\omega T + B \sin\omega T$$

$$B = \frac{1}{s\omega T}\left(x_f - x_i\, c\omega T\right)$$

$$x = x_i\, c\omega t + \frac{s\omega t}{s\omega T}\left(x_f - x_i\, c\omega T\right)$$

$$\dot{x} = - x_i\, s\omega t\, \omega + \frac{\omega\, c\omega t}{s\omega T}\left(x_f - x_i\, c\omega T\right)$$

$$S_{cl} = \int dt \left(\frac{m\dot{x}^2}{2} - \frac{1}{2} m\omega^2 x^2 \right) =$$

$$\frac{m\dot{x}\dot{x}}{2}$$

$$= \int dt \left(\cancel{-\frac{m\ddot{x}x}{2} - \frac{1}{2} m\omega^2 x^2} + \partial_t \left(\frac{m x\dot{x}}{2} \right) \right)$$

$$\frac{m\omega^2 x}{2}$$

$$\ddot{x} = -\omega^2 x$$

$$= m\frac{x\dot{x}}{2}\Big|_0^T = \frac{m x_f}{2}\dot{x}(T) - \frac{m x_i}{2}\dot{x}(0)$$

$$= \frac{m}{2}\Big(x_f\left(-\omega x_i\, s\omega T\right) + x_f\,\omega \frac{c\omega T}{s\omega T}\left(x_f - x_i\, c\omega T\right) -$$

$$- x_i \frac{\omega}{s\omega T}\left(x_f - x_i\, c\omega T\right)\Big)$$

$$= \frac{m}{2}\Big(-\omega\, s\omega T\, x_i x_f + \omega x_f^2 \frac{c\omega T}{s\omega T} - x_i x_f\,\omega \frac{c^2\omega T}{s\omega T} -$$

$$- \frac{x_i x_f \omega}{s\omega T} + x_i^2\,\omega \frac{c\omega T}{s\omega T}\Big)$$

$$= \frac{m}{2}\left[\frac{\omega x_i x_f}{s\omega T}\left(-s^2\omega T - c^2\omega T - 1\right) + \frac{\omega\, c\omega T}{s\omega T}\left(x_i^2 + x_f^2\right) \right]$$

$$S_{cl} = \frac{m\omega}{2} \operatorname{cotan}\omega T \, (x_i^2 + x_f^2) - \frac{m\omega \, x_i x_f}{\sin\omega T}$$

$$K = e^{\frac{im\omega}{2} \operatorname{cotan}\omega T \, (x_i^2 + x_f^2) - \frac{im\omega}{\sin\omega T} x_i x_f} \cdot \int_{y(0)=y(T)=0} \mathcal{D}y(t) \, e^{iS[y(t)]}$$

$$\int \mathcal{D}y(t) \, e^{i\int \frac{m}{2}\dot{y}^2 - \frac{1}{2}m\omega^2 \cancel{x}^2} = \int \mathcal{D}y(t) \, e^{-\frac{im}{2}\int y(\ddot{y} + \omega^2 y)}$$

$$= \int \mathcal{D}y(t) \, e^{-\frac{im}{2}\int y(\partial_t^2 + \omega^2) y} = \det{}^{-1/2}\left(+i\frac{m}{2}(\partial_t^2 + \omega^2)\right)$$

eigenvalue $\quad y = A \sin\left(\frac{n t \pi}{T}\right)$

$$= \prod_{n=1}^{\infty} \frac{1}{\sqrt{\frac{im}{2}\left(\omega^2 - \frac{n^2\pi^2}{T^2}\right)}} = \frac{1}{\left(-\frac{im\omega^2}{2}\right)^{N/2}} \prod_{n=1}^{\infty} \frac{1}{\left(-1 + \frac{n^2\pi^2}{\omega^2 T^2}\right)^{1/2}}$$

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right)$$

$$\cancel{\ln = \sum_{n=1}^{\infty} \ln\left(\frac{n^2\pi^2}{\omega^2 T^2} - 1\right)}$$

Use then:

$$\frac{\det\left(\partial_t^2+\omega^2\right)}{\det\left(\partial_t^2\right)} = \prod_{n=1}^{\infty} \frac{\omega^2 - \frac{n^2\pi^2}{T^2}}{-\frac{n^2\pi^2}{T^2}} = \prod_{n=1}^{\infty} \cancel{\omega^2} \left(1 - \frac{\omega^2 T^2}{n^2\pi^2}\right)$$

$$= \frac{\sin\left(\omega T\right)}{\omega T}$$

$$\det^{-1/2}\left(\partial_t^2+\omega^2\right) = \sqrt{\frac{\omega T}{\sin \omega T}} \; \underbrace{\det^{-1/2}\left(\partial_t^2\right)}_{\substack{\text{free particle} \\ K_{f.p.}(0t_f,\,0t_i)}}$$

$$= e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi T}}$$

$$K_{h.o.} = e^{-\frac{i\pi}{4}} \sqrt{\frac{m\omega}{2\pi \sin\omega T}} \; e^{\frac{im\omega}{2}\left[\cot\!\operatorname{an}(\omega T)\,(x_i^2+x_f^2) - \frac{2x_i x_f}{\sin\omega T}\right]}$$

path-integral result agrees.

Derivation

$$\langle x_f | e^{-iHT} | x_i \rangle = \int dx_1 - - dx_{N-1} \ \langle x_f | e^{-iH\Delta t} | x_{N-1}\rangle\langle x_{N-1}| - - e^{-iHt} | x_0 \rangle$$

$$= \int dx_1 - - dx_{N-1} \ K(x_f\, \Delta t; x_{N-1}, 0) - - \ K(x_1\, \Delta t; x_0, 0)$$

$$K(x_2\, \Delta t; x_1, 0) = \langle x_2 | e^{-i\frac{p^2}{2m}\Delta t + iV(x)\Delta t} | x_1 \rangle =$$

$$\simeq \langle x_2 | e^{-i\frac{p^2}{2m}\Delta t} e^{-iV(x)\Delta t} | x_1 \rangle =$$

to order Δt

$$= \int dp \ e^{-i\frac{p^2}{2m}\Delta t} e^{ip(x_2 - x_1)} e^{-iV(x_2)\Delta t}$$

$$= \int = \sqrt{\frac{2m\pi}{i\Delta t}} \ e^{-\frac{(x_2-x_1)^2}{4\frac{i\Delta t}{2m}} - iV(x_2)\Delta t}$$

$$= \sqrt{\frac{2m\pi}{i\Delta t}} \ e^{\frac{im(x_2-x_1)^2}{2\Delta t} - iV(x_2)\Delta t} = N_{\Delta t}\, e^{iS(x_2, x_1, \Delta t)}$$

$$K(x_f, T; x_i, 0) = \lim_{N\to\infty} N \int dx_1 - - dx_{N-1} \ e^{i\sum_j S(x_{j+1}, \Delta t; x_j\, 0)}$$

$$= \lim_{N\to\infty} N \int dx_1 - - dx_{N-1} \ e^{i\int L}$$

What if

$$H = \frac{p^2}{2m} f(x) + V(x)$$

$$\langle x_2 | e^{-i p^2 \frac{f(x)}{2m} \Delta t} | p \rangle$$

$$\int dp \; e^{-\frac{i}{2m} f(x) p^2 \Delta t} \; e^{i p (x_2 - x_1)} = \sqrt{\frac{\pi}{\frac{i}{2m} f(x_i)}} \; e^{\frac{i m}{2} \frac{(\Delta x)^2}{f \Delta t}}$$

$$\prod_i \frac{1}{\sqrt{f(x_i)}} = e^{\frac{1}{2} \sum_i \ln f(x_i)} = e^{\frac{1}{2} \frac{1}{\Delta t} \sum_i \ln f(x_i) \Delta t}$$

$$= e^{\frac{1}{2} \delta(0) \int dt \ln f(x(t))}$$

$\uparrow$ extra term.

$$S_{eff} = \int dt \, L(q, \dot{q}) - \frac{i}{2} \delta(0) \int \ln f(q) \, dt$$

$$\int \left[\sqrt{f(q)} \, dq \right] e^{iS}$$

Fermionic path integral.

Grassmann numbers

$$\theta\eta = -\eta\theta$$

$$\theta^2 = 0$$

$$f(\theta) = a + b\theta$$

$$\int d\theta\,(a+b\theta) = \int d\theta\,(a+b\theta+b\eta) \quad ; \quad \int d\theta\,(a+b\theta) = b$$

$$\int d\theta\, 1 = 0 \qquad \int d\theta\,\theta = 1$$

fermion zero mode very important.

$$\int d\theta_1 \cdots d\theta_n\, e^{\theta_i A_{ij}\theta_j} \cancel{= \int d\xi_1 \cdots d\xi_n\, e}$$

antisymmetric

diagonalize

complex $\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2)$

$$\int d\theta^* d\theta\,(\theta\theta^*) = 1 \qquad \int d\theta^* d\theta\, e^{-a\theta^*\theta} = \int d\theta^* d\theta\,(1 + a\theta\theta^*) = a$$

$$\int d\theta_i^* d\theta_j\, e^{-\theta_i^* A_{ij}\theta_j} = \int d\xi_i^* d\xi_i\, e^{-\xi_i^* a_i \xi_i} = \prod_i a_i = \det A$$

diagonalize

$$\int d\theta_i^* d\theta_j \; \theta_k \theta_l^* \, e^{-\theta_p^* B_{pq} \theta_q} =$$

$$\theta_k = U_{kl}\,\xi_l \qquad \theta_k^* = \xi_l^*\, U^+_{lk}$$

$$\prod_k d\theta_k = \prod_k \left(U_{kl}\, d\xi_l \right) = \det U \prod_l d\xi_l = \prod_l d\xi_l$$

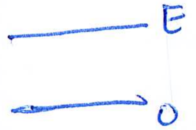
$$\int d\xi_l^* d\xi_l \quad U_{kk'}\,\xi_{k'}\,\xi^*_{l'}\,U^+_{l'l} \; e^{-\sum_p \xi_p^* \xi_p (\Lambda_B)_p}$$

$(\Lambda_B)_p = b_p$

$U^+ B U = \Lambda_B$ diag.

$$= \cancel{\int d\xi_l^* d\xi_l} \; U_{kk'} \frac{\delta_{k'l'}}{b_{k'}} U^+_{l'l} \; \det B =$$

$$= \det B \left(U \Lambda_B^{-1} U^+ \right)_{kl} = \det B \; B^{-1}_{kl}$$



$$\int_{\text{periodic}} d\theta(t)\; e^{i\int \bar\theta \dot\theta\, dt} = \int d\theta_i \; e^{\sum_j i(\theta_{j+1}-\theta_j)\,\theta_j \,\Delta t}$$

$$= \det(i\partial_t - E) = \prod_n \left(i\frac{2\pi n}{T} - E \right)$$

periodic.

sind $e^{i\frac{2\pi t n}{T}}$ $\qquad i\dot{\bar\theta}\theta - E\bar\theta\theta$

QM example.

$$\text{———}\ \omega/2$$

$$\text{———}\ -\omega/2$$

$$L = i\bar\theta\dot\theta - \omega\bar\theta\theta$$

$$S = \int dt\, \bar\theta\,(i\partial_t - \omega)\,\theta$$

$$\int_{0\to T} d\bar\theta d\theta\, e^{iS} \underset{\uparrow\ \text{antiperiodic}}{=} \det(-\partial_t - i\omega) = \prod_{n=-\infty}^{\infty}\left(-\frac{i2\pi}{T}(n+1/2) - i\omega\right)$$

$$e^{i(n+1/2)\frac{2\pi t}{T}} \to +\frac{2\pi}{T}(n+1/2)\,i \quad \text{eigenvalue of } \partial_t$$

$$= \prod_{n=-\infty}^{\infty}\left(-\frac{2\pi i}{T}(n+1/2)\right)\underbrace{\prod_{n=-\infty}^{\infty}\left(1+\frac{\omega T}{2\pi(n+1/2)}\right)}_{\cos\left(\frac{\omega T}{2}\right)}$$

$$\det(-\partial_t) = 2 \qquad \text{regularization.}$$

$$\det(-\partial_t - i\omega) = 2\cos\left(\frac{\omega T}{2}\right) = e^{\frac{i\omega T}{2}} + e^{-\frac{i\omega T}{2}}$$

with periodic. $\mathrm{Tr}\left((-)^F e^{-iHt}\right)$

$$\int d\bar\theta d\theta \; e^{iS} = \det(-\partial_t - i\omega) = \prod_{n=-\infty}^{\infty}\left(-i\frac{2\pi}{T}n - i\omega\right)$$

periodic

$$= \prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{2\pi i n}{T}\right)\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(1+\frac{\omega T}{2\pi n}\right)\quad \underbrace{(-i\omega)}_{n=0}$$

$$= -i\omega \prod_{n=-\infty}^{\infty}\left(-\frac{2\pi i n}{T}\right)\prod_{n=1}^{\infty}\left(1-\frac{\omega^2T^2}{4\pi^2n^2}\right)$$

$$= -i\cancel{\omega}\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{2\pi i n}{T}\right)\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\cancel{\omega} T}{2}}$$

$$= -2i\prod_{n=1}^{\infty}\left(-\frac{2\pi i n}{T}\right)^2\frac{\sin\left(\frac{\omega T}{2}\right)}{T} = \underset{(2\pi)}{-4\pi i \sin\left(\frac{\omega T}{2}\right)}$$

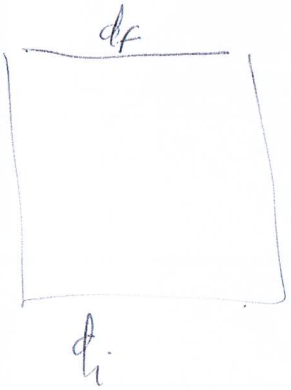
$$\prod_{n=1}^{\infty}\left(-\frac{4\pi^2n^2}{T^2}\right) = 4\pi\frac{T}{2}$$

$$\det(\partial_t^2)_{T/2} = \frac{4\pi T}{-i}$$

$$2i\,\frac{e^{-\frac{i\omega T}{2}} - e^{\frac{i\omega T}{2}}}{2i}$$

$$= -2i\sin\left(\frac{\omega T}{2}\right)$$

Path integral for scalar field



$$\int \mathcal{D}\phi \, e^{i\int_{-T}^{T} d^4x \, \mathcal{L}(\phi, \partial\phi)} = \langle \phi_f(x) | e^{-2iHT} | \phi_i(x) \rangle$$

$$T \to \infty(1-i\epsilon) \qquad e^{-2iH(\infty)(1-i\epsilon)} = e^{-2iH\infty - 2H\infty\cdot\epsilon} |\phi_i\rangle$$

only the vacuum contributes.

$$\int \mathcal{D}\phi \; \phi(x_1)\, \phi(x_2) \; e^{i\int_{-T}^{T} d^4x \, \mathcal{L}(\phi,\partial\phi)} =$$

$$= \int \mathcal{D}\phi_1(x_1) \int \mathcal{D}\phi_2(x_2) \int\limits_{\substack{\phi(x_1^0,\vec{x}) = \phi_1(\vec{x}) \\ \phi(x_2^0,\vec{x}) = \phi_2(\vec{x})}} \mathcal{D}\phi(x) \; e^{i\int_{-T}^{T} d^4x \, \mathcal{L}(\phi,\partial\phi)} \phi(x_1)\, \phi(x_2)$$

$$= \int \mathcal{D}\phi_1(x) \int \mathcal{D}\phi_2(x) \; \phi_1(\vec{x})\, \phi_2(\vec{x}) \, \langle \phi_f | e^{-iHT} | \phi_2 \rangle \langle \phi_2 | e^{-iH(x_2^0 - x_1^0)} | \phi_1(\vec{x}) \rangle \langle \phi_1(\vec{x}) | e^{-iH(T-x^0)} | \phi_i \rangle$$

$$= \langle \Omega | \hat{T}\{ \phi_H(x_2)\, \phi_H(x_1) \} | \Omega \rangle$$

$$\langle \Omega | \hat{T}\{ \phi_H(x_1) \cdots \phi_H(x_n) \} | \Omega \rangle = \frac{\int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) \, e^{iS[\phi]}}{\int \mathcal{D}\phi \, e^{iS[\phi]}}$$

Generating function

$$Z[j(x)] = \int \mathcal{D}\phi \, e^{iS[\phi] + i\int d^4x \, \phi(x) j(x)} = \langle \Omega | T\{ e^{i\int d^4x \, \phi(x) j(x)} \} | \Omega \rangle$$

$$\frac{\delta^n Z[j(x)]}{\delta j(x_n) \cdots \delta j(x_1)} = i^n \int \mathcal{D}\phi \, \phi(x_1) \cdots \phi(x_n) \, e^{iS[\phi] + i\int d^4x \, \phi j}$$

$$\boxed{\langle \Omega | T\{\phi(x_1) \cdots \phi(x_n)\} | \Omega \rangle = \frac{(-i)^n}{Z[j]} \frac{\delta^2 Z(j(x))}{\delta j(x_1) \cdots \delta j(x_n)} \Bigg|_{j=0}.}$$

$$Z(j) = e^{-iW(j)} \leftarrow \text{connected diagrams}$$

$$W(j) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n \, G_c(x_1 \cdots x_n) \, j(x_1) \cdots j(x_n)$$

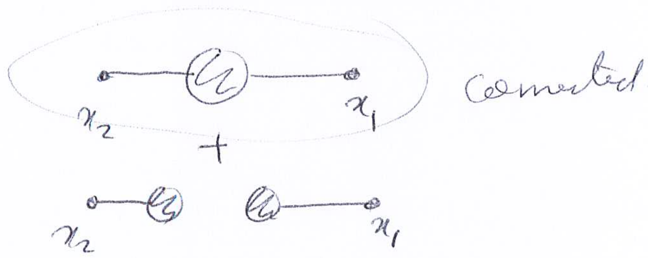
$$W = i \ln Z$$

$$\frac{\delta W}{\delta j} = i \frac{1}{Z} \frac{\delta Z}{\delta j} = -\langle \phi(x) \rangle_j$$

$$\frac{\delta^2 W}{\delta j(x_2) \delta j(x_1)} = i \frac{1}{Z} \frac{\delta^2 Z}{\delta j(x_2) \delta j(x_1)} - \frac{i}{Z^2} \frac{\delta Z}{\delta j(x_2)} \frac{\delta Z}{\delta j(x_1)}$$

$$\frac{\delta^2 W}{\delta j(x_2)\,\delta j(x_1)} = -i\,\langle\Omega|T\{\phi(x_2)\phi(x_1)\}|\Omega\rangle + i\,\langle\phi(x_1)\rangle\langle\phi(x_2)\rangle$$

$$= -i\left(\langle\phi(x_2)\phi(x_1)\rangle - \langle\phi(x_1)\rangle\langle\phi(x_2)\rangle\right)$$



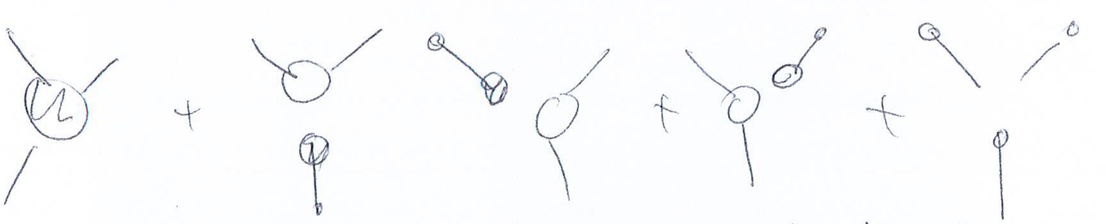
$\langle\phi(x_2)\rangle \qquad \langle\phi(x_1)\rangle$

$$\frac{\delta^3 W}{\delta j(x_3)\,\delta j(x_2)\,\delta j(x_1)} = i\,\frac{1}{Z}\,\frac{\delta^3 Z}{\delta j(x_3)\delta j(x_2)\delta j(x_1)} - \frac{i}{Z^2}\,\frac{\delta Z}{\delta j(x_3)}\,\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_1)}$$

$$+\frac{2i}{Z^3}\,\frac{\delta Z}{\delta j(x_3)}\,\frac{\delta Z}{\delta j(x_2)}\,\frac{\delta Z}{\delta j(x_1)} - \frac{i}{Z^2}\,\frac{\delta^2 Z}{\delta j_3\delta j_2}\,\frac{\delta Z}{\delta j_1} - \frac{i}{Z^2}\,\frac{\delta Z}{\delta j_2}\,\frac{\delta^2 Z}{\delta j_3\delta j_1}$$

$$= \langle\phi(x_3)\phi(x_2)\phi(x_1)\rangle - \langle\phi(x_3)\rangle\langle\phi_1\phi_2\rangle - \langle\phi_2\phi_3\rangle\langle\phi_1\rangle$$

$$-\langle\phi_2\rangle\langle\phi_3\phi_1\rangle + 2\,\langle\phi_1\rangle\langle\phi_2\rangle\langle\phi_3\rangle$$



$$\langle 123\rangle_c - \langle 3\rangle_c\langle 12\rangle_c + \cancel{\langle 3\rangle\langle 1\rangle\langle 2\rangle} - \langle 23\rangle\langle 1\rangle - \langle 2\rangle\langle 31\rangle - \langle 1\rangle\langle 2\rangle\langle 3\rangle$$

$$\frac{\delta^n W(j)}{\delta j(x_n) - - \delta j(x_1)} = i^{n+1} \langle \phi(x_1) - - \phi(x_n) \rangle_c$$

<u>Effective action</u>

$$\underbrace{\langle \Omega | \phi(x) | \Omega \rangle_j} = -\frac{\delta W}{\delta j} = \phi_{cl}(x)$$

$$\phi_{cl}(x; j) \qquad j[\phi_{cl}(x)]$$

$$\Gamma[\phi_{cl}(x)] \equiv -W(j) - \int d^4x \, j(x) \phi_{cl}(x)$$

↑ effective action.

$$\frac{\delta \Gamma[\phi_{cl}(x)]}{\delta \phi_{cl}(x)} = \cancel{\phi_{cl}} = -\frac{\delta W}{\delta j_y}\frac{\delta j_y}{\delta \phi_x} - j(x) - \int \phi_y \frac{\delta j_y}{\delta \phi_x}$$

$$= \int_y \phi_{cl}(y) \frac{\delta j_y}{\delta \phi_x} - j(x) - \int \phi_y \frac{\delta j_y}{\delta \phi_x} = -j(x)$$

$$\frac{\delta}{\delta \phi_{cl}} \left[\Gamma[\phi_{cl}] + \int_x j(x) \phi_{cl}(x) \right] = 0$$

$$\boxed{\frac{\delta S}{\delta \phi} = 0.} \qquad j=0 \quad \frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}} = 0$$

$$\frac{\delta \Gamma}{\delta \phi_{cl}} = -j$$

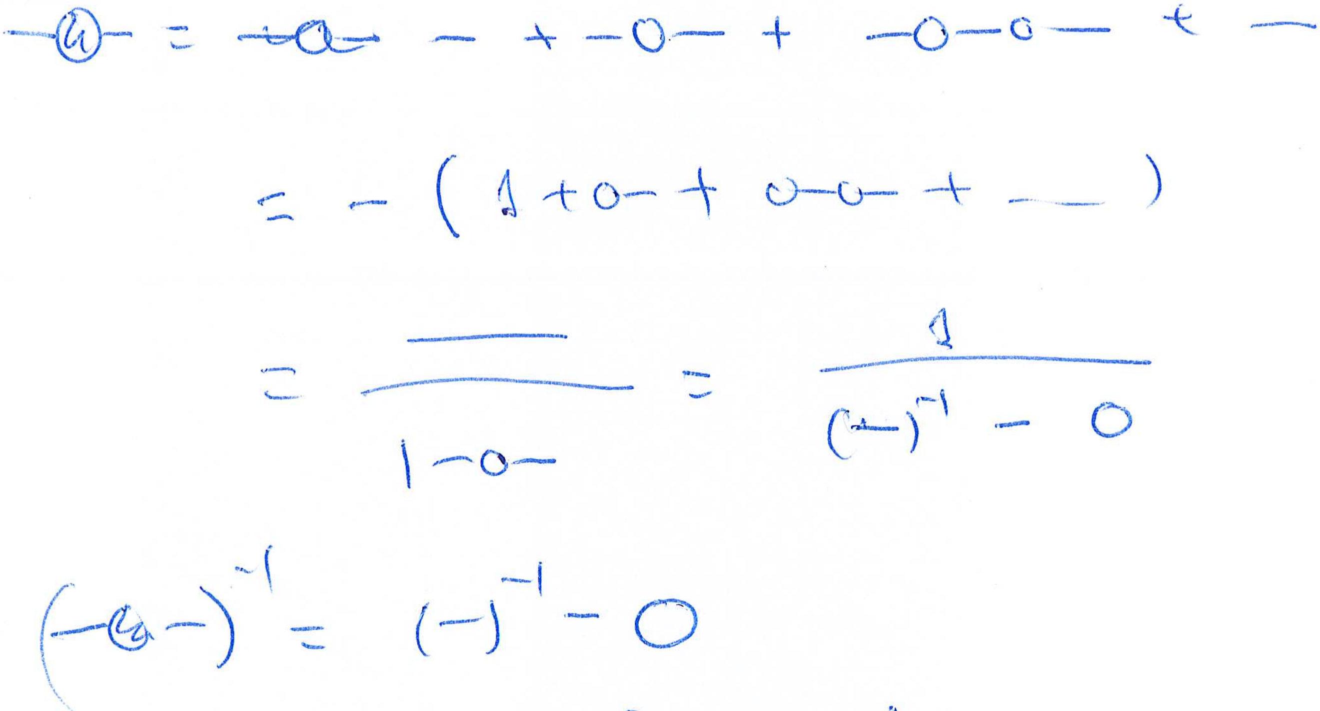
$$\frac{\delta}{\delta j(x)} \frac{\delta \Gamma}{\delta \phi_{cl}(y)} = -\delta^{(4)}(x-y)$$

$$\int_z \frac{\delta \phi_{cl}(z)}{\delta j(x)} \frac{\delta^2 \Gamma}{\delta \phi_{cl}(z)\, \delta \phi_{cl}(y)} = -\delta^{(4)}(x-y)$$

$$\int_z \underbrace{\frac{\delta^2 W}{\delta j(x)\, \delta j(z)}} \frac{\delta^2 \Gamma}{\delta \phi_{cl}(y)\, \delta \phi_{cl}(z)} = \delta^{(4)}(x-y)$$

$$-i \langle \phi(x) \phi(z) \rangle_c \quad \Gamma^{(2)}_{yz} = \delta^{(4)}$$

$$\Gamma^{(2)}(y,z) = i\, D^{-1}(x,z)$$



$$= -i\left(p^2 - m^2 - \Sigma(p^2)\right)$$

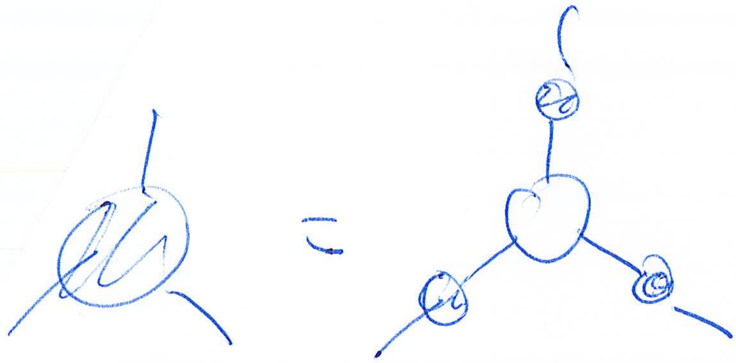
$$\frac{\delta^2 W}{\delta J \delta J} = \left(\frac{\delta^2 \Gamma}{\delta \phi \delta \phi}\right)^{-1}$$

$$\delta M^{-1} = -M^{-1} \delta M\, M^{-1}$$

$$\frac{\delta}{\delta J(z)} = \int d^4x \frac{\delta \phi}{\delta J(z)} \frac{\delta}{\delta \phi} = i \int d^4x\, D(z,x) \frac{\delta}{\delta \phi(x)}$$

$$\frac{\delta^3 W}{\delta J \delta J \delta J} = -i\left(\frac{\delta^2 \Gamma}{\delta \phi \delta \phi}\right)^{-1} \int D \frac{\delta^3 \Gamma}{\delta \phi \delta \phi \delta \phi} \cdot \left(\frac{\delta^2 \Gamma}{\delta \phi \delta \phi}\right)^{-1}$$

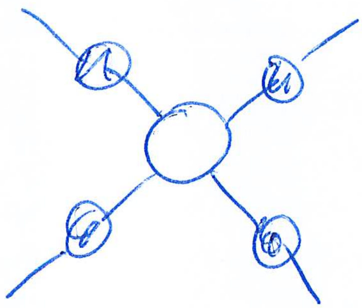
$$= -i\, D\, D \frac{\delta^3 \Gamma}{\delta \phi \delta \phi \delta \phi}\, D$$

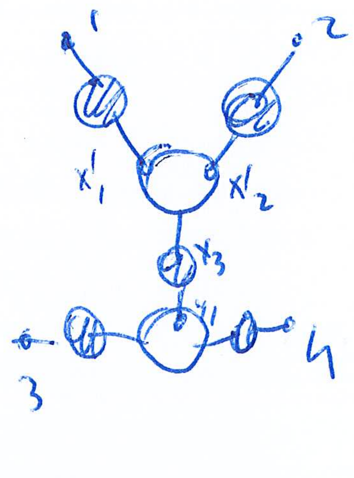


$$\frac{\delta^4 W}{\delta J_{x_4}\,\delta J_{x_3}\,\delta J_{x_2}\,\delta J_{x_1}} = -i\,\frac{\delta}{\delta J_{x_4}}\left(D_{x_3x_3'}D_{x_2x_2'}D_{x_1x_1'}\frac{\delta^3\Gamma}{\delta\phi_{x_3'}\,\delta\phi_{x_2'}\,\delta\phi_{x_1'}}\right)$$

$$= -i\left(D_{x_3y_1}D_{x_4y_2}D_{x_5y_3}\frac{\delta^3\Gamma}{\delta\phi_{y_1}\,\delta\phi_{y_2}\,\delta\phi_{y_3}}\; D_{\lambda_1 x_3'}\, D - \frac{\delta^3\Gamma}{\delta -} + \right.$$

$$\left. + DDDD\,\frac{\delta^4\Gamma}{\delta\phi\,\delta\phi\,\delta\phi\,\delta\phi}\right)$$





$+ (-) + (-)$

Feynman rules from path integral.

$$Z[j(x)] = \int \mathcal{D}\phi \; e^{iS[\phi] + i\int j(x)\phi(x)}$$

generating function.

example ϕ^4

$$Z[j(x)] = \int \mathcal{D}\phi \quad e^{i\int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right) + i\int j(x)\phi(x)}$$

$$= \int \mathcal{D}\phi \quad e^{-i\frac{\lambda}{4!}\int d^4x\,\phi^4} \quad e^{-\frac{i}{2}\int d^4x\,\phi(x)(\partial^2+m^2)\phi(x) + i\int j(x)\phi(x)}$$

$$= \int \mathcal{D}\phi \quad e^{-i\frac{\lambda}{4!}\int d^4x\,\frac{\delta^4}{\delta j(x)}} \quad e^{-\frac{i}{2}\int d^4x \;-\;-}$$

$$= e^{-i\frac{\lambda}{4!}\int d^4x\,(-i)^4\frac{\delta^4}{\delta j^4(x)}} \quad \mathcal{N} \quad e^{-\frac{1}{4i/2}\int_{xy} j(x)\,(\partial^2+m^2)^{-1}\,j(y)}$$

$$= e^{-i\frac{\lambda}{4!}\int d^4x\,\frac{\delta^4}{\delta j^4(x)}} \quad \mathcal{N} \; e^{\frac{i}{2}\int j(x)\,\Delta(x,y)\,j(y)}$$

$$\boxed{(\partial_x^2 + m^2)\,\Delta(x,y) = \delta^{(4)}(x-y)}$$

$$(-p^2+m^2)\,\Delta(p) = 1 \qquad \Delta(p) = -\frac{1}{p^2-m^2}$$

②

$$e^{-i\frac{\lambda}{4!}\int d^4x\,(?)\frac{\delta^4}{\delta j^4(x)}} \qquad e^{-\frac{1}{2}\int j(x)\,\Delta_F(x-y)\,j(y)}$$

$$\Delta_F \to \frac{i}{p^2-m^2}$$

$$(-i)\frac{\delta}{\delta j(x_n)} - - - (-i)\frac{\delta}{\delta j(x_1)} \qquad e^{-\frac{1}{2}\int j(x)\,\Delta_F(x-y)\,j(y)}$$

$$(-i)\frac{\delta}{\delta j(x_2)}\;\cancel{-\tfrac{1}{2}}\left(+i\int j(x)\,\Delta_F(x-x_1)\right) e^{-\frac{1}{2}\int j(x)\,\Delta_F(x-y)\,j(y)}$$

$$+\Delta_F(x_2-x_1) + i\int j(x)\,\Delta_F(x-x_1)\; i\int j(y)\,\Delta_F(y-x_2)$$

Gives Wick's theorem.

Effective action $\quad Z(j) = e^{iW} = \int \mathcal{D}\phi\; e^{iS[\phi] + \int j\phi}$

j chosen such that $\langle\phi\rangle = \phi_{cl}(x)$

take $\phi = \phi_{cl} + \eta \qquad Z(j) = \ln\Gamma(\phi_{cl}) = e^{iW - \int j\phi} = \int \mathcal{D}\phi\, e$

$$= e^{iS[\phi_{cl}]}\int \mathcal{D}\phi\; e^{i\frac{\delta S}{\delta\phi\delta\phi}\eta\eta + \frac{\partial^3 S}{\partial\phi\partial\phi\partial\phi}\eta\eta\eta + \cdots + \underbrace{\left(\frac{\partial S}{\partial\phi}\eta + j\eta\right)}_{j=-\partial S/\partial\phi} + i\cancel{\phi_{cl}} + \cancel{\delta j\,\phi_{cl}} + \delta j\,\eta)} \cancel{+ j\phi_{cl}}$$

Path integrals for gauge theories

$$S = \frac{1}{4}\int d^4x\; F_{\mu\nu}F^{\mu\nu}$$

$$= \frac{1}{2}\int d^4x\; \partial_\mu A_\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$$\cancel{= \frac{1}{2}\int d^4x\; A_\nu \left(\partial^2 A\, \partial_\mu\right)}$$

$$= -\frac{1}{2}\int d^4x\; A_\nu \left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\nu$$

$$\left(-\eta^{\mu\nu}p^2 + p^\mu p^\nu\right) A_{\nu\alpha} = \delta^\mu{}_\alpha$$

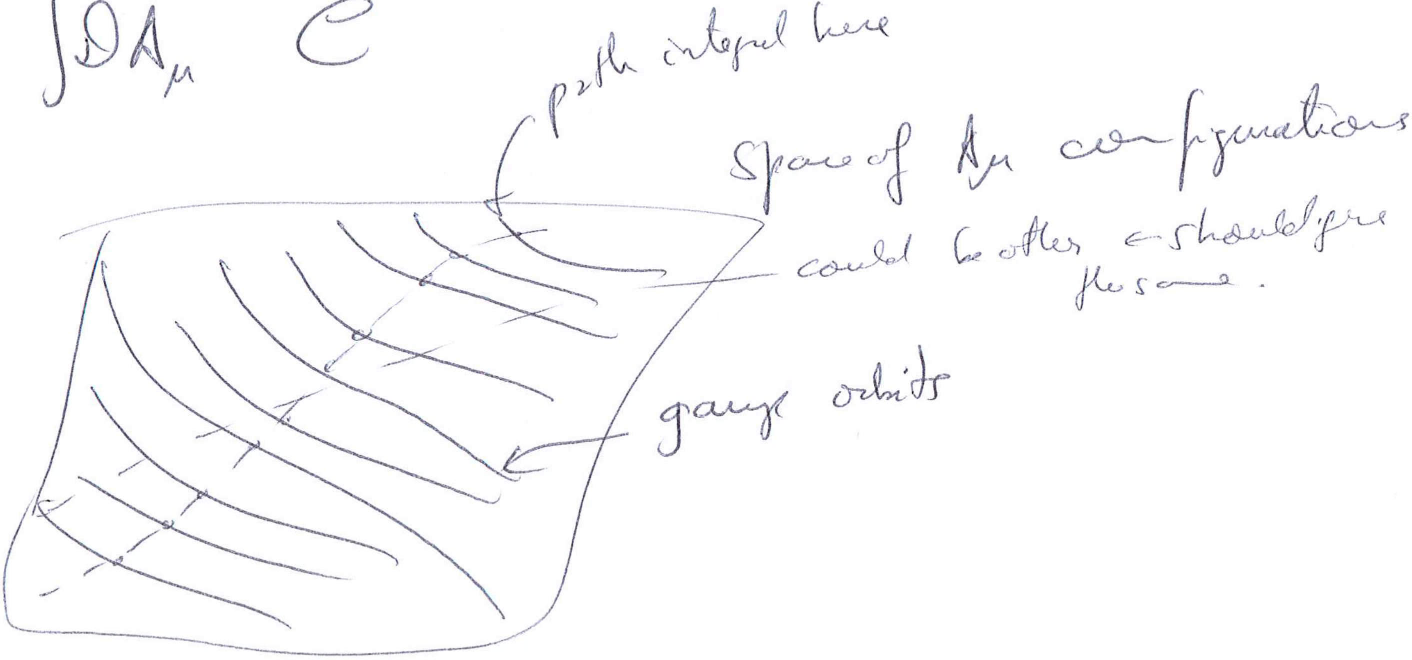
$$0 = -p^\mu p^2 + p^\mu p^2 \qquad = p_\nu$$

zero ~~mode~~ eigenvalue $\rightarrow$ **no inverse**

$A_\mu \rightarrow A_\mu + \partial_\mu \lambda$; extra degrees of freedom.

gauge fixing

$$\int \mathcal{D}A_\mu \; e^{iS[A_\mu]}$$



Gauge fixing $\quad \mathcal{F}(A_\mu) = 0 \qquad A^\lambda_\mu = A_\mu + \partial_\mu \lambda$

$$\Delta_f^{-1}[A_\mu] = \int \underbrace{\mathcal{D}\lambda(x)}_{\text{gauge transf.}} \; \delta(\mathcal{F}(A^\lambda_\mu))$$

$$\Delta_f^{-1}[A^{\tilde\lambda}_\mu] = \int \mathcal{D}\lambda(x) \; \delta(\mathcal{F}(A_\mu^{\lambda+\tilde\lambda}))$$

$$= \int \mathcal{D}\lambda(x)\, \delta(\mathcal{F}(A^\lambda_\mu)) = \Delta_f^{-1}[A_\mu]$$

gauge invariant functional

$$\int \mathcal{D}A_\mu \int \mathcal{D}\lambda \; \delta(\mathcal{F}(A_\mu^\lambda)) \, \Delta_f(A_\mu) \, e^{iS[A_\mu]} =$$

gauge transf λ^{-1}

$$= \int \mathcal{D}\lambda \int \mathcal{D}A_\mu \; \delta(\mathcal{F}(A_\mu)) \, \Delta_{FP}(A_\mu) \, e^{iS[A_\mu]}$$

Assuming measure & action are gauge inv.

Volume of gauge group $\to \infty$ but indep. of A_μ, etc.

b.c.

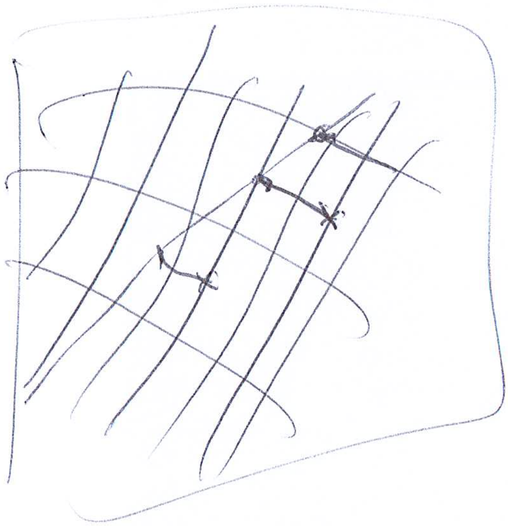
$$\int \mathcal{D}A_\mu \; \delta(\mathcal{F}(A_\mu)) \int \mathcal{D}\bar{c}\mathcal{D}c \; e^{\int_{xy} \bar{c}_x A_{FP} c_y} \, e^{iS[A_\mu]}$$

$$\delta(\mathcal{F}(A_\mu) - B_\mu)$$

$$\int e^{-\frac{i}{\xi}\int_x B^2(x)} \mathcal{D}B(x)$$

$$\int \mathcal{D}A_\mu \int \mathcal{D}\bar{c}\mathcal{D}c \; e^{i\int_{xy} \bar{c} A_{FP} c - \frac{i}{\xi}\int \mathcal{F}^2 + iS[A_\mu]}$$

$$\int \mathcal{D}\lambda \; \delta\left(\mathcal{F}(A_\mu^{\lambda})\right) = \int \mathcal{D}\lambda \; \delta\Big(\mathcal{F}\big(\underbrace{A_\mu^{\lambda_0}}_{A_\mu^{(0)}} + \partial\lambda\big)\Big)$$



$$= \int \mathcal{D}\lambda \; \delta\left(\mathcal{F}(A_\mu^{(0)\lambda})\right)$$

$$= \int \mathcal{D}\lambda \; \delta\left(\int_y \frac{\delta \mathcal{F}(x)}{\delta\lambda(y)} \lambda(y) + \cdots\right)$$

$$= \det{}^{-1}\left(\frac{\delta \mathcal{F}(x)}{\delta\lambda(y)}\right)$$

$$\Delta_{FP} \, \cancel{(A_\mu)} = \det\left(\frac{\delta \mathcal{F}(A_\mu(x))}{\delta\lambda(y)}\right)\Bigg|_{\mathcal{F}(A)=0}$$

$$= \int \mathcal{D}\bar{c}(x)\, \mathcal{D}c \; e^{\int_{xy} \bar{c}(x) \frac{\delta \mathcal{F}(x)}{\delta\lambda(y)} c(y)} \Bigg|_{\mathcal{F}(A)=0}$$

$$\zeta(x) = \int_y \frac{\delta \mathcal{F}(x)}{\delta\lambda(y)} \lambda(y)$$

$$\frac{\delta\zeta(x)}{\delta\lambda(y)} = \frac{\delta \mathcal{F}(x)}{\delta\lambda(y)} \qquad \int \mathcal{D}\zeta \; \det{}^{-1}\left(\frac{\delta \mathcal{F}(x)}{\delta\lambda(y)}\right) \delta(\zeta(x))$$

$$= \det\left(\frac{\delta \mathcal{F}(x)}{\delta\lambda(y)}\right)$$

examples

$\partial_\mu A^\mu = 0$.

$$\Delta_{FP}^{-1}(A_\mu) = \int \mathcal{D}\lambda(x)\ \delta(\partial_\mu A^\mu + \partial_\mu^2 \lambda) = \partial^{-2}$$

$$\Delta_{FP}[A_\mu] = \partial^2 \quad \text{indep. of } A_\mu.$$

$$\bar{c}\,\partial^2 c$$

$A_\mu A^\mu = 0$

$$\Delta_{FP}^{-1}[A_\mu] = \int \mathcal{D}\lambda(x)\ \delta((A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda))$$

$$A_\mu \to A_\mu + \partial_\mu \lambda \,/\, (A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda) = 0$$

$$A_\mu A^\mu + 2A^\mu \partial_\mu \lambda + \partial_\mu \lambda \partial^\mu \lambda = 0$$

$$\left.\frac{\delta F_i(A^\lambda(x))}{\delta\lambda(x)}\right|_{F=0} = \frac{\delta((A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda))^{(x)}}{\delta\lambda(y)} = 2(A^\mu + \partial_\mu \lambda)\,\partial^\mu \delta(x-y)$$

$$= 2A^{\mu}_{\lambda\mu}\, \partial_\mu \delta(x-y)$$

$$2\,\bar{c}\,\partial_\mu c\, A^\mu$$

$$\int \mathcal{D}A_\mu\; e^{2\int_x \bar{c}\,\partial_\mu c\, A^\mu}\;\delta(\partial_\mu A^\mu)\; e^{i S[A_\mu]}$$

$$\int \mathcal{D}A_\mu\; e^{2\int \bar{c}\,\partial_\mu c\, A^\mu + \frac{1}{\xi}\int (\partial_\mu A^\mu)(\partial_\nu A^\nu) + i S[A_\mu]}$$

(9)

$$S = \int \bar{c}\,\partial^2 c - \frac{i}{2\zeta}\int (\partial A)^2 - \frac{i}{4}\int F_{\mu\nu}F^{\mu\nu}$$

$\partial_\mu A_\mu \partial_\nu A_\nu$

$$+ \frac{i}{2\zeta}\int A_\mu\,\partial_{\mu\nu} A_\nu - \frac{i}{2}\int \partial_\mu A_\nu\,(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\frac{i}{2\zeta}\int A_\mu\,\partial_{\mu\nu} A_\nu + \frac{i}{2}\int A_\nu\,\partial^2 A_\nu - A_\nu\,\partial_{\mu\nu} A_\mu$$

$$\frac{i}{2}\int A_\nu\left(\partial^2 A_\nu - \left(1-\frac{1}{\zeta}\right)\partial_\nu\partial_\mu A_\mu\right)$$

$\underline{\zeta = 1}$

$$\frac{i}{2}\int A_\nu\,\partial^2 A_\nu \qquad -\frac{i}{2}\int \phi\,\partial^2\phi$$

scalar field

$$\langle A_\mu A_\nu\rangle = \frac{-i\,\eta_{\mu\nu}}{p^2 + i\varepsilon}$$

$$a\,\eta^{\nu\rho} + b\,p^\nu p^\rho$$

$$\left(-p^2\eta_{\mu\nu} + \left(1-\frac{1}{\zeta}\right)p_\mu p_\nu\right)\Delta_{\nu\rho} = \delta_{\mu\rho}$$

$$\left(-a p^2\,\delta_\mu{}^\rho - b p^2\,p_\mu p^\rho + \left(1-\frac{1}{\zeta}\right)a\,p_\mu p^\rho + \left(1-\frac{1}{\zeta}\right)p^2 b\,p_\mu p^\rho\right.$$

$$= \delta_\mu{}^\rho$$

$$a = -1/p^2 \qquad -b p^2 + a(1-1/\zeta) + (1-1/\zeta)p^2 b = 0$$

$$b\left(\cancel{1}-\tfrac{1}{\xi}-\cancel{1}\right)p^2 = \frac{1}{p^2}\left(1-\tfrac{1}{\xi}\right)$$

$$b = \frac{1}{p^4}\,\xi\left(\frac{1}{\xi}-1\right) = \frac{1-\xi}{p^4}$$

$$\boxed{\langle A_\mu A_\nu\rangle = \frac{-i}{p^2+i\varepsilon}\left(\eta_{\mu\nu} + \frac{(1-\xi)}{p^2}\,p_\mu p_\nu\right)}$$

$\xi = 0$ Landau gauge

To quantize we fixed the gauge.

~~However~~ then there is no gauge symmetry to restrict the counterterms ?!

BRST symmetry (abelian case)

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\not{D}-m)\psi + \frac{\xi}{2}B^2 + B\,\mathcal{F}(A) + \bar{c}\,\mathcal{M}c$$

$$\delta\mathcal{F} = \mathcal{M}\underbrace{\delta\lambda}_{\text{gauge parameter}} = \frac{\delta\mathcal{F}}{\delta A(x)}\partial_\mu\lambda$$

example $\mathcal{F} = \partial_\mu A^\mu$ $\qquad \delta\mathcal{F} = \partial^2\lambda = \partial^\mu\partial_\mu\lambda$

$$\mathcal{M} = \partial^2$$

$$\left.\begin{aligned}\delta A_\mu &= \zeta\partial_\mu c\\ \delta\psi &= i\zeta c\psi\end{aligned}\right\}\ \text{gauge transf. w/ parameter } \zeta\psi$$

fermionic (constant).

$-\frac{1}{4}F^2 + \bar{\psi}(i\not{D}-m)\psi$ is gauge inv.

$$\delta(B\,\mathcal{F}(A)) = B\,\mathcal{M}\zeta c = \zeta B\mathcal{M}c$$

take $\delta B = 0$

$$\delta(\mathcal{M}c) = \delta\left(\frac{\delta\mathcal{F}}{\delta A(x)}\partial_\mu c\right) = \underbrace{\frac{\delta^2\mathcal{F}}{\delta A(x)\delta(A(y))}}_{\text{sym}}\underbrace{\zeta\partial_\mu c\,\partial_\mu c}_{\text{antisym}} + \frac{\delta\mathcal{F}}{\delta A(x)}\partial_\mu\underbrace{\delta c}_{0}$$

$$= 0$$

$$\delta(\bar{c}\mathcal{M}c) = \delta\bar{c}\,\mathcal{M}c \quad \Rightarrow \quad \boxed{\delta\bar{c} = -\zeta B}$$

Thus:

$$\boxed{\begin{aligned} \delta A_\mu &= \zeta \partial_\mu c \\ \delta \psi &= i \zeta c \psi \\ \delta c &= 0 \\ \delta B &= 0 \\ \delta \bar{c} &= -\zeta B \end{aligned}}$$

$$Q A_\mu = \partial_\mu c \qquad Q^2 A_\mu = 0$$

$$Q \psi = i c \psi \qquad Q^2 \psi = i \underbrace{c\, i c}_{\text{fermions}} \psi = 0$$

$$Q \bar{c} = -B \qquad Q^2 \bar{c} = 0$$

$$Q c = 0 \qquad Q^2 c = 0$$

$$Q B = 0 \qquad Q^2 B = 0$$

$$\Rightarrow \boxed{Q^2 = 0}$$

Fermionic symmetry.

cohomology $\qquad |\psi\rangle$ <u>states</u>

$Q|\psi\rangle$ are such that $Q(Q|\psi\rangle) = 0$

Look for solutions of $Q|\psi\rangle = 0$ modulo trivial $Q|\psi\rangle$ solutions.

assume all possible pol.

$$A_\mu |0\rangle = \int e^{ikx} \sum_{\sigma=1}^{4} \epsilon_\mu^\sigma |1^\gamma_{k,\sigma}\rangle \quad ; \quad \partial_\mu c|0\rangle = \int e^{ikx} k_\mu |1^c_k\rangle$$

$$Q A_\mu |0\rangle = \int e^{ikx} \sum_\sigma \epsilon_\mu^\sigma Q |1^\gamma_{k,\sigma}\rangle = \int e^{ikx} k_\mu |1^c_k\rangle$$

$$\left(\sum_\sigma \epsilon_\mu^\sigma Q |1^\gamma_{k,\sigma}\rangle = k_\mu |1^c_k\rangle \right) \epsilon_\mu^{\sigma'} \mathcal{O}^\mu \Rightarrow Q |1^\gamma_{k,\sigma}\rangle = (\epsilon_\mu^{*\sigma} k^\mu) |1^c_k\rangle$$

if $(k\epsilon) = 0 \qquad Q|1^\gamma_{k,\sigma}\rangle = 0$

WT, ST, BRST identities

$$S = -\frac{1}{4}F^2 + \bar\psi(i\not{D}-m)\psi + \frac{\xi}{2}B^2 + B\partial A + \bar c \partial c$$

$$\int \mathcal{D}[A B c \bar c]\, e^{i\int d^4x\, S + i\int d^4x\,(j_\mu A^\mu + \bar\eta c + \bar c \eta + \bar K c\psi + \bar\psi c K + j_B B + \bar J_\psi \psi + \bar\psi J_\psi)}$$

change of variables : BRST.

$$\text{"}\ i\int d^4x \left(j_\mu \zeta c - \zeta B\eta + i\bar J_\psi \zeta c\psi - i\bar\psi c\zeta J_\psi\right) G_c = 0\ \text{"}$$

$$i\int d^4x \left(\zeta j_\mu \frac{\delta}{\delta\bar\eta} - \zeta\eta\frac{\delta}{\delta j_B} + i\bar J_\psi \zeta\frac{\delta}{\delta \bar K} - i\zeta\frac{\delta}{\delta K} J_\psi\right) G_c = 0$$

$G_c \to \Gamma$

$$i\int d^4x \left(\zeta\frac{\delta\Gamma}{\delta A_\mu} c - \zeta\frac{\delta\Gamma}{\delta\bar c} B + i\frac{\delta\Gamma}{\delta\bar\psi}\zeta\frac{\delta\Gamma}{\delta\bar K} - i\zeta\frac{\delta\Gamma}{\delta K}\frac{\delta\Gamma}{\delta\psi}\right) = 0$$

$$i\int d^4x \left(\frac{\delta\Gamma}{\delta A_\mu(x)} c(x) - \frac{\delta\Gamma}{\delta\bar c(x)} B(x) + i\frac{\delta\Gamma}{\delta\bar\psi(x)}\frac{\delta\Gamma}{\delta\bar K(x)} - i\frac{\delta\Gamma}{\delta K(x)}\frac{\delta\Gamma}{\delta\psi(x)}\right) = 0$$

$i S \to i\bar c\partial^2 c$ gauge $\partial_\mu A^\mu$

$$i\partial^2\frac{\delta G}{\delta\bar\eta} + \eta G = 0 \qquad i\partial^2 c + \frac{\delta\Gamma}{\delta\bar c} = 0 \qquad \Gamma = i\bar c\partial^2 c$$

$B \to B + \delta B$

$$e^{-\frac{F^2}{4\frac{\xi}{2}i}} \quad e^{-\frac{i}{2\xi}(\partial A)^2}$$

$$\int (\xi B + \partial_\mu A_\mu) G_c = 0$$

$$S = -\frac{1}{2\xi}(\partial A)^2$$

$$B = -\frac{1}{\xi}\partial_\mu A^\mu$$

$$i\int d^4x \left(\delta_c^g \Gamma + \underbrace{\frac{\delta \Gamma}{\delta \bar{c}}}_{\partial^2 c} \frac{1}{\xi}\partial_\mu A^\mu\right) = 0$$

$$\delta_c^g \Gamma + \int \frac{1}{\xi}\partial_\mu A^\mu \partial^2 c = 0$$

$$\delta\left(\frac{1}{2\xi}\int (\partial_\mu A^\mu)^2\right) = \frac{1}{\xi}\partial_\mu A^\mu \partial^2 c$$

$$\delta_c^g \left(\Gamma + \frac{1}{2\xi}\int (\partial A)^2\right) = 0.$$

Γ is gauge invariant except for the gauge fixing term!

Γ gauge inv.

$\Gamma \sim \quad \bar\psi\psi \quad \bar\psi\psi A \quad \bar\psi\psi AA \quad - -$

$(\bar\psi\psi)(\bar\psi\psi) \quad -\lambda \quad -$

$\delta\psi = i\alpha\psi \qquad \delta\bar\psi = -i\alpha\bar\psi \qquad \delta A = \partial_\mu \alpha.$

\# of ψ, $\bar\psi$ preserved

\# A decreases by 1.

So we can look at

$$\Gamma = \int d^4x\, d^4y\ \bar\psi(x)\, S^{-1}(x-y)\, \psi(y) + \int d^4x\, d^4y\, d^4z\ \bar\psi(x)\, \Gamma^\mu(x-z, y-z)\, \psi(y)\, A_\mu(z) + - -$$

$$\delta\Gamma = \int d^4x\, d^4y\ \left(-i\alpha(x)\,\bar\psi(x)\, S^{-1}(x-y)\,\psi(y) + i\,\bar\psi(x)\, S^{-1}(x-y)\,\alpha(y)\psi(y)\right)$$

$$+ \int d^4x\, d^4y\, d^4z\ \bar\psi(x)\, \partial_\mu\alpha(z)\, \Gamma^\mu(x-z, y-z)\, \psi(y) + - -$$

$=0$

$$\boxed{-\partial^z_\mu \Gamma^\mu(x-z, y-z) = i\,\delta(z-x)\, S^{-1}(x-y) - i\,\delta(z-y)\, S^{-1}(x-y)}$$

$$\Gamma^\mu(x,y) = \int d^4p\, d^4q\; e^{-ipx+iqy}\, \Gamma^\mu(p,q)$$

$$\int d^4p\, d^4q\, (-\partial^z_\mu)\, e^{-ip(x-z)+i(y-z)q}\, \Gamma^\mu(p,q) =$$

$$e^{i(p-q)z}$$

$$= \int d^4p\, d^4q\, (-i)\,(p-q)_\mu\, \Gamma^\mu(p,q)\, e^{-ip(x-z)+iq(y-z)}$$

$$S^{-1}(x-y) = i\int d^4p\; e^{-ip(x-y)}\, S^{-1}(p)$$

$$\delta(z-x) = \int d^4q\; e^{iq(z-x)}$$

$$= i\int d^4p\, d^4q \left(e^{iq(z-x)+ip(x-y)}\, S^{-1}(p) - e^{iq(z-y)+ip(x-y)}\, S^{-1}(p) \right)$$

$x \to x+z \qquad y \to y+z$

$$-i\int d^4p\, d^4q\, (p-q)_\mu \Gamma^\mu(p,q)\; e^{-ip(x-\cancel{z})+iqy} =$$

$$e^{-i(q+p)y+ipx}$$

$$= i\int d^4p\, d^4q\; e^{-iqx+ip(x-y)}\, S^{-1}(p) - i\int d^4p\, d^4q\; e^{-iqy+ip(x-y)}\, S^{-1}(p)$$

$$e^{-i(q+p)x+ipy}\quad S^{-1}(p)$$

$$-i(p-q)_\mu \Gamma^\mu(p,q) = i\left(i\, S^{-1}(q) - i\, S^{-1}(\cancel{q}+p)\right)$$

$$-i\, k_\mu \Gamma^\mu(p+k,p) = i\left(i\, S^{-1}(p) - i\, S^{-1}(+p+k)\right) = S^{-1}(p+k) - S^{-1}(p)$$