2/2

PHYS 662 — Quantum Field Theory I

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Student Number: [redacted] **Assignment:** Homework 1

Problem 1 For a harmonic oscillator in usual single-particle quantum mechanics, use the Schroedinger formalism to compute the propagator:

$$K(x_f, t_f; x_i, t_i) = \langle x_f| e^{-\frac{i}{\hbar}H(t_f-t_i)} |x_i\rangle\,, \quad H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

In order to do that recall that

$$\langle x|n\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{2}$$

where $H_n(y)$ are Hermite polynomials that satisfy the identity ($|\mathrm{Re}\,\alpha| < \frac{1}{2}$)

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(x) H_n(y) = \frac{1}{2}\frac{1}{\sqrt{\frac{1}{4}-\alpha^2}} e^{x^2+y^2-\frac{(x+y)^2}{2(1+2\alpha)}-\frac{(x-y)^2}{2(1-2\alpha)}} \tag{3}$$

Compare the propagator with the exponential of the classical action. Can you explain the result using the path integral formalism?

Answer 1 To compute the propagator, it will be useful to let the Hamiltonian operator act on the eigenstates use the orthogonality of the eigenstates. Remember that Hermite polynomials are real. In order to do this we can use the unitary identity,

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1} \tag{4}$$

Inserting this yields

$$\begin{aligned}
K(x_f, t_f; x_i, t_i) &= \sum_{n,k=0}^{\infty} \langle x_f|k\rangle \langle k| e^{-\frac{i}{\hbar}H(t_f-t_i)} |n\rangle \langle n|x_i\rangle = \sum_{n,k=0}^{\infty} e^{-\frac{i}{\hbar}E_n(t_f-t_i)} \langle x_f|k\rangle \langle k|n\rangle \langle n|x_i\rangle \\
&= \sum_{n,k=0}^{\infty} e^{-\frac{i}{\hbar}E_n(t_f-t_i)} \langle x_f|k\rangle \delta_{kn} \langle n|x_i\rangle = \sum_{n=0}^{\infty} e^{-\frac{i}{\hbar}E_n(t_f-t_i)} \langle x_f|n\rangle \langle n|x_i\rangle \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{m\omega(x_f^2+x_i^2)}{2\hbar}} \sum_{n=0}^{\infty} \frac{e^{-\frac{i}{\hbar}E_n(t_f-t_i)}}{2^n} \frac{1}{n!} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x_f\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x_i\right)
\end{aligned}$$

where $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$. After plugging this we see

$$\frac{e^{-\frac{i}{\hbar}E_n(t_f-t_i)}}{2^n} = e^{-\frac{i}{2}\omega(t_f-t_i)}\left(\frac{e^{-i\omega(t_f-t_i)}}{2}\right)^n \tag{5}$$

from here we can identify α in the given formula for Hermite polynomials as

$$\alpha = \frac{e^{-i\omega(t_f - t_i)}}{2} \tag{6}$$

This guarantees

$$|\mathrm{Re}\,\alpha| < \frac{1}{2} \tag{7}$$

for $t_f - t_i \neq \pi n/\omega$. Thus, we can use the given formula. After using these results, finally we find

$$\begin{aligned}
K(x_f, t_f; x_i, t_i) &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{m\omega(x_f^2+x_i^2)}{2\hbar}} e^{-\frac{i}{2}\omega(t_f-t_i)} \sum_{n=0}^{\infty} \alpha^n \frac{1}{n!} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_f\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_i\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{m\omega(x_f^2+x_i^2)}{2\hbar}} e^{-\frac{i}{2}\omega(t_f-t_i)} \frac{1}{\sqrt{1-4\alpha^2}} e^{\frac{m\omega}{\hbar}(x_i^2+x_f^2) - \frac{m\omega(x_i+x_f)^2}{2\hbar(1+2\alpha)} - \frac{m\omega(x_i-x_f)^2}{2\hbar(1-2\alpha)}} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{i}{2}\omega(t_f-t_i)} \frac{1}{\sqrt{1-e^{-2i\omega(t_f-t_i)}}} e^{\frac{m\omega}{2\hbar}(x_i^2+x_f^2) - \frac{m\omega(x_i+x_f)^2}{2\hbar(1+2\alpha)} - \frac{m\omega(x_i-x_f)^2}{2\hbar(1-2\alpha)}} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{1}{\sqrt{e^{i\omega(t_f-t_i)} - e^{-i\omega(t_f-t_i)}}} e^{\frac{m\omega}{2\hbar}(x_i^2+x_f^2) - \frac{m\omega(x_i+x_f)^2}{2\hbar(1+2\alpha)} - \frac{m\omega(x_i-x_f)^2}{2\hbar(1-2\alpha)}} \\
&= \left(\frac{m\omega}{2\pi i\hbar \sin(\omega(t_f-t_i))}\right)^{\frac{1}{2}} e^{\frac{m\omega}{2\hbar}(x_i^2+x_f^2) - \frac{m\omega(x_i+x_f)^2}{2\hbar(1+2\alpha)} - \frac{m\omega(x_i-x_f)^2}{2\hbar(1-2\alpha)}} \\
&= \left(\frac{m\omega}{2\pi i\hbar \sin(\omega(t_f-t_i))}\right)^{\frac{1}{2}} e^{\frac{m\omega}{2\hbar}(x_i^2+x_f^2) - \frac{m\omega(x_i^2+x_f^2-4x_ix_f\alpha)}{\hbar(1-4\alpha^2)}} \\
&= \left(\frac{m\omega}{2\pi i\hbar \sin(\omega(t_f-t_i))}\right)^{\frac{1}{2}} e^{\frac{m\omega}{2\hbar}(x_i^2+x_f^2) - \frac{m\omega(x_i^2+x_f^2-4x_ix_f\alpha)}{\hbar(1-e^{-2i\omega(t_f-t_i)})}} \\
&= \left(\frac{m\omega}{2\pi i\hbar \sin(\omega(t_f-t_i))}\right)^{\frac{1}{2}} \exp\left\{\frac{m\omega}{2\hbar}\left[(x_i^2+x_f^2) - \frac{(e^{i\omega(t_f-t_i)}(x_i^2+x_f^2) - 2x_ix_f)}{i\sin(\omega(t_f-t_i))}\right]\right\} \\
&= \left(\frac{m\omega}{2\pi i\hbar \sin(\omega(t_f-t_i))}\right)^{\frac{1}{2}} \exp\left\{\frac{im\omega}{2\hbar}\left[\frac{\cos(\omega(t_f-t_i))(x_i^2+x_f^2) - 2x_ix_f)}{\sin(\omega(t_f-t_i))}\right]\right\} \checkmark
\end{aligned}$$

Let us write classical action to make the comparison,

$$S = \int_{t_i}^{t_f} dtL = \int_{t_i}^{t_f} dt\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2\right) \tag{8}$$

This identity will go over the exponential, this motivates us to evaluate the action integral to make the comparison. Consider the equation of motion,

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{x}} - \frac{\mathrm{d}L}{\mathrm{d}x} = \ddot{x} + \omega^2 x = 0 \tag{9}$$

which has the solution

$$x = A\sin(\omega t) + B\cos(\omega t) \tag{10}$$

from the boundary conditions $x_i = A\sin(\omega t_i) + B\cos(\omega t_i)$ and $x_f = A\sin(\omega t_f) + B\cos(\omega t_f)$ we find the coefficients,

$$A = \frac{x_f\cos(\omega t_i) - x_i\cos(\omega t_f)}{\sin(\omega(t_f - t_i))} \tag{11}$$

$$B = \frac{x_i\sin(\omega t_f) - x_f\sin(\omega t_i)}{\sin(\omega(t_f - t_i))}. \tag{12}$$

Inserting them gives the solution as

$$\begin{aligned} x &= \frac{x_f\cos(\omega t_i) - x_i\cos(\omega t_f)}{\sin(\omega(t_f - t_i))}\sin(\omega t) + \frac{x_i\sin(\omega t_f) - x_f\sin(\omega t_i)}{\sin(\omega(t_f - t_i))}\cos(\omega t) \\ &= \frac{x_i\sin(\omega(t_f - t)) + x_f\sin(\omega(t - t_i))}{\sin(\omega(t_f - t_i))} \end{aligned}$$

Thus,

$$\dot{x} = \omega\frac{x_f\cos(\omega(t - t_i)) - x_i\cos(\omega(t_f - t))}{\sin(\omega(t_f - t_i))} \tag{13}$$

Now we can write the action in terms of t, to evaluate

$$\begin{aligned} S = \frac{1}{2}m\omega^2\int_{t_i}^{t_f} dt & \left[\frac{x_f\cos(\omega(t - t_i)) - x_i\cos(\omega(t_f - t))}{\sin(\omega(t_f - t_i))}\right]^2 \\ & - \left[\frac{x_i\sin(\omega(t_f - t)) + x_f\sin(\omega(t - t_i))}{\sin(\omega(t_f - t_i))}\right]^2 \end{aligned}$$

$$\begin{aligned} S = \frac{m\omega^2}{2\sin^2(\omega(t_f - t_i))}\int_{t_i}^{t_f} & dt\,[x_f\cos(\omega(t - t_i)) - x_i\cos(\omega(t_f - t))]^2 \\ & - [x_i\sin(\omega(t_f - t)) + x_f\sin(\omega(t - t_i))]^2 \end{aligned}$$

$$\begin{aligned} S = \frac{m\omega^2}{2\sin^2(\omega(t_f - t_i))}\int_{t_i}^{t_f} & dt x_f^2\cos^2(\omega(t - t_i)) + x_i^2\cos^2(\omega(t_f - t)) - 2x_i x_f\cos(\omega(t - t_i))\cos(\omega(t_f - t)) \\ & - x_i^2\sin^2(\omega(t_f - t)) - x_f^2\sin^2(\omega(t - t_i)) - 2x_i x_f\sin(\omega(t_f - t))\sin(\omega(t - t_i)) \end{aligned}$$

$$\begin{aligned} S = \frac{m\omega^2}{2\sin^2(\omega(t_f - t_i))}\int_{t_i}^{t_f} & dt[x_f^2(\cos^2(\omega(t - t_i)) - \sin^2(\omega(t - t_i))) \\ & + x_i^2(\cos^2(\omega(t_f - t)) - \sin^2(\omega(t_f - t))) \\ & - 2x_i x_f\cos(\omega(2t - t_f - t_i))] \end{aligned}$$

Lets just focus on the integral for the moment

$$\begin{aligned} I &= \int_{t_i}^{t_f} dt[x_f^2\cos(2\omega(t - t_i)) + x_i^2\cos(2\omega(t_f - t)) - 2x_i x_f\cos(\omega(2t - t_f - t_i))] \\ &= \frac{1}{2\omega}(x_f^2 + x_i^2)\sin(2\omega(t_f - t_i)) - \frac{2x_i x_f}{\omega}\sin(\omega(t_f - t_i)) \\ &= \frac{1}{\omega}(x_f^2 + x_i^2)\sin(\omega(t_f - t_i))\cos(\omega(t_f - t_i)) - \frac{2x_i x_f}{\omega}\sin(\omega(t_f - t_i)) \end{aligned}$$

Inserting this gives us the action as

$$S = \frac{m\omega}{2\sin(\omega(t_f - t_i))}((x_f^2 + x_i^2)\cos(\omega(t_f - t_i)) - 2x_i x_f)$$

After regarding this, we see that

$$K(x_f, t_f; x_i, t_i) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega(t_f - t_i))}\right)^{\frac{1}{2}} \exp\left\{\frac{im\omega}{2\hbar}\left[\frac{\cos\bigl(\omega(t_f - t_i))(x_i^2 + x_f^2) - 2x_i x_f\bigr)}{\sin(\omega(t_f - t_i))}\right]\right\} \tag{14}$$

$$K(x_f, t_f; x_i, t_i) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega(t_f - t_i))}\right)^{\frac{1}{2}} e^{\frac{iS}{\hbar}} \tag{15}$$

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In path integral formalism, the propagator is

$$K(x_f, t_f; x_i, t_i) = \int_{x_1}^{x_N} \mathcal{D}[x(t)] \exp\left[i \int_{t_1}^{t_N} dt \frac{L_{classical}(x, \dot{x})}{\hbar}\right] \tag{16}$$

Variation around the classical trajectory, $x \to x_{cl} + \delta x$ gives,

$$K(x_f, t_f; x_i, t_i) = e^{i\frac{S_{cl}}{\hbar}} \int \mathcal{D}[y(t)] \exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f} dt \frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right) \tag{17}$$

Where $y = x_{cl}\delta x$ and $\dot{y} = \dot{x}_{cl}\delta\dot{x}$. We see the appearance of the classical action in the propagator by means of the path integral formulation which is consistent with our previous results. A quick comparison of the path integral formulation and the propagator tells us

$$\int \mathcal{D}[y(t)] \exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f} dt \frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right) = \left(\frac{m\omega}{2\pi i\hbar\sin(\omega(t_f - t_i))}\right)^{\frac{1}{2}} \tag{18}$$

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2.

a) With $\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2 \phi_a \phi_a$

$$= \frac{1}{2}\dot{\phi}_a \dot{\phi}_a - \frac{1}{2}(\vec{\nabla}\phi_a)\cdot(\vec{\nabla}\phi_a) - \frac{1}{2}m^2 \phi_a \phi_a$$

the equation of motion: $\frac{\delta L}{\delta \phi_a} = \partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi_a)}$

$$-m^2 \phi_a = \ddot{\phi}_a - \nabla^2 \phi_a$$

$$\ddot{\phi}_a = \nabla^2 \phi_a - m^2 \phi_a$$

the Hamiltonian, $p_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = \dot{\phi}_a$

$$H = \int (p_a \dot{\phi}_a - \mathcal{L})\, d^3x$$

$$= \int \left(\frac{1}{2}\dot{\phi}_a \dot{\phi}_a + \frac{1}{2}(\vec{\nabla}\phi_a)\cdot(\vec{\nabla}\phi_a) + \frac{1}{2}m^2 \phi_a \phi_a\right) d^3x$$

the momentum $\vec{P}$, $P^i = T^{i0}$

With $T^{ij} = \frac{\delta L}{\delta \partial_j \phi_a}\partial^i \phi_a - \eta^{ij} L$

$$\Rightarrow T^{i0} = \int \frac{\delta \mathcal{L}}{\delta \dot{\phi}_a}\partial^i \phi_a \, d^3x$$

$$= \int \partial^i \phi_a (\dot{\phi}_a)\, d^3x$$

$$\Rightarrow \vec{P} = \int -\dot{\phi}_a \vec{\nabla}\phi_a \, d^3x$$

b) $\phi_a' = R_{ab}\phi_b$

the infinitesimal $SO(N)$: $\phi_a' = \phi_a + \epsilon_{ab}\phi_b$

$\Rightarrow R_{ab} = 1 + \epsilon_{ab}$

Orthogonal R: $R^T R = \mathbb{1} \Rightarrow \epsilon^T + \epsilon = 0$

$$\epsilon_{ab} = -\epsilon_{ba}$$

$$R = \lim_{n\to\infty}(R^{\frac{1}{n}})^n = \lim_{n\to\infty}(1 + \frac{A}{n} + 0)^n = e^A \quad \text{and} \quad A_{ba} = -A_{ab}$$

$$\delta\phi_c = \epsilon_{cb}\phi_b = \frac{1}{2}\epsilon_{cb}\phi_b - \frac{1}{2}\epsilon_{bc}\phi_b$$

$$= \frac{1}{2}(\epsilon_{ab}\delta_{ac}\phi_b - \epsilon_{ab}\delta_{bc}\phi_a)$$

the current $j^\mu_{ab} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_c)}\Delta\phi_c$

$$= \partial^\mu \phi_c(\delta_{ac}\phi_b - \delta_{bc}\phi_a)$$

$$= (\partial^\mu \phi_a)\phi_b - (\partial^\mu \phi_b)\phi_a$$

the charge $Q_{ab} = \int j^0_{ab}\, d^3x$

$= \int (\dot\phi_a \phi_b - \dot\phi_b \phi_a)\, d^3x$

then

$$\frac{d}{dt} Q_{ab} = \int (\ddot\phi_a \phi_b + \dot\phi_a \dot\phi_b - \dot\phi_b \dot\phi_a - \ddot\phi_b \phi_a)\, d^3x$$

$$= \int (\ddot\phi_a \phi_b - \ddot\phi_b \phi_a)\, d^3x$$

With EOM,

$$\ddot\phi_a \phi_b - \ddot\phi_b \phi_a = (\nabla^2 \phi_a)\phi_b - m^2 \phi_a \phi_b - (\nabla^2 \phi_b)\phi_a + m^2 \phi_a \phi_b$$

$$= (\nabla^2 \phi_a)\phi_b - (\nabla^2 \phi_b)\phi_a$$

$$= \vec\nabla \cdot (\phi_b \vec\nabla \phi_a - \phi_a \vec\nabla \phi_b)$$

$$\Rightarrow \frac{d}{dt} Q_{ab} = \int_V \vec\nabla (\phi_b \vec\nabla \phi_a - \phi_a \vec\nabla \phi_b)\, d^3x$$

$$= \int_{\partial V} (\phi_b \vec\nabla \phi_a - \phi_a \vec\nabla \phi_b)\, d^3x$$

$$= 0$$

c)

$$\phi_a(\vec x) = \int d^3k \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2w_k}} \left(C_{\vec k a} e^{i(\vec k\cdot\vec x - w_k t)} + C^\dagger_{\vec k a} e^{-i(\vec k\cdot\vec x - w_k t)} \right)$$

$$\Pi_a(\vec x) = -i \int d^3k \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{w_k}{2}} \left(C_{\vec k a} e^{i(\vec k\cdot\vec x - w_k t)} - C^\dagger_{\vec k a} e^{-i(\vec k\cdot\vec x - w_k t)} \right)$$

Then

$$[\phi_a(\vec x), \phi_b(\vec y)] = \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{2\sqrt{w_k w_{k'}}} \Big(e^{i(\vec k\cdot\vec x + \vec k'\cdot\vec y - w_k t - w_{k'} t)} [C_{\vec k a}, C_{\vec k' b}] + e^{i(\vec k\cdot\vec x - \vec k'\cdot\vec y - w_k t + w_{k'} t)} [C_{\vec k a}, C^\dagger_{\vec k' b}]$$
$$+ e^{-i(\vec k\cdot\vec x - w_k t + \vec k'\cdot\vec y - w_{k'} t)} [C^\dagger_{\vec k a}, C^\dagger_{\vec k' b}] + e^{i(\vec k'\cdot\vec y - \vec k\cdot\vec x - w_{k'} t + w_k t)} [C^\dagger_{\vec k a}, C_{\vec k' b}]$$

$$= 0$$

Same way

$$[\Pi_a(\vec x), \Pi_b(\vec y)] = 0$$

$$[\Pi_a(\vec x), \phi_b(\vec y)] = -i \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2} \left(e^{i\vec k(\vec x - \vec y)} \delta_{ab} + e^{i\vec k(\vec y - \vec x)} \delta_{ab} \right)$$

$$= -i\, \delta(\vec x - \vec y)\, \delta_{ab}$$

$$[\phi_a(\vec x), \Pi_b(\vec y)] = i\, \delta(\vec x - \vec y)\, \delta_{ab}$$

d) $H = \frac{1}{2} \int d^3x \left(\Pi_a \Pi_a + \vec\nabla\phi_a \cdot \vec\nabla\phi_a + m^2 \phi_a \phi_a \right)$

$$= \frac{1}{2} \int d^3k \frac{1}{(2\pi)^3} w_k \left(C_{k,a} C^\dagger_{k,a} + C^\dagger_{k,a} C_{k,a} \right)$$

$$\vec p = -\int \Pi_a \vec\nabla \phi_a\, d^3x$$

$$= \frac{1}{2} \int d^3k \frac{1}{(2\pi)^3} \vec k \left(C_{k,a} C^\dagger_{k,a} + C^\dagger_{k,a} C_{k,a} \right)$$

$$Q_{ab} = \int (\Pi_a \phi_b - \Pi_b \phi_a)\, d^3x$$
$$= \frac{i}{2}\int d^3k \cdot \frac{1}{(2\pi)^3} \left(c_{k,a} c^\dagger_{k,b} - c^\dagger_{k,a} c_{k,b} \right)$$

Then

$$[Q_{ab}, H] = \frac{i}{4}\iint \frac{W_k}{(2\pi)^6}\, d^3k\, d^3k' \left[c_{k,a} c^\dagger_{k,b} - c^\dagger_{k,a} c_{k,b} \,,\; c_{k',a} c^\dagger_{k',a} + c^\dagger_{k',a} c_{k',a} \right]$$

$$= 0$$

$\Rightarrow$ Q_{ab} is conserved

3.

a)

Fourier transform:

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \Delta_F(p)$$
$$= -\int \frac{d^4p}{(2\pi)^4}\, \frac{i\, e^{-i(p_0 x_0 - \vec{p}\cdot\vec{x})}}{p_0^2 - \vec{p}^2 - m^2 + i\epsilon}$$

denote $W_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}$

$$\Rightarrow \quad \Delta_F(x) = -\int \frac{d^3p}{(2\pi)^4} \left(i e^{i\vec{p}\cdot\vec{x}} \right) \int dp_0 \frac{e^{-ip_0x_0}}{p_0^2 - W_p^2 + i\epsilon}$$
$$= -\int \frac{d^3p}{(2\pi)^4} \left(i e^{i\vec{p}\cdot\vec{x}} \right) \int dp_0 \frac{e^{-ip_0x_0}}{(p_0 - W_p + i\epsilon)(p_0 + W_p - i\epsilon)}$$

if $x_0 > 0$,

$$\Delta_F(x) = -\int \frac{d^3p}{(2\pi)^4} \left(i e^{i\vec{p}\cdot\vec{x}} \right) \frac{2\pi i}{(W_p - i\epsilon) \times 2}\, e^{-iW_p x_0}$$
$$= +\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} \cdot \frac{1}{2W_p}\, e^{-iW_p x_0}$$

if $x_0 < 0$,

$$\Delta_F(x) = +\int \frac{d^3p}{(2\pi)^4} \left(i e^{i\vec{p}\cdot\vec{x}} \right) \frac{2\pi i}{2 \times (-W_p + i\epsilon)}\, e^{+iW_p x_0}$$
$$= +\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} \cdot \frac{1}{2W_p}\, e^{iW_p x_0}$$

$$\Rightarrow \quad \Delta_F(x) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\, \frac{e^{-iW_p|x_0|}}{2W_p}$$

denote $d^3p = |\vec{p}|^2\, dp\, \sin\theta\, d\theta\, d\phi$

$t = \frac{W_p}{m}$

time like : $\Delta_F(x) = \frac{im}{2(2\pi)^2 x_0} H_1^{(1)}(m|x_0|)$ ← Hankel function

space like : $\Delta_F(x) = \frac{m}{(2\pi)^2 |\vec{x}|} K_1^{(1)}(m|\vec{x}|)$ ← modified Bessel function

$$\implies \mathcal{U}\hat{\phi}_a\mathcal{U}^{-1} = \cos(\theta A)\hat{\phi}_a(\mathbf{y}) + \sin(\theta A)\hat{\phi}_b(\mathbf{y}).$$

As an infinitesimal change, we get

$$\mathcal{U}\hat{\phi}_a\mathcal{U}^{-1} \approx \hat{\phi}_a(\mathbf{y}) + \theta A_{ab}\hat{\phi}_b(\mathbf{y}).$$

Alternatively, you could use $\mathcal{U} = \exp(-i\theta A Q_{ab})$ instead and consider $\mathcal{U}^{-1}\hat{\phi}_a\mathcal{U}$, the answer is the same.

3 Problem 3

$$\Delta_F(p) = -\frac{i}{p^2 - m^2 + i\epsilon}$$

3.1 Part A

The Fourier transform is given as

$$\begin{aligned}\Delta_F(x) &= \frac{-i}{(2\pi)^4}\int d^4p\, \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon} \\ &= \frac{-i}{(2\pi)^4}\int d^3\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}}\int dp^0\, \frac{e^{-ip^0x^0}}{(p^0)^2 - |\mathbf{p}|^2 - m^2 + i\epsilon},\end{aligned}$$

We consider two separate cases here for the p^0 integral, if $x^0 > 0$ or if $x^0 < 0$. We use a contour integral to solve this:

1. If $x^0 > 0$, then $x^0 = |x^0|$, and we integrate in the lower half plane on a semicircle. The poles are given by $p^0 = \pm\sqrt{|\mathbf{p}|^2 + m^2 - i\epsilon}$, with the positive root in the lower half plane. Denoting $a = \sqrt{|\mathbf{p}|^2 + m^2 - i\epsilon}$, then the pole at $p^0 = a$ has a residue

$$\text{Res}_a = \lim_{p^0\to a}\frac{e^{-ip^0x^0}}{(p^0)^2 - a^2}(p^0 - a) = \frac{e^{-ia|x^0|}}{2a}.$$

   So the integral simplifies using the residue formula (place a minus sign on the right hand side since we are integrating clockwise):

$$\int dp^0\, \frac{e^{-ip^0x^0}}{(p^0)^2 - |\mathbf{p}|^2 - m^2 + i\epsilon} = \pi i\frac{e^{-ia|x^0|}}{-a}.$$

2. If $x^0 < 0$, then $x^0 = -|x^0|$, and we integrate in the upper half plane instead, using a similar semicircle contour. The poles are the same, but this time we have $p^0 = -a$ as the relevant pole. So the residue at $p = -a$ is (we don't place a minus sign since this is now a counterclockwise contour)

$$\text{Res}_{-a} = \lim_{p^0\to a}\frac{e^{-ip^0x^0}}{(p^0)^2 - a^2}(p^0 + a) = \frac{e^{-ia|x^0|}}{-2a}.$$

   Thus the integral is

$$\int dp^0\, \frac{e^{-ip^0x^0}}{(p^0)^2 - |\mathbf{p}|^2 - m^2 + i\epsilon} = \pi i\frac{e^{-ia|x^0|}}{-a}.$$

In general we conclude that

$$\int dp^0 \, \frac{e^{-ip^0x^0}}{(p^0)^2 - a^2} = \pi \frac{e^{-ia|x^0|}}{ia} = -\pi i \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}},$$

This simplifies the integral as follows. Use spherical coordinates on $\mathbf{p}$:

$$\begin{aligned}
\Delta_F(x) &= \frac{-i}{(2\pi)^4} \int d^3\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} (-\pi i) \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}} = \frac{-\pi}{(2\pi)^4} \int d^3\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}} \\
&= \frac{-\pi}{(2\pi)^3} \int d|\mathbf{p}|\, d\theta\, \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}} e^{i|\mathbf{p}||\mathbf{x}|\cos\theta} |\mathbf{p}|^2 \sin\theta \\
&= \frac{-\pi}{(2\pi)^3} \int d|\mathbf{p}|\, \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}} |\mathbf{p}|^2 \frac{2\sin(|\mathbf{p}||\mathbf{x}|)}{|\mathbf{p}||\mathbf{x}|} = \frac{-1}{(2\pi)^2} \int d|\mathbf{p}|\, \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}} |\mathbf{p}|^2 \frac{\sin(|\mathbf{p}||\mathbf{x}|)}{|\mathbf{p}||\mathbf{x}|}.
\end{aligned}$$

✓

We consider this integral in two cases, if x is spacelike or timelike.

1. If x is timelike, then $s = x^2 > 0$. By Lorentz invariance of the integral, we can apply the transformation $x^\mu \to (\sqrt{s}, 0, 0, 0)$, where $s = x^2 = (x^0)^2 - |\mathbf{x}|^2$. Therefore we let $x^0 \to \sqrt{s}$ and $|\mathbf{x}| \to 0$ in the integral:

$$\Delta_F(x) \to \frac{-1}{(2\pi)^2} \int d|\mathbf{p}|\, \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}} |\mathbf{p}|^2.$$

Now apply the substitution $t = \frac{1}{m}\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}$, so $dt = d|\mathbf{p}|\,|\mathbf{p}|/(mt)$. Note that we can write $|\mathbf{p}| = m\sqrt{t^2-1+i\epsilon/m^2}$. Also replace instances of ϵ/m^2 with ϵ since we are letting $\epsilon \to 0$:

$$\frac{-1}{(2\pi)^2} \int d|\mathbf{p}|\, \frac{e^{-i|x^0|\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}}}{\sqrt{|\mathbf{p}|^2+m^2-i\epsilon}} |\mathbf{p}|^2 = \frac{-m^2}{(2\pi)^2} \int_{1-i\epsilon}^{\infty-i\epsilon} dt\, e^{-itm\sqrt{s}} \sqrt{t^2-1+i\epsilon}$$

$$\to \frac{-m^2}{(2\pi)^2} \int_{1-i\epsilon}^{\infty-i\epsilon} dt\, e^{-itm\sqrt{s}} \sqrt{t^2-1+i\epsilon} = \frac{-m^2}{(2\pi)^2} \frac{-i\pi}{2m\sqrt{s}} H_1^{(1)}(m\sqrt{s}) \boxed{= \frac{im}{8\pi\sqrt{s}} H_1^{(1)}(m\sqrt{s}).}$$

✓

Here we use the following integral representation:

$$H_\nu^{(1)}(x) = -i \frac{2(-\frac{x}{2})^\nu}{\pi^{\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)} \int_1^{\infty-i\epsilon} e^{-ixt}(t^2-1)^{\nu-\frac{1}{2}}\, dt; \quad \mathrm{Re}\,\nu > 1/2, \quad x > 0.$$

This may be obtained from taking the following integral representation for K_ν and shifting the contour by $-i\epsilon$, taking $\epsilon \to 0$ at the end:

$$K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_1^\infty e^{-zt}\left(t^2-1\right)^{\nu-\frac{1}{2}}\, dt; \quad \mathrm{Re}(\nu+\tfrac{1}{2}) > 0, \quad |\arg z| < \frac{\pi}{2}.$$

Now use the identity $H_\nu^{(1)}(x) = \frac{2}{\pi} i^{-\nu-1} K_\nu(-ix) = \frac{2}{\pi} i^{\nu-1} K_\nu(ix)$ for real x:

$$H_\nu^{(1)}(z) = \frac{2i^{\nu-1}}{\pi} \frac{\left(\frac{iz}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_1^{\infty-i\epsilon} e^{-izt}\left(t^2-1\right)^{\nu-\frac{1}{2}}\, dt.$$

$$\Longrightarrow \int_{1-i\epsilon}^{\infty-i\epsilon} e^{-ixt}\sqrt{t^2-1+i\epsilon}\, dt \to -\frac{i\pi}{2x} H_1^{(1)}(x).$$

This integral would not converge as given, but if we suppose that we are still perturbing by $i\epsilon$, then this integral holds as $\sqrt{s}$ is now pushed slightly off of the real line. So the integral works here.

2. If x is spacelike, then $s < 0$. Therefore we have the same integral result in two ways. We can use the Lorentz transformation $x^\mu \to (0, \mathbf{x})$ or simple analytic continuation to determine the answer.

$$\Delta_F(x) = \frac{-1}{(2\pi)^2\sqrt{-s}} \int d|\mathbf{p}| \, \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 + m^2 - i\epsilon}} \sin\big(|\mathbf{p}|\sqrt{-s}\big).$$

$$\to \frac{im}{8\pi\sqrt{s}} H_1^{(1)}(m\sqrt{s}) = \frac{-m}{8\pi\sqrt{-s}} H_1^{(1)}(-im\sqrt{-s}) \boxed{= \frac{-m}{4\pi^2\sqrt{-s}} K_1(m\sqrt{-s})}$$

Note that there is also a Dirac delta centered around $s = 0$, which we can see for lightlike separations. Let $m \to 0$ and $\epsilon \to 0$, and we can see the integral:

$$\Delta_F(x) \to \frac{-1}{(2\pi)^2|\mathbf{x}|} \int d|\mathbf{p}| \, e^{-i|x^0||\mathbf{p}|} \sin(|\mathbf{p}||\mathbf{x}|) = \frac{i}{8\pi^2|\mathbf{x}|} \int d|\mathbf{p}| \, e^{-i|x^0||\mathbf{p}|} \left[e^{i|\mathbf{p}||\mathbf{x}|} - e^{-i|\mathbf{p}||\mathbf{x}|}\right]$$

$$= \frac{i}{8\pi|\mathbf{x}|} \left[\delta(|x^0| - |\mathbf{x}|) - \delta(|x^0| + |\mathbf{x}|)\right] = \frac{i}{4\pi}\delta(s).$$

Therefore in total we have

$$\Delta_F(x) = \begin{cases} \frac{i}{4\pi}\delta(s) + \frac{im}{8\pi\sqrt{s}} H_1^{(1)}(m\sqrt{s}) & s \geq 0 \\ \frac{-m}{4\pi^2\sqrt{-s}} K_1(m\sqrt{-s}) & s < 0. \end{cases}$$

Note: This integral appears to not be solved explicitly too often. I tried comparing my results to other derivations, but my answer differs by a couple minus signs every time. The other factors all check out, though.

3.2 Part B

We can talk about the large lightlike and spacelike separation, and near the light cone.

1. For large space-like separation ($s < 0$), we have the formula $\Delta_F(x) = \frac{-m}{2\pi^2\sqrt{-s}} K_1(m\sqrt{-s})$. The modified Bessel function K_1 asymptotically behaves as $K_1(z) \sim \sqrt{\frac{2}{\pi z}} e^{-z}$. Plugging this into the formula, we have

$$\Delta_F(x) \sim \frac{-m}{2\pi^2\sqrt{-s}} \sqrt{\frac{2}{\pi m\sqrt{-s}}} e^{\overset{-m}{\cancel{im}}\sqrt{-s}}.$$

Recall that in the formula we have $\sqrt{-s}$ is $|\mathbf{x}|$ more or less, and if we keep increasing $|x|$ then this term dominates over any $(x^0)^2$ contribution:

$$\implies \Delta_F(x) \sim \frac{-m}{2\pi^2} \sqrt{\frac{2}{\pi|\mathbf{x}|^3}} e^{-m|\mathbf{x}|}.$$

So $\Delta_F(x)$ essentially behaves proportional to $|\mathbf{x}|^{-3/2}e^{-m|\mathbf{x}|}$.

2. For large time-like separation ($s > 0$), we use the similar asymptotic expansion

$$H_1^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - 3\pi/4)}$$

$$\implies \Delta_F(x) \sim \frac{im}{8\pi^2\sqrt{s}}\sqrt{\frac{2}{\pi z}}e^{i(z-3\pi/4)} = \sqrt{\frac{2}{\pi z}}e^{i(z-3\pi/4)}$$

$$\implies \Delta_F(x) \sim \frac{e^{-i\pi/4}m}{8\pi^2\sqrt{s}}\sqrt{\frac{2}{\pi m\sqrt{s}}}e^{im\sqrt{s}}.$$

Recall that $\sqrt{s} = |x^0|$ for this, and we can essentially ignore any spatial contributions. We get that $\Delta_F(x)$ is proportional to $m^{1/2}|x^0|^{-3/2}e^{im|x^0|}$.