662, Homework II, (3 problems)

Problem 1

The Lagrangian (density) for a Dirac fermion

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{0.1}$$

is invariant under the transformation

$$\psi(x) \to e^{iq\alpha}\psi(x) \tag{0.2}$$

a) Compute the corresponding conserved current (electromagnetic current).

b) Write the current in terms of oscillators and check that electron and positron have opposite charges.

c) Consider the transformation $\psi(x) \to e^{i\alpha\gamma^5}\psi(x)$. Check that this is a symmetry of the Lagrangian when $m = 0$ but not if $m \neq 0$. Write the corresponding conserved current and compute its divergence in the $m \neq 0$ case using the Dirac equation.

Problem 2

In the chiral representation[1] a Dirac spinor can be written as

$$\psi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \tag{0.3}$$

where $\xi_{L,R}$ are two component spinors that do not mix under Lorentz transformations.

a) Show that the Dirac equation mixes both components.

b) Show that if $m = 0$ one can set *e.g.* $\xi_R = 0$ and still satisfy the Dirac equation (Weyl spinor).

[1] This is the representation used in class where $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$.

c) Show that ξ_L^* (conjugate) transforms (after an appropriate change of basis) as ξ_R and therefore it can be identified with ξ_R (Majorana fermion).

d) Using the result of **c)** write a massive Dirac equation for just ξ_L (Majorana fermion). Show that this equation is not invariant under the charge symmetry $\xi_L \to e^{iq\alpha}\xi_L$ of problem 1) and therefore this fermion has no charge ($q = 0$).

Problem 3

Consider solutions to the Dirac equation of the form $\psi(x) = u(p)e^{-ipx}$ with $p^2 = m^2$ where $u(p)$ solves the algebraic equation

$$(\gamma^\mu p_\mu - m)u(p) = 0 \tag{0.4}$$

If $\hat{p}$ is a unit vector along $\vec{p}$, define the helicity operator as

$$h = \hat{p}.\vec{S} = \frac{1}{2}\begin{pmatrix} \hat{p}.\vec{\sigma} & 0 \\ 0 & \hat{p}.\vec{\sigma} \end{pmatrix} \tag{0.5}$$

where $\hat{p}.\vec{\sigma} = \hat{p}_i \sigma^i$.

a) Find the two linearly independent solutions $u^{1,2}$ that are also eigenvectors of the helicity operator and normalize them such that

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs}, \qquad \bar{u} = u^\dagger\gamma^0 \tag{0.6}$$

b) Repeat the same for solutions of the form $\psi(x) = v(p)e^{ipx}$ but normalize them now as

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs}, \tag{0.7}$$

c) Check the identities

$$\bar{u}^r(p)v^s(p) = 0 \tag{0.8}$$

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \gamma^\mu p_\mu + m \tag{0.9}$$

$$\sum_{s=1,2} v^s(p)\bar{v}^s(p) = \gamma^\mu p_\mu - m \tag{0.10}$$