Large N-limit

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 + \frac{1}{2}r\phi_a^2 + \frac{u}{4!}(\phi_a^2)^2$$

$$\rho = \frac{1}{N}\langle\phi_a^2\rangle$$

$$1 = N\int d\rho\, \delta(\phi_a^2 - N\rho) = \frac{N}{4\pi i}\int d\rho\, d\xi\; e^{-\frac{i}{2}\xi(\phi_a^2 - N\rho)}$$

$$Z = \int \mathcal{D}\phi\,\mathcal{D}\rho\,\mathcal{D}\xi\; e^{-S[\phi] - \frac{i}{2}\int \xi(\phi_a^2 - N\rho)d^dx}$$

$$\phi_1 = \sigma\sqrt{N} \qquad \phi_a = \pi_a \qquad a = 2 \dots N$$

Direction 1 along ϕ.

$$\phi_b^2 \sim N \quad \phi_1^2 \sim N$$

$$Z = \int \mathcal{D}\sigma\,\mathcal{D}\pi_a\,\mathcal{D}\xi\,\mathcal{D}\rho\; e^{-S[\sigma,\pi_a] - \int d^dx\,\frac{i\xi}{2}(\phi_a^2 - N\rho)}$$

$$\mathcal{L} = \frac{N}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\pi_a)^2 + \frac{N}{2}r\rho + \frac{uN^2}{4!}\rho^2$$

$$Z = \int \mathcal{D}\sigma\,\mathcal{D}\pi_a\,\mathcal{D}\xi\,\mathcal{D}\rho\; e^{-\int d^dx\left\{\frac{N}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\pi_a)^2 + \frac{N}{2}r\rho + \frac{uN^2}{4!}\rho^2 + \frac{i\xi}{2}(N\sigma^2 + \pi_a^2) - \frac{i\xi}{2}N\rho\right.}$$

Define $\bar{u} = uN \qquad N \to \infty \qquad \bar{u}$ fixed $(u \sim 1/N)$

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\xi\, \mathcal{D}\rho\; e^{-N\int d^dx \left\{ \frac{1}{2}(\nabla\sigma)^2 + \frac{1}{2} r\rho + \frac{\bar{u}}{4!}\rho^2 + \frac{i\xi}{2}\sigma^2 - \frac{i\xi}{2}\rho \right\}}$$

$$\times \int \mathcal{D}\pi_a\; e^{-\int d^dx \left\{ \frac{1}{2}(\nabla\pi_a)^2 + \frac{i\xi}{2}\pi_a^2 \right\}}$$

$$\int dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}} \qquad \int dx_j\, e^{-x_i A_{ij} x_j} =$$

$$= \int dx_j\, e^{-x_i S_{ii'} A_{i'j'} \underbrace{S_{j'j} x_j}_{\eta_{j'}}} = \int d\eta_j\, e^{-\eta_i A_i \eta_j} = \frac{\pi^{n/2}}{(\lambda_1 \cdots \lambda_n)^{1/2}}$$

diagonalize

$$= \pi^{n/2} \det{}^{-1/2} A$$

$$\int \mathcal{D}\pi_a\; e^{-\int d^dx \left\{ \frac{1}{2}(\nabla\pi_a)^2 + \frac{i\xi}{2}\pi_a^2 \right\}} = \left(\det{}^{-1/2} \left(-\nabla^2 + \frac{i\xi}{2}\right) \right)^{N-1}$$

$$= e^{-\frac{N-1}{2} \mathrm{Tr} \ln \left(-\nabla^2 + \frac{i\xi}{2}\right)}$$

$$Z = \int \mathcal{D}\sigma\, \mathcal{D}\xi\, \mathcal{D}\rho\; e^{-N\int d^dx \left\{ \frac{1}{2}(\nabla\sigma)^2 + \frac{1}{2} r\rho + \frac{\bar{u}}{4!}\rho^2 + \frac{i\xi}{2}\sigma^2 - \frac{i\xi}{2}\rho + \frac{1}{2}\mathrm{Tr}\ln\left(-\nabla^2 + \frac{i\xi}{2}\right) \right\}}$$

$N \to \infty$ is a classical limit of this action.

③

$$S = \frac{1}{g}\int_x \left[\frac{(\nabla\sigma)^2}{2} + \frac{i\xi}{2}\sigma^2 + \frac{1}{2}r\rho + \frac{\bar{u}}{4!}\rho^2 - \frac{i\xi}{2}\rho\right] + \frac{1}{2}\mathrm{Tr}\ln\left(-\nabla^2 + \frac{i\xi}{g}\right)$$

$$-\nabla^2\sigma + i\xi\,\sigma = 0$$

$$\frac{r}{2} + \frac{\bar{u}}{12}\rho - \frac{i\xi}{2} = 0 \quad \leftarrow \text{ ξ purely imaginary at saddle point}$$

$$\int\left(\frac{i\sigma^2}{2} - \frac{i}{2}\rho\right) + \frac{1}{2}\frac{\delta}{\delta\xi(x)}\mathrm{Tr}\ln\left(-\nabla^2 + \frac{i\xi}{g}\right) = 0$$

<u>ground state</u> constant fields.

$i\xi = \lambda$ (since ξ purely imag.)

$$\lambda\sigma = 0 \begin{cases} \lambda = 0 \quad \sigma \neq 0 & \text{ferromagnetic phase} \\ \sigma = 0 \quad \lambda \neq 0 & \leftarrow \sigma \text{ gets a mass.} \end{cases}$$

$$\mathrm{Tr}\ln\left(-\nabla^2 + \frac{\lambda}{g}\right) = V\int\frac{d^dk}{(2\pi)^d}\ln(k^2 + \lambda)$$

$\int d^dx$ eqn for λ.

$$\int_x \frac{\delta}{\delta\xi(x)} \to \underset{\text{constant}}{\frac{\delta}{\delta\xi}}$$

$$\frac{\sigma^2}{2} - \frac{\rho}{2} + \frac{1}{2}\frac{\delta}{\delta\lambda}\mathrm{Tr}\ln(-\nabla^2 + \lambda) = 0$$

$$\sigma^2 - \rho + \frac{\partial}{\partial \lambda} \int \frac{d^d k}{(2\pi)^d} \ln(k^2 + \lambda) = 0.$$

$$\sigma^2 - \rho + \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + \lambda} = 0$$

Ferromagnetic phase

$\lambda = 0, \quad \sigma \neq 0$

$$\frac{r}{2} + \frac{\bar{u}}{12} \rho = 0 \qquad \rho = -\frac{6r}{\bar{u}}$$

$$\sigma^2 = -\frac{6r}{\bar{u}} - \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = -\frac{6}{\bar{u}} (r - r_c)$$

$$r_c = -\frac{\bar{u}}{6} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \qquad < 0$$ position of transition moves.

$r < r_c \Rightarrow \sigma^2 > 0$ ✓ this phase is possible and actually has lower energy.

$r > r_c$ not possible $\Rightarrow \lambda \neq 0 \quad \sigma = 0$.

$$r_c = -\frac{\bar{u}}{6} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = -\frac{\bar{u}}{6} \frac{\Omega_d}{(2\pi)^d} \int_0^\Lambda k^{d-3} dk = -\frac{\bar{u}}{6} \frac{\Omega_d}{(2\pi)^d} \frac{k^{d-2}}{d-2} \Big|_0^\Lambda$$

$$= -\frac{\bar{u}}{6} \frac{\Omega_d}{(2\pi)^d} \frac{\Lambda^{d-2}}{d-2}$$

$d \leq 2$ logarithmically divergent $\Rightarrow$ no phase transition $r_c \to -\infty$.

·) Paramagnetic phase. $\lambda \neq 0 \quad \sigma = 0$

$$-\frac{6}{\bar{u}}(\lambda - r) + \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+\lambda} = 0$$

λ real.

Define $\lambda = m^2$

$$-\frac{6}{\bar{u}}(m^2 - r) + \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} = 0$$

$$S = \int \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \underbrace{\frac{1}{2} m^2 \sigma^2}_{\text{massive.}}$$

$$\underbrace{\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2}}_{\frac{\Omega_d}{(2\pi)^d} \int_0^{\Lambda} \frac{k^{d-1} dk}{k^2+m^2}} = \frac{6}{\bar{u}}(m^2 - r)$$

determines m^2 "gap equation".

$\boxed{d=3}$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{(k^2+m^2)} \simeq \frac{\Lambda}{2\pi^2} - \frac{m}{4\pi}$$

$$-\frac{6}{\bar{\mu}}(m^2 - r) + \frac{\Lambda}{2\pi^2} - \frac{m}{4\pi} = 0$$

$$\frac{6r}{\bar{u}} + \frac{\Lambda}{2\pi^2} = \frac{m}{4\pi}$$

$$m = 4\pi \times \frac{6}{\bar{\mu}}(r - r_c)$$

$\boxed{\nu = 1}$

$$r_c = -\frac{\bar{u}}{6} \frac{\Lambda}{2\pi^2}$$

$\eta = 0$

free theory for σ.

Other dimensions.

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} = \frac{\Omega_d}{(2\pi)^d} \int_0^\Lambda \frac{k^{d-1}\,dk}{k^2+m^2}$$

Take $2<d<4$ $\qquad 1<d-1<3$

$$\int_0^\Lambda \frac{k^{d-3}(k^2+m^2-m^2)\,dk}{k^2+m^2} = \int_0^\Lambda k^{d-3}\,dk - m^2 \int_0^\Lambda \frac{k^{d-3}\,dk}{k^2+m^2}$$

$$= \frac{k^{d-2}}{d-2}\Big|_0^\Lambda - m^2 \int_0^\infty \frac{k^{d-3}\,dk}{k^2+m^2}$$

$$= \frac{\Lambda^{d-2}}{d-2} - \cancel{m^2}\, m^{d-3+1\cancel{+2}} \int_0^\infty \frac{x^{d-3}\,dx}{1+x^2}$$

$$= \frac{\Lambda^{d-2}}{d-2} - m^{d-2} I_{d-3}$$

$$\frac{6}{\bar\mu}(\lambda - r) = \frac{\Lambda^{d-2}}{d-2} - \lambda^{\frac{d-2}{2}} I_{d-3}$$

$$\frac{6\lambda}{\bar\mu} = \frac{6r}{\bar\mu} + \frac{\Lambda^{d-2}}{d-2} - \lambda^{\frac{d-2}{2}} I_{d-3}$$

$2<d<4 \qquad 0<d-2<2$

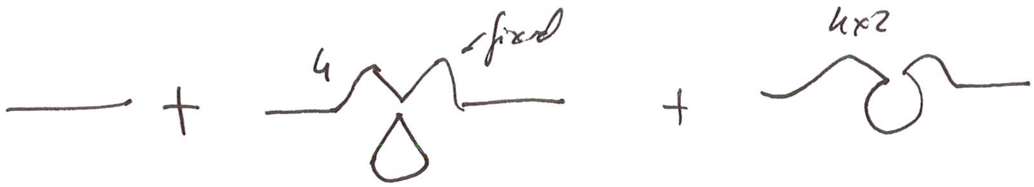
$0<\frac{d-2}{2}<1$

$$\Rightarrow\ r - r_c = \frac{\bar\mu}{6} I_{d-3} \underbrace{\lambda^{\frac{d-2}{2}}}_{\xi^{-(d-2)}}$$

$$\xi \sim (r-r_c)^{-\frac{1}{d-2}} \qquad \boxed{\nu = \frac{1}{d-2}}$$

large-N diagrammatic approach.

Propagator.



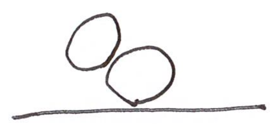
$$-\frac{u}{6}N = -\frac{\bar{u}}{6} \qquad\qquad -\frac{u}{6}\times 2$$

$$\underbrace{\qquad\qquad\qquad\qquad}$$

$$-\frac{u}{6}(N+2) \;\to\; -\frac{\bar{u}}{6} \quad \text{large } N.$$

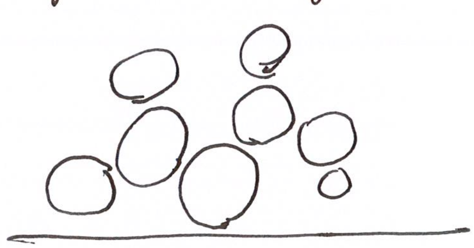
Higher loops:

$u^2 N = \frac{\bar{u}^2}{N}$ goes away $N \to \infty$.



$u^2 N^2$

in general we get things like



for

Also

$\Delta(p) =$  $= \underline{\Delta_0(p)}\left(1 + \Sigma\Delta_0 + \Sigma\Delta_0\Sigma\Delta_0 + \dots\right)$

$\Sigma =$ 

$$= \frac{\Delta_0}{1 - \Delta_0\Sigma} = \frac{1}{\Delta_0^{-1} - \Sigma(p^2)}$$

$$\Delta^{-1}(p) = p^2 + m^2 - \Sigma(p^2)$$

But $\Sigma(p^2) =$  so it is indep. of p^2

It is a <u>constant</u>

$\Delta^{-1}(p) = p^2 + m^2 - \Sigma$: Free propagator ($\eta = 0$) with modified mass.

To sum all diagrams we get a self-consistent condition.

 =  ← full propagator.

$$\Sigma = -\frac{\bar{u}}{6}\int \frac{d^dk}{(2\pi)^d}\,\Delta(k) = -\frac{\bar{u}}{6}\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2 + \underbrace{m^2 - \Sigma}_{\lambda \text{ (before)}}}$$

$m^2 - \Sigma = \lambda \Rightarrow \Sigma = m^2 - \lambda = r - \lambda$ (notation before)

$$\Rightarrow \boxed{r - \lambda = -\frac{\bar{u}}{6}\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2 + \lambda}}$$

same as before (gap-equation)

$\boxed{d=3}$

$$I_1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+\lambda} = \frac{4\pi}{8\pi^3} \int_0^{\infty} \frac{k^2 dk}{k^2+\lambda} = \frac{1}{2\pi^2} \left(\Lambda - \lambda \int_0^{\Lambda} \frac{dk}{k^2+\lambda} \right)$$

$$\int_0^{\Lambda/\sqrt{\lambda}} \frac{\sqrt{\lambda}\, dx}{\lambda(1+x^2)} = \frac{1}{\sqrt{\lambda}} \operatorname{atan}\left(\frac{\Lambda}{\sqrt{\lambda}}\right)$$

$k = \sqrt{\lambda}\, x$

$$I_1 = \frac{\Lambda}{2\pi^2} - \frac{\sqrt{\lambda}}{2\pi^2} \operatorname{atan}\left(\frac{\Lambda}{\sqrt{\lambda}}\right)$$

$\Lambda \to \infty \qquad I_1 = \dfrac{\Lambda}{2\pi^2} - \dfrac{\sqrt{\lambda}}{4\pi}$

$$I_2 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2+\lambda)^2} = -\frac{\partial}{\partial\lambda} I_1 = \frac{1}{2\pi^2} \frac{\partial}{\partial\lambda} \left(\sqrt{\lambda} \operatorname{atan}\left(\frac{\Lambda}{\sqrt{\lambda}}\right)\right)$$

$$= \frac{1}{2\pi^2} \left(\frac{1}{2\sqrt{\lambda}} \operatorname{atan}\left(\frac{\Lambda}{\sqrt{\lambda}}\right) + \sqrt{\lambda} \frac{1}{1+\Lambda^2/\lambda} \left(-\frac{1}{2}\lambda^{-3/2}\right) \right)$$

$$= \frac{1}{4\pi^2} \frac{1}{\sqrt{\lambda}} \operatorname{atan}\left(\frac{\Lambda}{\sqrt{\lambda}}\right) - \frac{1}{4\pi^2} \frac{1}{\lambda+\Lambda^2} \xrightarrow[\Lambda\to\infty]{} \frac{1}{8\pi} \frac{1}{\sqrt{\lambda}}$$

$$r-\lambda = -\frac{\bar{u}}{6}\left(\frac{\Lambda}{2\pi^2}-\frac{\sqrt{\lambda}}{4\pi}\right)$$

$$\underbrace{r+\frac{\bar{u}\Lambda}{12\pi^2}}_{-r_c} = \lambda + \frac{\bar{u}}{24\pi}\sqrt{\lambda}$$

$$r-r_c = \frac{\bar{u}}{24\pi}\sqrt{\lambda} + \lambda$$

$\lambda \to 0$ $\quad$ $r \to r_c$

$$\boxed{r-r_c = \frac{\bar{u}}{24\pi}\sqrt{\lambda}}$$

$\xi = 1/\sqrt{\lambda}$ $\qquad$ $\xi^{-1} = \frac{24\pi}{\bar{\mu}}(r-r_c)$ $\qquad$ $(r>r_c)$

$\xi = \frac{\bar{u}}{24\pi}\frac{1}{r-r_c}$ $\qquad$ t^{-1} $\qquad$ $\boxed{\nu=1}$

$\boxed{\eta=0}$ (free propagator)