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Lecture notes for QFT I (662)

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Abstract: These are the notes for the lectures. They contain what is explained in class and can be used to refresh your memory or to stay up to date if you miss a class. They do *not* replace the book since they have much less information. Also take into account that the actual lectures might run a little behind schedule.

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1. Lecture 1

1.1 Quantum field theory

Newtonian physics is a theory of particles and forces between them. Further insight was gained later by the work of several people among which Faraday, Maxwell and Einstein are particularly important. It was found that forces are due to fields that can exist in space (and time) independently of the particles. They are described by giving a quantity (*e.g.* the electric field $\vec{E}(\vec{x},t)$) at each point in space and have their own dynamics as evidenced by the propagation of electromagnetic waves. With the advent of quantum mechanics it was realized again by Einstein, following Planck's ideas that fields should be associated with particles, the "quantum" of electromagnetic radiation was named the photon. Later, as emphasized by Yukawa, it became clear that to any interaction one should associate a mediator particle, for example Yukawa predicted the existence of the π meson as the mediator of the nuclear force. Thus we are lead to the important notion that:

$$\boxed{\text{Particles and fields are equivalent notions}} \tag{1.1}$$

The elementary electromagnetic process is the emission or absorption of a photon by a charged particle. This is the particle approach in which the state of the electromagnetic field is described as a multi photon state (for example saying how many photons there are with each momentum). The (equivalent) field approach is to replace the field (*e.g.* $\vec{E}(\vec{x},t)$) at each point in space by operators defined at each point of space and obeying commutation relations analogous to the Heisenberg commutation relation $[x,p]=i\hbar$. Such commutation relations are realized by operators acting on multiparticle states thus providing the relation between the particle and field approaches.

Quantum field theory is therefore a quantum theory of fields and particles. It was initially developed as the appropriate way to find a quantum theory that is relativistically invariant. However it can be applied to many other areas of physics such as condensed matter, many body physics, finite temperature systems, phase transitions, etc.

In its simplest form, and this perspective is the one that we are going to develop mainly in this course, it is a theory of (a few) weakly interacting particles.

Oftentimes the Hamiltonian of a system does not describe a system of weakly interacting particles. In that case the problem is to see if it is possible to rewrite the system in terms of weakly interacting excitations. In that case the same methods apply. Thus, it is important to notice that the "particles" do not have to be elementary, they can be composite objects that, at the energies we analyze behave as weakly interacting. In condensed matter such weakly interacting objects are sometimes called quasi-particles but generically in QFT they are all called particles.

1.2 Typical QFT setup

Since we are interested in weakly interacting particles the natural computational tool is perturbation theory. As we know from quantum mechanics this requires the following steps:

a) **Identify unperturbed Hamiltonian H_0.** In this case the unperturbed Hamiltonian is found by first finding the single particle eigenstates and then considering H_0 to be the Hamiltonian of such particles assuming that they do not interact. That is, the eigenstates of H_0 are simply found by filling the single particle eigenstates according to the statistics of the particles (*i.e.* bosons or fermions).

b) **Identify the symmetries of the problem** The main guide to find the single particle eigenstates and afterwards the interactions is to identify the symmetries of the problem. In the case of a relativistic theory the basic symmetry is Poincare symmetry (Lorentz transformations plus translations) and therefore the single particle eigenstates are usually labeled by their spacial momentum $\vec{p} = \hbar\vec{k}$, energy $\epsilon_{\vec{k}}$ and projection of angular momentum along the momentum $\vec{J}\cdot\hat{p}$. Therefore a single particle eigenstate is given as

$$|\psi\rangle = |\epsilon_k, \vec{k}, \sigma, a\rangle \tag{1.2}$$

where $\sigma = -s \ldots s$ with s the spin of the particle and a indicates any other quantum numbers (such as the charge). The relativistic energy is given by $\epsilon_k = \sqrt{\vec{k}^2 + m^2}$ and therefore is usually omitted.

c) **Identify interactions.** In principle the interactions are given by all possible interactions that respect all the symmetries of the problem, each with an arbitrary strength (or coupling constant as it is called). Again the first guide is relativistic invariance that, as we shall see, is very powerful. Besides that it is necessary to compare with the experiment to see what interactions are actually present and their strengths. If some terms do not appear, or the strengths of different interactions are related, it generically means that there are more symmetries that we have failed to identify. To compare with experiment we have to do some calculation. This is the next step.

d) **Use perturbation theory to compute some quantity that can be compared with an experiment.** In relativistic quantum field it is quite difficult to compute bound states. Even in classical mechanics, the propagation time introduces difficulties since the force felt by a charge depends on the position of the other charges at previous times (retarded time) and not at their present positions (in fact as we know the notion of simultaneity also depends on the observer). This problem is eliminated by considering states with photons but that leads to divergences in perturbation theory. The usual method is to find an effective potential or find poles in the scattering amplitude.

   For all those reasons the standard tool is time-dependent perturbation theory and the quantities that are usually computed are transition rates, namely mean life of particles and scattering cross section (using the Fermi golden rule).

e) **Renormalization.** A problem that arises in quantum field theory is that standard perturbation theory give divergent contributions at each order. They are dealt with by systematically modifying the Hamiltonian at each order. The terms that are added are called counter terms and cancel the divergent contributions. Such procedure obviously requires some justification that we are going to discuss in more detail later on. For the moment let us just mention that the divergences appear from sums over intermediate states of arbitrarily large energy. In actual physical reality we cannot presume to know the physics at arbitrarily large energies. In particular, at large energies gravity becomes strong and there is not known consistent quantum field theory of gravity. String theory can consistently quantize gravity but it is not known if it describes reality. From this point of view the theory looks hopeless since such large energy physics is not known. On the other hand, on physical grounds, we do not expect such ignorance to prevent us to study low energy processes such as electron interactions or condensed matter physics. The answer is the renormalization group idea that provides a physi-

cal foundation to the renormalization procedure. The result is that low energy physics is determined by a finite number of parameters (masses and coupling constants) that can be easily measured but are not simply related to the parameters in the Hamiltonian. Such theories are called *normalizable*. The counter term procedure can be seen as a way to find the correct relation between the parameters in the Hamiltonian and the physically measurable parameters. This can be further justified by showing it gives the correct result in field theories that, as a theoretical model, are valid and under control at arbitrarily large energies. Finally, it is also possible to consider the case where an infinite number of parameters are necessary to define the theory (so called *non-renormalizable* theories). In those cases it is necessary to expand physical quantities in powers of the energy. At each order in such expansion only a finite number of parameters appear making the theory predictable.

To summarize, although the justification can be somewhat involved, there is a straight-forward procedure of introducing counter terms that gives the correct physical answer.

2. Occupation number formalism / Second quantization / Fock space

Since we are going to deal with systems of bosons and fermions it is important to describe the simplest formalism to deal with particle statistics. This is the occupation number formalism. In this formalism a basis is defined by describing how many particles occupy each single particle state. In the case of bosons we can put an arbitrary number of particles in each state, in the case of fermions at most one. For simplicity we are going to label the single particle states by their energy $|\epsilon_i\rangle$. A generic state in the basis is given by

$$|\psi\rangle = |n_1, n_2, \ldots, n_i, \ldots\rangle \tag{2.1}$$

where n_i is the number of particles in state $|\epsilon_i\rangle$. For bosons $n_i = 0 \ldots \infty$ and for fermions $n_i = 0, 1$. A generic state is a linear combination

$$|\Psi\rangle = \sum_{n_1 \ldots} A(n_1, n_2, \ldots)\ |n_1, n_2, \ldots\rangle \tag{2.2}$$

Since the particles are identical this gives all the available information on the state. It is now natural to define creation an annihilation operators. For bosons they act independently on the occupation number of each single particle state as:

$$a_i^\dagger |n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle \tag{2.3}$$

$$a_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle \tag{2.4}$$

With such normalization in the definition is is easy to show that

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{2.5}$$

$$a_i^\dagger a_i|\ldots, n_i, \ldots\rangle = n_i|\ldots, n_i, \ldots\rangle \tag{2.6}$$

and therefore the Hamiltonian is

$$H_0 = \sum_i \epsilon_i a_i^\dagger a_i \tag{2.7}$$

implying that the eigenstates of H_0 are

$$H_0|n_1, n_2, \ldots, n_i, \ldots\rangle = \left(\sum_j n_j \epsilon_j\right)|n_1, n_2, \ldots, n_i, \ldots\rangle \tag{2.8}$$

It is also conveninet to define those states as

$$|n_1, n_2, \ldots, n_i, \ldots\rangle = \prod_i \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}}|0\rangle \tag{2.9}$$

where $|0\rangle$ is the vaccum or empty state with no bosons.

The case of fermions is slightly more subtle. Suppose we define

$$c_i^\dagger|0_i\rangle = |1_i\rangle \tag{2.10}$$

$$c_i^\dagger|1_i\rangle = 0 \tag{2.11}$$

$$c_i|0_i\rangle = 0 \tag{2.12}$$

$$c_i|1_i\rangle = |0_i\rangle \tag{2.13}$$

Then the "commutation" relation becomes an anticommutation relation

$$\{c_i, c_i^\dagger\} = c_i c_i^\dagger + c_i^\dagger c_i = 1 \tag{2.14}$$

as can be checked by acting on the two states $|0_i\rangle$ and $|1_i\rangle$. However, when acting on different occupation numbers we find that such operators commute $[c_i, c_j^\dagger] = 0$ (for $i \neq j$). It is more convenient to define operators such that

$$\{c_i, c_j^\dagger\} = \delta_{ij} \tag{2.15}$$

In order to do that we can introduce some sings in the definition of the non-vanishing case

$$c_i^\dagger|n_1, n_2, \ldots, 0_i \ldots\rangle = (-)^{\sum_{j<i} n_j}|n_1, n_2, \ldots, 1_i \ldots\rangle \tag{2.16}$$

$$c_i|n_1, n_2, \ldots, 1_i \ldots\rangle = (-)^{\sum_{j<i} n_j}|n_1, n_2, \ldots, 0_i \ldots\rangle \tag{2.17}$$

It is easy to check that now the operators anti-commute. A simple way to take this into account is to define the states as

$$|n_1, n_2, \ldots, n_i, \ldots\rangle = \prod_i (c_i^\dagger)^{n_i}|0\rangle \tag{2.18}$$

Remember that $n_i = 0, 1$. If we now act with $c_j^\dagger$ for example, $c_j^\dagger$ has to "jump" over all previous $c_{i<j}^\dagger$ before it gets to its correct spot. Each time it "jumps" we get a minus from the anti-commutation relation.

The occupation number is once again given by

$$n_i = c_i^\dagger c_i \tag{2.19}$$

and the Hamiltonian

$$H_0 = \sum_i \epsilon_i c_i^\dagger c_i \tag{2.20}$$

Suppose now that we have a more complicated Hamiltonian such as

$$H = H_0 + H_1 + H_2 = \sum_a \frac{p_a^2}{2m} + \sum_a U(r_a) + \frac{1}{2}\sum_{ab} V(\vec{r}_a - \vec{r}_b) \tag{2.21}$$

where $U(r_a)$ is an external potential and $V(\vec{r}_a - \vec{r}_b)$ is a two particle interaction, for example Coulomb:

$$V(\vec{r}_a - \vec{r}_b) = \frac{e^2}{|\vec{r}_a - \vec{r}_b|} \tag{2.22}$$

How do we write such Hamiltonian in the occupation number formalism? Naturally we have to assure that the Hamiltonian has the same matrix elements between states in both formalisms.

Let us assume that the single particle states that we consider are eigenstates of momentum with wave-function

$$\psi_{\vec{k}}(\vec{x}) = \langle \vec{x}|\vec{k}\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}\vec{x}} \tag{2.23}$$

Since we are dealing with continuous $\vec{k}$ let us normalize the creation and annihilation operators such that

$$\{c_{\vec{k}'}^\dagger, c_{\vec{k}}\} = \delta^{(3)}(\vec{k}' - \vec{k}) \tag{2.24}$$

with the single particle states defined as

$$|1_{\vec{k}}\rangle = c_{\vec{k}}^\dagger|0\rangle \tag{2.25}$$

The first term in the Hamiltonian is simply:

$$H_0 = \int d^3\vec{k}\ \frac{\hbar^2\vec{k}^2}{2m}\ c^\dagger_{\vec{k}} c_{\vec{k}} \tag{2.26}$$

For example

$$H_0|1_{\vec{k}}\rangle = \int d^3\vec{k}' \frac{\hbar^2\vec{k}'^2}{2m} c^\dagger_{\vec{k}'} c_{\vec{k}'} c^\dagger_{\vec{k}}|0\rangle \tag{2.27}$$

$$= \int d^3\vec{k}' \frac{\hbar^2\vec{k}'^2}{2m} c^\dagger_{\vec{k}'}\delta^{(3)}(\vec{k}-\vec{k}')|0\rangle - \int d^3\vec{k}' \frac{\hbar^2\vec{k}'^2}{2m} c^\dagger_{\vec{k}'} c^\dagger_{\vec{k}} c_{\vec{k}'}|0\rangle \tag{2.28}$$

$$= \frac{\hbar^2\vec{k}^2}{2m} c^\dagger_{\vec{k}}|0\rangle = \frac{\hbar^2\vec{k}^2}{2m}|1_{\vec{k}}\rangle \tag{2.29}$$

and the same if we have more fermions with different momenta. The second term, namely the external potential can only have non-vanishing matrix elements between states where at most one particle has changed state. Namely, $U(r_a)$ can only change the state of particle a. Computing the matrix element

$$\langle\vec{k}'|U(\vec{r})|\vec{k}\rangle = \frac{1}{(2\pi)^3}\int d^3\vec{r}\ e^{-i(\vec{k}'-\vec{k}).\vec{x}}\ U(\vec{x}) = U_{\vec{k}'-\vec{k}} \tag{2.30}$$

and taking into account that the operator that moves a particle from state $|\vec{k}\rangle$ to $|\vec{k}'\rangle$ is $c^\dagger_{\vec{k}'}c_{\vec{k}}$ we find that the second term can be written as

$$H_1 = \int d^3\vec{k} d^3\vec{k}'\ U_{\vec{k}'-\vec{k}}\ c^\dagger_{\vec{k}'} c_{\vec{k}} \tag{2.31}$$

Finally the last term can only change the state of at most two particles. Defining

$$V_{\vec{k}'_1\vec{k}'_2\vec{k}_1\vec{k}_2} = \frac{1}{(2\pi)^3}\int d^3r_1 d^3r_2\ e^{-i(\vec{k}'_1-\vec{k}_1).\vec{r}_1} e^{-i(\vec{k}'_2-\vec{k}_2).\vec{r}_2} V(\vec{r}_1-\vec{r}_2) \tag{2.32}$$

we find

$$H_2 = \frac{1}{2}\int d^3\vec{k}_1 d^3\vec{k}'_1 d^3\vec{k}_2 d^3\vec{k}'_2\ V_{\vec{k}'_1\vec{k}'_2\vec{k}_1\vec{k}_2} c^\dagger_{\vec{k}'_2} c^\dagger_{\vec{k}'_1} c_{\vec{k}_1} c_{\vec{k}_2} \tag{2.33}$$

Overall the Hamiltonian reads

$$H = \int d^3\vec{k}\frac{\hbar^2\vec{k}^2}{2m} c^\dagger_{\vec{k}} c_{\vec{k}} + \int d^3\vec{k} d^3\vec{k}'\ U_{\vec{k}'-\vec{k}}\ c^\dagger_{\vec{k}'} c_{\vec{k}} + \frac{1}{2}\int d^3\vec{k}_1 d^3\vec{k}'_1 d^3\vec{k}_2 d^3\vec{k}'_2\ V_{\vec{k}'_1\vec{k}'_2\vec{k}_1\vec{k}_2} c^\dagger_{\vec{k}'_2} c^\dagger_{\vec{k}'_1} c_{\vec{k}_1} c_{\vec{k}_2} \tag{2.34}$$

Now we can define creation and annihilation operators in position space

$$\psi^\dagger(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\ \int d^3k\ e^{i\vec{k}.\vec{x}} c^\dagger_{\vec{k}} \tag{2.35}$$

$$\psi(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\ \int d^3k\ e^{i\vec{k}.\vec{x}} c_{\vec{k}} \tag{2.36}$$

that obey

$$\{\psi^\dagger(\vec{x}), \psi(\vec{y})\} = \delta^3(\vec{x} - \vec{y}) \tag{2.37}$$

The Hamiltonian can then be written as

$$H = \int d^3\vec{x}\psi^\dagger(\vec{x}) \left(-\frac{\hbar^2\nabla^2}{2m}\right) \psi(\vec{x}) + \int d^3\vec{x} U(\vec{x})\psi^\dagger(\vec{x})\psi(\vec{x}) + \frac{1}{2}\int d^3x d^3y V(\vec{x}-\vec{y})\psi^\dagger(\vec{y})\psi^\dagger(\vec{x})\psi(\vec{x})\psi(\vec{y}) \tag{2.38}$$

But this looks precisely as a quantum field theory. Namely operators associated with each point in space ($\psi(\vec{x})$, $\psi^\dagger(\vec{x})$). The "classical" Hamiltonian would be

$$H = \int d^3\vec{x}\psi^*(\vec{x}) \left(-\frac{\hbar^2\nabla^2}{2m}\right) \psi(\vec{x}) + \int d^3\vec{x} U(\vec{x})\psi^*(\vec{x})\psi(\vec{x}) + \frac{1}{2}\int d^3x d^3y V(\vec{x}-\vec{y})\psi^*(\vec{y})\psi^*(\vec{x})\psi(\vec{x})\psi(\vec{y}) \tag{2.39}$$

for a complex field $\psi(\vec{x})$. Such a classical field has no obvious physical meaning. However if we had done the same computation for bosons, for large occupation numbers, the operators a_i, $a_i^\dagger$ become classical amplitudes of a harmonic oscillator and the field $\psi(\vec{x})$ is an actual measurable classical field. In fact this is how the electromagnetic field arises from the particle point of view.

Finally let us just mention that the computation can be obviously done in reverse. Starting from the classical Hamiltonian for $\psi(\vec{x})$ one can quantize it and go back to the particle picture. This is the procedure that we are going to use later on. However, to help ensure relativistic invariance it turns out to be convenient to derive the Hamiltonian from a Lagrangian.