Classical Field theory

Preliminaries:

Units

$c = 3\times 10^8\, m/s \rightarrow c=1$ time measure in m = length

$\hbar c = 197$ MeV fm $\rightarrow \hbar c = 1$ energy measured in m^{-1} = length^{-1}

mass $\rightarrow \ell^{-1}$

Only unit left → units of length. We can eliminate length if we use gravity

$$m_1 a = G_N \frac{m_1 m_2}{r^2} \qquad G_N = \frac{a r^2}{m} = \frac{m/s^2 . m^2}{kg} = \frac{m/s^2 . m^2\, m^2/s^2}{MeV} = \frac{m^5/s^4}{MeV}$$

Newton's constant

$$[G_N] = \frac{\ell^5/\ell^4}{\ell^{-1}} = \ell^2 \Rightarrow \sqrt{G_N} = \ell_p$$ is a unit of length. $\ell_p \sim 10^{-35} m$

Not useful unless we are studying quantum gravity

↳ in usual units $[G_N] = \frac{m^4}{s^4}\frac{m^2}{MeV.m} \rightarrow \frac{c^4 m^2}{\hbar c} = \frac{c^3 \ell_p^2}{\hbar}$

$$\rightarrow \boxed{\ell_p = \sqrt{\frac{\hbar G_N}{c^3}}}$$

Special relativity: $x^\mu = (t, x_1, x_2, x_3) \rightarrow SO(3,1)$ symmetry.

interval $\Delta s^2 = +c^2 \Delta t^2 - \Delta x^2 \underset{c=1}{\rightarrow} +\Delta t^2 - \Delta x^2$

metric $$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \Delta s^2 = \eta_{\mu\nu} x^\mu x^\nu$$; sum : 0 -- 3 implicit.

A field is a quantity function of space and time.

$$\phi_a = \phi_a(\vec{x}, t)$$

Lagrangian density : $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$ $a=1 \ldots N$: several fields

action : $S = \int d^4x \; \mathcal{L}(\phi_a, \partial_\mu \phi_a)$ $\quad (L = \int d^3x \; \mathcal{L})$

Minimal action principle, Euler-Lagrange eq. of motion.

$$\phi_a \to \phi_a + \delta\phi_a \quad ; \quad \partial_\mu \phi_a \to \partial_\mu \phi_a + \partial_\mu \delta\phi_a$$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu \delta\phi_a \right) \underset{\uparrow}{=} 0$$

extremality condition

$$= \int d^4x \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) +$$

$$+ \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a = 0 \quad (\text{For any } \delta\phi_a)$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0$$

$$\oiint \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \underbrace{n^\mu dA}_{\text{area differential}} = 0$$

Boundary is at $t = t_i \; (\to -\infty)$ $\quad t_f \; (t_f \to +\infty)$

and at $|\vec{x}| \to \infty$ (or could be a box, etc.)

We take $\delta\phi_a \to 0$ at the bdy. by fixing the value of ϕ, namely we find and extremum for given values of the field.

For example $\phi_a(|\vec{x}| \to \infty) \to 0$, and fix $\phi(\vec{x}, t_{i,f})$

We could also fix $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} n^\mu = 0$ at bdy. (Neumann bdy cond.)

Hamiltonian

momentum $\Pi_a(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$ (canonical momentum conjugate to ϕ_a, not to be confused with usual momentum associated w/ translations)

$\Pi_a(\vec{x})$

momentum

~~density~~

$$H = \int d^3x \, \mathcal{H}(\vec{x}) = \int d^3x \, \Pi_a(\vec{x}) \dot{\phi}_a(x) - \int \mathcal{L} \, d^3x$$

Hamiltonian density

$\dot{\phi} = \partial_t \phi_a(x) =$

$= \frac{\partial}{\partial x^0} \phi_a(x)$

Noether's theorem:

Symmetries are associated with conserved quantities.

Conserved quantities are locally conserved, in QFT.

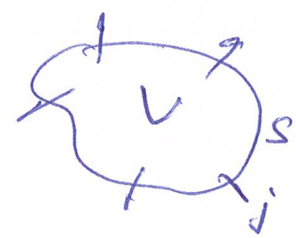
↑ charge cannot "teleport", it can flow ⇒ current.

$$\partial_\mu j^\mu = 0 \qquad Q = \int d^3x \; j^0(\vec{x})$$

$$\partial_0 Q = \int d^3x \, \partial_0 j^0 = -\int d^3x \, \partial_i j^i = -\oint dn^i j^i$$

↑ loss of charge = flow through surface.

if $j\big|_{\text{surface}} = 0 \Rightarrow \partial_0 Q = 0$



A symmetry is a variation of the fields that does not change the action, or equivalently changes the Lagrangian by a total derivative.

$$\phi_a \to \phi_a + \underbrace{\delta\phi_a}_{\text{particular variation now.}}$$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\partial_\mu \delta\phi_a = \partial_\mu J^\mu$$

using e.o.m. ⇒ on classical solution $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \delta\phi_a - J^\mu \right) = 0$.

<u>conserved current</u> $$j^\mu = \frac{\partial h}{\partial \partial_\mu \phi_a} \delta \phi_a - \mathcal{J}^\mu$$

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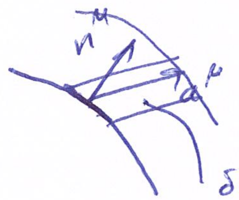
Translation Invariance.

$x^\mu \to x^\mu + a$



$\delta V = dA \cdot n^\mu a_\mu$

$$\int_V d^4x\, \mathcal{L} = \int_{\tilde V} d^4x$$

$$\int d^4x\, \mathcal{L}(\phi_a(x), \partial_\nu \phi_a(x), x) = \int d^4x\, \mathcal{L}(\phi_a(x^\mu + a^\mu), \partial_\nu \phi(x^\mu + a^\mu), x^\mu + a^\mu) -$$

$$- \oint\!\!\oint_S \mathcal{L}\, a^\mu n_\mu\, dA = \int d^4x\, \mathcal{L} + \int d^4x \left(\frac{\partial h}{\partial \phi_a} \partial_\mu \phi_a + \frac{\partial h}{\partial \partial_\nu \phi_a} \partial_{\mu\nu} \phi_a\, a^\mu + \frac{\partial h}{\partial x^\mu} \right) -$$

small a

$$- \int d^4x \frac{d\mathcal{L}}{dx^\mu} a^\mu$$

$$\Rightarrow \quad \frac{d\mathcal{L}}{dx^\mu} = \frac{\partial h}{\partial \phi_a} \partial_\mu \phi_a + \frac{\partial h}{\partial(\partial_\nu \phi_a)} \partial_{\mu\nu} \phi_a + \frac{\partial h}{\partial x^\mu}$$ (can also be written directly)

if $\frac{\partial h}{\partial x^\mu} = 0$ $\Rightarrow$ using eq. of motion we get

$$\frac{d\mathcal{L}}{dx^\mu} = \partial_\nu \frac{\partial h}{\partial \partial_\nu \phi_a} \partial_\mu \phi_a + \frac{\partial}{\partial(\partial_\nu \phi_a)} \partial_{\mu\nu} \phi_a$$

$$\frac{d}{dx^\nu} \left(\frac{\partial h}{\partial(\partial_\nu \phi_a)} \partial_\mu \phi_a - \eta^{\mu\nu} \mathcal{L} \right) = 0$$

$$T^{\mu\nu} = \frac{\partial h}{\partial(\partial_\nu \phi_a)} \partial_\mu \phi_a - \eta^{\mu\nu} \mathcal{L}$$ conserved $\boxed{\partial_\mu T^{\mu\nu} = 0}$

$E = \int d^3x\, T^{00}$ ← energy density

$P^i = \int d^3x\, T^{0i}$ ← mom. density

T^{i0} : flow of energy in direction i

T^{ij} flow of P^j in direction i.

Example $SO(N)$, $SU(N)$ scalar fields.

$\phi_{a=1\ldots N}$ real or $\phi_{a=1\ldots N}$ complex.

$$\mathcal{L} = + \cancel{\tfrac{1}{2}}\, \partial_\mu \phi_a^* \partial^\mu \phi_a - \cancel{\tfrac{1}{2}}\, m^2 \phi_a^* \phi_a$$ (for real drop $*$)

e.o.m.

$$\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a^*} = -\cancel{\tfrac{1}{2}}\, \partial^\mu \phi_a \qquad \frac{\delta \mathcal{L}}{\delta \phi_a^*} = + \cancel{\tfrac{1}{2}}\, m^2 \phi_a$$

$$\partial_\mu \partial^\mu \phi_a = \cancel{+}\, m^2 \phi_a \qquad \boxed{\partial_\mu \partial^\mu \phi_a + m^2 \phi_a = 0}$$

$m^2 = 0$ gives $-\partial_0 \partial_0 \phi_a + \nabla^2 \phi = 0$ wave equation.

$$\Pi_a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a^*} = \partial^0 \phi_a = \dot{\phi}_a \qquad \Pi_a^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = \dot{\phi}_a^*$$

$$\mathcal{H} = \Pi_a \dot{\phi}_a^* + \Pi_a^* \dot{\phi}_a - \partial_\mu \phi_a^* \partial^\mu \phi_a + m^2 \phi_a^* \phi_a$$

$$= \Pi_a^* \Pi_a + \nabla \phi_a^* \nabla \phi_a + m^2 \phi_a^* \phi_a$$

$$T_{\mu\nu} = \partial_\mu \phi_a \partial_\nu \phi_a^\dagger + \partial_\mu \phi_a^\dagger \partial_\nu \phi_a - \eta_{\mu\nu} \partial_\alpha \phi_a^\dagger \partial^\alpha \phi_a + \eta_{\mu\nu} m^2 \phi_a^\dagger \phi_a$$

$$T_{00} = \underbrace{\dot{\phi}_a \dot{\phi}_a^*}_{\text{Kinetic energy}} + \underbrace{(\nabla \phi_a^* \nabla \phi_a)}_{\text{energy in gradient, fields "prefers" to be uniform.}} + \underbrace{m^2 \phi_a^\dagger \phi_a}_{\text{energy associated with increasing value of the field}} = \mathcal{H}.$$

$$T_{0i} = \dot{\phi}_a \nabla_i \phi_a^* + \dot{\phi}_a^* \nabla_i \phi_a = \overline{\Pi}_a \partial_i \phi_a^\dagger + \Pi_a \partial_i \phi_a = T_{i0}$$

$$P_i = \int (\Pi_a \partial_i \phi_a + \overline{\Pi}_a \partial_i \phi_a^*)\, d^3x$$

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$SU(N)$ symmetry $\quad \phi_a \to U_{ab} \phi_b$ does not change Lagrangian
$\uparrow$ unitary $\qquad J^\mu = 0.$

$$\delta\phi_a = i\, \epsilon_{ab} \phi_b$$

$\delta\phi_a^\dagger = -i\, \epsilon_{ab}^* \phi_b^*$ $\uparrow$ hermitian.

$$j^\mu = i\, \partial_\mu \phi_a^* \epsilon_{ab} \phi_b + i\, \partial_\mu \phi_a (-\overbrace{\epsilon_{ab}^*}^{\epsilon_{ba}} \phi_b^\dagger)$$

$$= i\epsilon_{ab} (\partial_\mu \phi_a^* \phi_b - \phi_a^\dagger \partial_\mu \phi_b)$$

$$\boxed{j^\mu_{ab} = i(\partial_\mu \phi_a^* \phi_b - \phi_a^\dagger \partial_\mu \phi_b)}$$ $SU(N)$ conserved current.

Quantization

classical solutions

$$\partial_0^2 \phi - \nabla^2 \phi + m^2 \phi = 0 .$$

$$\phi = \xi_k(t)\, e^{i\vec{k}\vec{x}} \quad \Rightarrow \quad \ddot{\xi}(t)\, e^{i\vec{k}\vec{x}} + \vec{k}^2 \xi e^{i\vec{k}\vec{x}} + m^2 \xi e^{i\vec{k}\vec{x}} = 0$$

$$\ddot{\xi}_k(t) + \omega_k^2 \xi_k(t) = 0 \quad ; \quad \boxed{\omega_k^2 = m^2 + \vec{k}^2}$$

harmonic oscillator $\quad \xi = \xi_0\, e^{\pm i\omega_k t}$

$\Rightarrow$ set of decoupled harmonic osc.

Decouple them by doing change of (canonical) variables

$$\phi_a = \int \frac{d^3k}{(2\pi)^{3/2}}\, \xi_{k,a}(t)\, e^{i\vec{k}\vec{x}} \qquad \phi_a^* = \int \frac{d^3k}{(2\pi)^{3/2}}\, \xi^*_{k,a}(t)\, e^{-i\vec{k}\vec{x}}$$

$$\dot{\phi}_a = \int \frac{d^3k}{(2\pi)^{3/2}}\, \dot{\xi}_{k,a}(t)\, e^{i\vec{k}\vec{x}} \qquad \dot{\phi}_a^* = \int \frac{d^3k}{(2\pi)^{3/2}}\, \dot{\xi}^*_{k,a}(t)\, e^{-i\vec{k}\vec{x}}$$

$$\mathcal{L} = \frac{1}{(2\pi)^3} \int d^3k\, d^3k'\; e^{i(\vec{k}-\vec{k}')x}\; \dot{\xi}_{k,a} \dot{\xi}^*_{k',a} \; -$$

$$- \frac{1}{(2\pi)^3} \int d^3k\, d^3k'\; e^{i(\vec{k}-\vec{k}')\vec{x}}\; (ik_j).(-ik'_j)\, \xi_{k,a} \xi^*_{k',a}$$

$$- \frac{1}{(2\pi)^3} \int d^3k\, d^3k'\; e^{i(\vec{k}-\vec{k}')\vec{x}}\; m^2\, \xi_{k,a} \xi^*_{k',a} .$$

$$L = \int d^3x\, \mathcal{L} \quad ; \quad \int d^3x\, e^{i(\vec{k}-\vec{k}')\vec{x}} = (2\pi)^3\, \delta(\vec{k}-\vec{k}')$$

$$L = \int d^3k \quad \dot{\xi}_{k,a} \dot{\xi}^{*}_{k,a} - \underbrace{(\vec{k}^2+m^2)}_{\omega_k^2} \xi_{k,a} \xi^{*}_{k,a}$$

take $\xi_{k,a} = \frac{1}{\sqrt{2}} (\eta_{k,a} + i \zeta_{k,a})$

$$L = \int d^3k \left(\frac{1}{2} \dot{\eta}^2_{k,a} - \frac{1}{2} \omega^2_{k,a} \eta^2_{k,a} + \frac{1}{2} \dot{\zeta}^2_{k,a} - \frac{1}{2} \omega_k^2 \zeta^2_{k,a} \right)$$

Lagrangian for a set of $2N$ harmonic osc. for each given $\vec{k}$.

As usual define $\left(a = \sqrt{\frac{m\omega}{2\hbar}} (x + \frac{i}{m\omega} p) \right)$ $m=1,\ \hbar=1$

$$a_{k,a} = \sqrt{\frac{\omega_k}{2}} \left(\eta_{k,a} + \frac{i}{\omega_k} \dot{\eta}_{k,a} \right) ; \quad a^{+}_{k,a} = \sqrt{\frac{\omega_k}{2}} \left(\eta_{k,a} - \frac{i}{\omega_k} \dot{\eta}_{k,a} \right)$$

$$b_{k,a} = \sqrt{\frac{\omega_k}{2}} \left(\zeta_{k,a} + \frac{i}{\omega_k} \dot{\zeta}_{k,a} \right) ; \quad b^{+}_{k,a} = \sqrt{\frac{\omega_k}{2}} \left(\zeta_{k,a} - \frac{i}{\omega_k} \dot{\zeta}_{k,a} \right)$$

<u>or</u>

$$\eta_{k,a} = \frac{1}{\sqrt{2\omega_k}} (a_{k,a} + a^{+}_{k,a}) \qquad \zeta_{k,a} = \frac{1}{\sqrt{2\omega_k}} (b_{k,a} + b^{+}_{k,a})$$

$$\dot{\eta}_{k,a} = -i \sqrt{\frac{\omega_k}{2}} (a_{k,a} - a^{+}_{k,a}) \qquad \dot{\zeta}_{k,a} = -i \sqrt{\frac{\omega_k}{2}} (b_{k,a} - b^{+}_{k,a})$$

$$\xi_{k,a} = \frac{1}{2\sqrt{\omega_k}} (a_{k,a} + i b_{k,a} + a^{+}_{k,a} + i b^{+}_{k,a})$$

$$\dot{\xi}_{k,a} = -i \frac{\sqrt{\omega_k}}{2} (a_{k,a} + i b_{k,a} - (a^{+}_{k,a} + i b^{+}_{k,a}))$$

$$[a_{k,a}, a^\dagger_{k',b}] = \delta(k-k')\,\delta_{ab}$$

$$[a_{k,a} + i b_{k,a},\ a^\dagger_{k',b} - i b^\dagger_{k',b}] = \delta(k-k')\,\delta_{ab}\,(1+1) = 2\,\delta(k-k')\,\delta_{ab}$$

$$c_{k,a} = \frac{1}{\sqrt{2}}(a_{k,a} + i b_{k,a}) \Rightarrow [c_{k,a}, c^\dagger_{k',b}] = \delta(k-k')\,\delta_{ab}$$

$$d_{-k,a} = \frac{1}{\sqrt{2}}(a_{k,a} - i b_{k,a}) \Rightarrow [c_{k,a}, d^\dagger_{k',b}] = 0$$

$$[d_{k,a}, d^\dagger_{k',b}] = \delta(k-k')\,\delta_{ab}.$$

$$\boxed{\xi_{k,a} = \frac{1}{\sqrt{2\omega_k}}\left(c_{k,a} + d^\dagger_{-k,a}\right); \quad \dot{\xi}_{k,a} = -i\sqrt{\frac{\omega_k}{2}}\left(c_{k,a} - d^\dagger_{-k,a}\right)}$$

these are now quantum operators!

Hilbert space is space of occupation numbers.

$$\boxed{\begin{aligned} \phi_a &= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(c_{k,a}\, e^{i\vec{k}\vec{x}} + d^\dagger_{k,a}\, e^{-i\vec{k}\vec{x}}\right) \\ \Pi_a &= \dot{\phi}^*_a = i\int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left(c^\dagger_{k,a}\, e^{-i\vec{k}\vec{x}} - d_{k,a}\, e^{i\vec{k}\vec{x}}\right) \end{aligned}}$$

Notice no t dependence Schrödinger picture

Fields are now operators acting on Hilbert space of multiparticle states. $|n^{(1,2)}_{k,a} \ldots\rangle$

occupation number of oscillator $k,a \to c$

$k,a \to d$

◦) Fundamental commutation relation

$$[\Pi_a(\vec{x}), \phi_b(\vec{y})] = \frac{i}{(2\pi)^3}\int d^3k\, d^3k' \, \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\Big(e^{-ikx+ik'y} \underbrace{[c^\dagger_{k,a}, c_{k',b}]}_{-\delta_{ab}\delta_{kk'}} -$$

$$- e^{ikx-ik'y} \underbrace{[d_{k,a}, d^\dagger_{k',b}]}_{\delta_{kk'}\delta_{ab}} \Big)$$

$$= \frac{i}{(2\pi)^3}\int d^3k \, \frac{1}{2}\left(-e^{-ikx+ik'y} - e^{ikx-ik'y}\right)\delta_{ab}$$

$$= -i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

◦◦) Hamiltonian

$$H = \int d^3x \;\; \Pi_a\overline{\Pi}_a + \nabla\phi_a \nabla\phi^*_a + m^2 \phi^*_a \phi_a$$

$$\Pi_a\overline{\Pi}_a = \frac{1}{(2\pi)^3}\int d^3k\, d^3k' \, \frac{\sqrt{\omega_k\omega_{k'}}}{2}\left(c^\dagger_{ka}e^{-ikx} - d_{k,a}e^{ikx}\right)\left(c_{k'a}e^{ik'x} - d^\dagger_{k',a}e^{-ik'x}\right)$$

$$\nabla\phi_a\nabla\phi^*_a = \frac{1}{(2\pi)^3}\int d^3k\, d^3k' \, \frac{1}{2\sqrt{\omega_k\omega_{k'}}}\left(i\vec{k}\, c_{k,a}e^{i\vec{k}\vec{x}} + i\vec{k}\, d^\dagger_{k,a}e^{-i\vec{k}\vec{x}}\right)\left(-ik'c^\dagger_{k',a}e^{-ik'x} + ik'd_{k',a}e^{ik'x}\right)$$

$$\int d^3x\,(\Pi_a\overline{\Pi}_a + \nabla\phi_a\nabla\phi^*_a) = \int d^3k \, \frac{\omega_k}{2}\left(c^\dagger_{ka}c_{ka} - c^\dagger_{ka}d^\dagger_{-k,a} - d_{k,a}c_{-k,a} + d_{k,a}d^\dagger_{k,a}\right)$$

$$+ \int d^3k \, \frac{1}{2\omega_k}\left(k^2 c_{k,a}c^\dagger_{k,a} + k^2 c_{k,a}d_{-k,a} + k^2 d^\dagger_{k,a}c^\dagger_{-k,a} + k^2 d^\dagger_{k,a}d_{k,a}\right)$$

$$\phi^*_a\phi_a = \frac{1}{(2\pi)^3}\int d^3k\, d^3k' \, \frac{1}{2\sqrt{\omega_k\omega_{k'}}}\left(c_{k,a}e^{ikx} + d^\dagger_{k,a}e^{-ikx}\right)\left(c^\dagger_{k',a}e^{-ik'x} + d_{k',a}e^{ik'x}\right)$$

$$\int d^3x\, m^2 \phi_a \phi_a^* = \int d^3k\, \frac{m^2}{2\omega_k} \left(c_{k,a} c^\dagger_{k,a} + c_{k,a} d_{-k,a} + d^\dagger_{k,a} c^\dagger_{-k,a} + d^\dagger_{k,a} d_{k,a} \right)$$

$$H = \int d^3k\, \frac{\omega_k}{2} \left(c^\dagger_{ka} c_{ka} - \cancel{c^\dagger_{k,a} d^\dagger_{-k,a}} - \cancel{d_{k,a} c_{-k,a}} + d_{k,a} d^\dagger_{k,a} \right)$$

$k \leftrightarrow -k$.

$$+ \int d^3k\, \frac{\overset{\omega_k^2}{m^2 + k^2}}{2\omega_k} \left(c_{k,a} c^\dagger_{k,a} + \cancel{c_{k,a} d_{-k,a}} + \cancel{d^\dagger_{k,a} c^\dagger_{-k,a}} + d^\dagger_{k,a} d_{k,a} \right)$$

$$= \int d^3k\, \frac{\omega_k}{2} \left(c^\dagger_{ka} c_{ka} + c_{ka} c^\dagger_{ka} + d_{ka} d^\dagger_{ka} + d^\dagger_{k,a} d_{k,a} \right)$$

$$c^\dagger_{ka} c_{ka} = n^{(+)}_{ka} \qquad d^\dagger_{ka} d_{ka} = n^{(-)}_{ka}$$

$$c_{ka} c^\dagger_{ka} = c^\dagger_{ka} c_{ka} + [c_{k,a}, c^\dagger_{ka}] = c^\dagger_{ka} c_{ka} + \delta_{aa} \delta(k-k)$$

Volume $\delta(0) = \frac{V}{(2\pi)^3}$

$$H = \int d^3k \sum_a (n^{(+)}_{ka} + n^{(-)}_{ka})\, \omega_k + \underbrace{N \int d^3k\, \omega_k\, \delta(0)}_{\infty}$$

∞ energy of vacuum.

∞ energy Density

$$\boxed{H_f = \int d^3k \sum_a (n^{(+)}_{ka} + n^{(-)}_{ka})\, \omega_k}$$

states are $| n^{(+)}_{ka} \,--\, \rangle$

System of bosons with energy $\omega_k = \sqrt{\vec{k}^2 + m^2}$

and we can put arbitrary # of them in each state.

Lowest energy : vacuum $n_{k,a} = 0$

charges

$$j^\mu_{ab} = i\left(\partial_\mu \phi_a^* \phi_b - \phi_a^* \partial_\mu \phi_b\right)$$

$$Q_{ab} = i\int d^3x \left(\overline{\pi}_a \phi_b - \phi_a^* \overline{\pi}_b\right)$$

$$= i\int d^3x \frac{i}{(2\pi)^3}\int d^3k\, d^3k' \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\left(c^\dagger_{k,a} e^{-ikx} - d_{k,a} e^{ikx}\right)\left(c_{k',b} e^{ik'x} + d^\dagger_{k',b} e^{-ik'x}\right)$$

$$+ i\int d^3x \frac{i}{(2\pi)^3}\int d^3k\, d^3k' \frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\left(c^\dagger_{k',a} e^{-ik'x} + d_{k',a} e^{-ik'x}\right)\left(c_{k,b} e^{ikx} - d^\dagger_{k,b} e^{-ikx}\right)$$

$$= -\int d^3k \frac{1}{2}\left(c^\dagger_{k,a} c_{k,b} + \cancel{c^\dagger_{k,a} d^\dagger_{-k,b}} - \cancel{d_{k,a} c_{-k,b}} - d_{k,a} d^\dagger_{k,b}\right)$$

$$-\int d^3k \frac{1}{2}\left(c^\dagger_{k,a} c_{k,b} - \cancel{c^\dagger_{k,a} d^\dagger_{-k,b}} + \cancel{d_{k,a} c_{-k,b}} - d_{k,a} d^\dagger_{-k,b}\right)$$

$$= -\int d^3k \left(c^\dagger_{k,a} c_{k,b} - d_{k,a} d^\dagger_{k,b}\right)$$

$$Q_{ab} = \int d^3k \left(-c^\dagger_{k,a} c_{k,b} + d_{k,a} d^\dagger_{k,b}\right)$$

interchange up to ∞.

$$Q^{el}_{aa} = \int d^3k \left(+n^{(+)}_{k,a} - n^{(-)}_{k,a}\right)$$

c & d carry opposite charges.

Also momentum $\quad P^i = \int d^3x\, T^{i0}$

$$P^i = \sum_a \int d^3k\, \vec{k}^i \left(n^{(+)}_{k,a} + n^{(-)}_{k,a}\right)$$

each particle has momentum $\vec{k}$ and energy $\omega_k = \sqrt{\vec{k}^2 + m^2}$

Heisenberg picture.

$$\phi_H(x,t) = e^{iHt}\,\phi_s(x)\,e^{-iHt}$$

$$i\frac{\partial\phi}{\partial t} = [\phi, H]$$

$\langle\psi_i|\phi_H|\psi_i\rangle =$
$= \langle\psi_i|\,e^{iHt}\,\phi_s\,e^{-iHt}\,|\psi_i\rangle$
$= \langle\psi(t)|\,\phi_s\,|\psi(t)\rangle$

$$e^{iHt}\,a^{\dagger}_{k,a}\,e^{-iHt}\,|E\rangle = e^{-iEt}\,e^{iHt}\,|E+\omega_k\rangle = e^{i\omega_k t}\,\underbrace{|E+\omega_k\rangle}_{a^{\dagger}_{k,a}|E\rangle}$$

↑ eigenstate of energy.

$$e^{iHt}\,a^{\dagger}_{k,a}\,e^{-iHt} = e^{i\omega_k t}\,a^{\dagger}_{k,a}$$

$$e^{iHt}\,a_{k,a}\,e^{-iHt} = e^{-i\omega_k t}\,a_{k,a}$$

$$\boxed{\begin{aligned}\phi_a(x,t) &= \int\frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(c_{k,a}\,e^{-i\omega_k t+i\vec{k}\vec{x}} + d^{\dagger}_{k,a}\,e^{i\omega_k t-i\vec{k}\vec{x}}\right)\\ \Pi_a(x,t) &= i\int\frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\left(c^{\dagger}_{k,a}\,e^{i\omega_k t-ikx} - d_{k,a}\,e^{-i\omega_k t+ikx}\right)\end{aligned}}$$

$$-\omega_k t + ikx = k_\mu x^\mu \qquad k_\mu = (\omega_k, \vec{k}) \qquad x^\mu = (t, \vec{x})$$

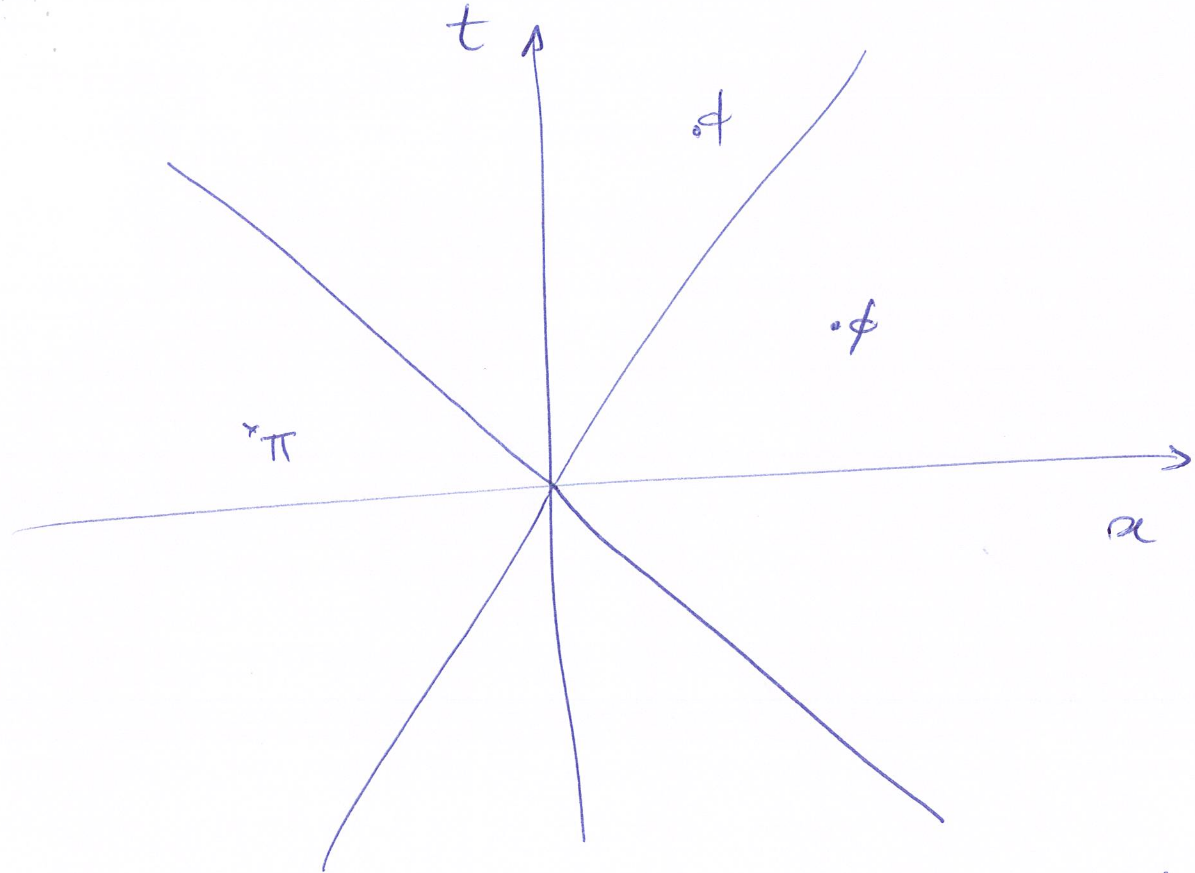
Notice we can do now

$$\dot{\phi}_a(x,t) = -i\int\frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}}\left(c_{k,a}\,e^{-i\omega_k t+i\vec{k}\vec{x}} - d^{\dagger}_{k,a}\,e^{i\omega_k t-i\vec{k}\vec{x}}\right)$$

$$= \overline{\Pi}_a$$

(as it should, in some sense we don't need Π_a any more)

Also $\partial_\mu\partial^\mu\phi_a(\vec{x}) + m^2\phi_a(\vec{x}) = 0$



In the Heisenberg picture we have operators ϕ_a, π_a at each point of space time. Very useful to study space-time symmetries, causal properties, etc.

This is the "field theory" point of view.

Operators are represented by acting on a Hilbert space of particles occupation numbers $|n_{k_1}, \ldots\rangle$

The Hilbert space is the "particle" point of view.

For curved space time and conformal theories the "field theory" point of view is better.

Commutators

$$[\phi_a(x,t), \phi_b^+(y,t')] = \frac{1}{(2\pi)^3}\int \frac{d^3k\, d^3k'}{2\sqrt{\omega_k\omega_{k'}}}\left[c_{ka}e^{-i\omega t+ikx} + d_{ka}^+ e^{i\omega t-ikx}, c_{k'b}^+ e^{i\omega' t'-ik'y} + d_{k'b} e^{-i\omega' t'+ik'y}\right]$$

$$= \frac{\delta_{ab}}{2(2\pi)^3}\int\frac{d^3k\,d^3k'}{\sqrt{\omega_k\omega_{k'}}}\left\{\delta(k-k')\, e^{-i\omega_k t+ikx+i\omega' t'-ik'y} - \delta(k-k')\, e^{i\omega t-ikx-i\omega' t'+ik'y}\right\}$$

$$= \frac{\delta_{ab}}{2(2\pi)^3}\int\frac{d^3k}{\omega_k}\left[e^{-i\omega_k(t-t')+ik(x-y)} - \underbrace{e^{i\omega_k(t-t')-ik(x-y)}}_{k\to -k}\right]$$

$\uparrow$ c-number, not operator.

$$= \frac{\delta_{ab}}{2(2\pi)^3}\int\frac{d^3k}{\omega_k}\, e^{i\vec{k}(\vec{x}-\vec{y})}\left[e^{-i\omega_k(t-t')} - e^{i\omega_k(t-t')}\right]$$

Complex integral   assume $t>0$

$$\frac{1}{2\pi i}\int_{\mathcal{C}} dk_0\, \frac{e^{-ik_0 t}}{k_0^2-\omega^2} = -\cancel{\frac{2\pi i}{2\pi i}}\left(\frac{1}{2\omega}e^{-i\omega t} + \frac{e^{i\omega t}}{-2\omega}\right)$$

orientation

$\lfloor k_0$

$$= \boxed{\frac{e^{i\omega t}}{2\omega} - \frac{e^{-i\omega t}}{2\omega}}$$

$$\frac{1}{(k_0-\omega)(k_0+\omega)}$$

(and $=0$ if $t<0$)

$$e^{-i(k_0+i\eta)t} = e^{-ik_0 t+\eta t}$$

for $t>0$ we close at $\eta<0$.

$$\int_{\mathcal{C}} = \int_{\mathbb{R}} \frac{\qquad}{k_0^2-\omega^2-i\varepsilon\, sg(k_0)}$$

$$k_0 = \pm\sqrt{\omega^2-i\varepsilon\, sg(k_0)} = \pm\left(\omega - \frac{i\varepsilon\, sg(k_0)}{2\omega}\right)$$

$$= \begin{cases}\omega - i\varepsilon\\ \omega - i\varepsilon\end{cases}$$

$$[\phi_a(x,t), \phi_b^\dagger(y,t')]_{t>t'} = -\frac{1}{2\pi i}\frac{\delta_{ab}}{(2\pi)^3}\int d^4k\, \frac{e^{i\vec{k}(\vec{x}-\vec{y})-i\omega_k(t-t')}}{k_0^2-\omega_k^2-i\varepsilon\, sg(k_0)}$$

$$[\phi_a(x,t), \phi_b^\dagger(y,t')]_{t<t'} = \frac{1}{2\pi i}\frac{\delta_{ab}}{(2\pi)^3}\int d^4k\, \frac{e^{-i\vec{k}(\vec{x}-\vec{y})+i\omega_k(t-t')}}{k_0^2-\omega_k^2-i\varepsilon\, sg(k_0)}$$

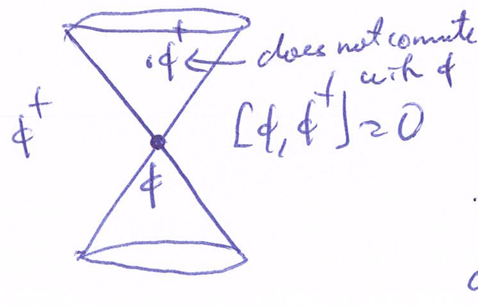
$$[\phi_a(x,t), \phi_b^\dagger(y,t')] = \delta_{ab}\,\Delta_c(x-y, t-t')$$

$$\Delta_c(\vec{x}-\vec{y}, t-t') = -\frac{1}{2\pi i}\frac{\cancel{\delta_{ab}}}{(2\pi)^3}\int d^4k\, \frac{e^{\mp ik(x-y)}}{k^2-i\varepsilon\, sg(k_0)}\,\theta(t-t') + \frac{1}{2\pi i}\frac{1}{(2\pi)^3}\int d^4k\, \frac{e^{ik(x-y)}}{k^2-i\varepsilon\, sg(k_0)}\,\theta(t'-t)$$

It is Lorentz inv. : $sg(t-t')$ is invariant inside the light-cone but not outside; so we have to be careful.

if $t=t'$ $\quad [\phi_a(x,t), \phi_b^\dagger(y,t)] = 0$

$\Rightarrow$ It is zero outside the light-cone. $\rightarrow$ by cancellation of $cc^\dagger$ and $dd^\dagger$ terms $\Rightarrow$ we need antiparticles (same mass opposite charge)



Inside the light-cone time ordering is invariant.

$\Rightarrow$ all space-like separated commutators are zero : causality. $(\Pi = \dot{\phi}^*)$.

Wightman functions

$$\langle 0|\phi^\dagger_a(x)\,\phi_b(y)|0\rangle = \int\frac{d^3k}{(2\pi)^{3/2}}\int\frac{d^3k'}{(2\pi)^{3/2}}\frac{1}{2\sqrt{\omega_k\omega_{k'}}}\langle 0|\, d_{k,a}\, e^{-i\omega_k t+i\vec{k}\vec{x}}\; d^\dagger_{k',b}\, e^{i\omega_{k'}t'-i\vec{k}'\vec{y}}\,|0\rangle$$

$$= \delta_{ab}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\,e^{-i\omega_k(t-t')+i\vec{k}(\vec{x}-\vec{y})} = \delta_{ab}\int\frac{d^3k}{(2\pi)^3}\frac{e^{-ik(x-y)}}{2\omega_k}$$

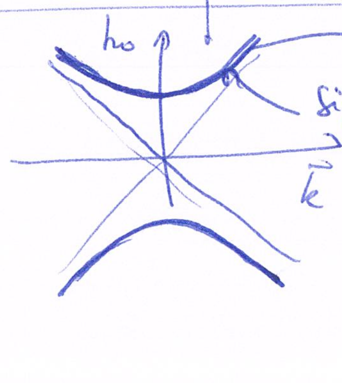
$$\int\frac{d^3k}{(2\pi)^3}\frac{e^{-i\omega_k t}}{2\omega_k} = \int dk_0\int\frac{d^3k}{(2\pi)^3}\frac{\delta(k_0-\omega_k)}{2k_0}\,e^{-ik_0t}$$

$$\boxed{\delta(k_0^2-\omega_k^2) = \frac{1}{2\omega_k}\left[\delta(k_0-\omega_k)+\delta(k_0+\omega_k)\right]}$$

$$= \int\frac{d^4k}{(2\pi)^3}\,\delta(k_0^2-\omega_k^2)\,\Theta(k_0)\,e^{-ik_0t}$$

$$\langle 0|\phi^\dagger_a(x)\,\phi_b(y)|0\rangle = \delta_{ab}\int\frac{d^4k}{(2\pi)^3}\,\underbrace{\delta(k_0^2-\omega_k^2)}_{\delta(k_0^2-\vec{k}^2-m^2)}\,\Theta(k_0)\,e^{-ik(x-y)}$$

$$\boxed{\langle 0|\phi^\dagger_a(x)\,\phi_b(y)|0\rangle = \delta_{ab}\int\frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\,\Theta(k_0)\,e^{-ik(x-y)}}$$



$k^2=m^2$ (mass-shell)

sign of k_0 is Lorentz inv.

$$\Theta(k_0) = \begin{cases} 1 & k_0>0 \\ 0 & \text{otherwise.} \end{cases}$$

Retarded propagator:

$$D_{abR}(x,y) = \Theta(x_0^0 - y_0^0)\, \langle 0 | [\phi_a(x), \phi_b^\dagger(y)] | 0 \rangle$$

$$= \Theta(x^0 - y^0)\, \frac{\delta_{ab}}{(2\pi)^3} \int \frac{d^3k}{2\omega_k}\, e^{i\vec{k}(\vec{x}-\vec{y})} \left(e^{-i\omega_k(t-t')} - e^{i\omega_k(t-t')} \right)$$

$$= i\delta_{ab} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - i\varepsilon\, sg(k_0)}$$

Advanced prop.

$$D_{abA}(x,y) = -\Theta(y^0 - x^0)\, \langle 0 | [\phi_a(x), \phi_b^\dagger(y)] | 0 \rangle$$

$$= +i\,\delta_{ab} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - i\varepsilon\, sg(k_0)}$$

$$[\phi_a(x), \phi_b^\dagger(y)] = D_R - D_A$$

Feynman prop.

$$\langle 0 | \underset{\substack{\uparrow \\ \text{time-order}}}{\hat{T}} \{ \phi_a(x)\, \phi_b^\dagger(y) \} | 0 \rangle = \begin{cases} \langle 0 | \phi_a(x)\, \phi_b^\dagger(y) | 0 \rangle & \text{if } t_x > t_y \\ \langle 0 | \phi_b^\dagger(y)\, \phi_a(x) | 0 \rangle & \text{if } t_x < t_y . \end{cases}$$

It is Lorentz invariant because ops. commute when space-like separated (in which Time ordering is ill-defined).

$$\langle 0|\, \phi_a(x)\, \phi_b^\dagger(y)\, |0\rangle = \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2\omega_k}\, \delta_{ab}\, e^{-ikx+iky} \qquad t_x > t_y$$

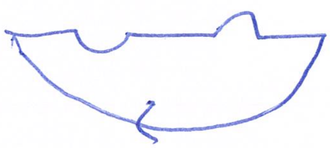
$$\langle 0|\, \phi_b^\dagger(y)\, \phi_a(x)\, |0\rangle = \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2\omega_k}\, \delta_{ab}\, e^{-iky+ikx} \qquad t_x < t_y$$

$$\cancel{\int \frac{dk_0}{2\pi i}\, \frac{1}{k_0^2-\omega_k^2+i\varepsilon}} \qquad \int \frac{dk_0}{2\pi i}\, \frac{e^{-ik_0(t_x-t_y)}}{k_0^2-\omega_k^2+i\varepsilon} = \begin{cases} -\dfrac{1}{2\omega_k}\, e^{-i\omega_k(t_x-t_y)} & t_x > t_y \\ -\dfrac{1}{2\omega_k}\, e^{i\omega_k(t_x-t_y)} & t_x < t_y \end{cases}$$

$$e^{-i(k_0+i\eta)(t_x-t_y)} = e^{-ik_0(t_x-t_y)}\, e^{\eta(t_x-t_y)}$$

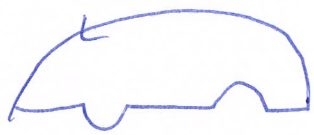
$$k_0 = \pm\sqrt{\omega_k^2 - i\varepsilon} = \pm\, \omega_k \sqrt{1 - i\varepsilon/\omega_k^2} = \pm\omega_k \left(1 - \frac{i\varepsilon}{2\omega_k^2}\right) = \pm\omega_k \mp \frac{i\varepsilon}{2\omega_k}$$

$t_x > t_y$



$$= -\frac{\cancel{2\pi i}}{\cancel{2\pi i}}\, \frac{e^{-i\omega_k(t_x-t_y)}}{2\omega_k}$$

$t_x < t_y$



$$= \frac{\cancel{2\pi i}}{\cancel{2\pi i}}\, \frac{e^{i\omega_k(t_x-t_y)}}{-2\omega_k}$$

$$\boxed{\langle 0|\, \hat{T}\left\{\phi_a(x)\, \phi_b^\dagger(y)\right\} |0\rangle = \delta_{ab} \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2-m^2+i\varepsilon}\, e^{-ik(x-y)}}$$

Can be analytically continued to Euclidean

Since $\partial_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0$

one might expect.

$$(\partial_x^2 + m^2) \langle 0 | \hat{T}\{\phi(x), \phi(y)\} | 0 \rangle \stackrel{?}{=} 0 \quad \underline{\text{No}}.$$

$$\downarrow$$

$$(\partial_0^2 - \partial_{\vec{x}}^2 + m^2)$$

$$\partial_0 \langle 0 | \hat{T}\{\phi(x), \phi(y)\} | 0 \rangle = \langle 0 | \hat{T}\{\partial_0 \phi(x), \phi(y)\} | 0 \rangle$$

$$+ \underbrace{\delta(x^0 - y^0)}_{\text{jump in the function.}} \langle 0 | \underbrace{\phi_a(x) \phi_b^\dagger(y) - \phi_b^\dagger(y) \phi_a(x)}_{[\phi_a(x), \phi_b^\dagger(y)]} | 0 \rangle$$

$[\phi_a(x), \phi_b^\dagger(y)] = 0$ when $x^0 = y^0$.

$$\partial_0 \langle 0 | \hat{T}\{\phi(x), \phi(y)\} | 0 \rangle = \langle 0 | \hat{T}\{\pi(x), \phi(y)\} | 0 \rangle$$

$$\partial_0^2 \langle 0 | \hat{T}\{\phi(x) \phi(y)\} | 0 \rangle = \langle 0 | \hat{T}\{\partial_t^2 \phi(x) \phi(y)\} | 0 \rangle$$

$$+ \delta(x^0 - y^0) \langle 0 | \underbrace{[\pi(x), \phi(y)]}_{-i\delta^{(3)}(\vec{x} - \vec{y})} | 0 \rangle$$

$$(\partial_0^2 - \partial_{\vec{x}}^2 + m^2) \underbrace{\langle 0 | \hat{T}\{\phi(x) \phi(y)\} | 0 \rangle}_{\text{Green's function.}} = -i\, \delta^{(4)}(x - y)$$

Same for retarded and advanced.

Indeed

$$(\partial_\mu \partial^\mu + m^2) \langle 0 | \hat{T} \{ \phi_a(x) \, \phi_b^\dagger(y) \} | 0 \rangle = \delta_{ab} \int \frac{d^4k}{(2\pi)^3} \, \frac{(-k^2+m^2)\, i}{k^2 - m^2 + i\varepsilon} \, e^{-ik(x-y)}$$

$$= -i\,\delta_{ab} \int \frac{d^4k}{(2\pi)^4} \, e^{-ik(x-y)} =$$

$$= -i\,\delta_{ab}\, \delta^{(4)}(x-y).$$

Same for advanced & retarded.

$$[\phi_a(\vec{x},t), \phi_b^\dagger(\vec{y},t')] = \frac{\delta_{ab}}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\, e^{i\vec{k}(\vec{x}-\vec{y})}\left[e^{-i\omega_k(t-t')} - e^{i\omega_k(t-t')}\right]$$

$$= -\frac{2i\delta_{ab}}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\, e^{i\vec{k}(\vec{x}-\vec{y})} \sin(\omega_k(t-t'))$$

take z axis along $(\vec{x}-\vec{y})$

$$= -\frac{2i\delta_{ab}}{(2\pi)^3}\, 2\pi \int \frac{k^2 dk}{2\omega_k}\, \underbrace{s\theta d\theta}_{d\mu}\, e^{ik|\vec{x}-\vec{y}|\cos\theta} \sin(\omega_k(t-t'))$$

$$\int_{-1}^{1} d\mu\, e^{ik|\vec{x}-\vec{y}|\mu} = \left.\frac{e^{ik|\vec{x}-\vec{y}|\mu}}{ik|\vec{x}-\vec{y}|}\right|_{-1}^{1} = \frac{2\left(e^{ik|\vec{x}-\vec{y}|} - e^{-ik|\vec{x}-\vec{y}|}\right)}{2ik|\vec{x}-\vec{y}|}$$

$\mu = \cos\theta$

$$= \frac{2\sin(k|\vec{x}-\vec{y}|)}{k|\vec{x}-\vec{y}|}$$

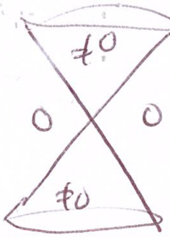
$$= -\frac{8\pi i\delta_{ab}}{(2\pi)^3}\int_0^\infty \frac{k\,dk\, \sin(k|\vec{x}-\vec{y}|)}{2\omega_k|\vec{x}-\vec{y}|} \sin(\omega_k|t-t'|)\, \mathrm{sign}(t-t')$$

$$= \frac{i\delta_{ab}}{\pi^2}\frac{1}{r}\frac{\partial}{\partial r}\int_0^\infty \frac{dk}{2\omega_k}\cos(\overset{b}{kr}) \sin(\omega_k\underbrace{|t-t'|}_{p})\, sg(t-t')$$

$r = |\vec{x}-\vec{y}|$ $\qquad a = m$

GR: $$\int_0^\infty \frac{\sin(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}}\cos bx\, dx = \begin{cases} \frac{\pi}{2} J_0(a\sqrt{p^2-b^2}) & 0<b<p \\ 0 & b>p>0 \end{cases} \quad a>0.$$

$$= \frac{i\delta_{ab}}{2\pi^2}\frac{1}{r}\frac{\partial}{\partial r}\begin{cases} \frac{\pi}{2} J_0\left(m\sqrt{|t-t'|^2-|\vec{x}-\vec{y}|^2}\right) sg(t-t') & 0<|\vec{x}-\vec{y}|<|t-t'| \\ 0 & |\vec{x}-\vec{y}|>|t-t'|>0. \end{cases}$$



$$= \delta_{ab} \frac{i}{4\pi} sg(t) \frac{1}{r}\frac{\partial}{\partial r} J_0\left(m\sqrt{t^2-r^2}\right) \qquad 0<r<t.$$

putting $t'=0$.

$$\frac{\partial}{\partial r} J_0\left(m\sqrt{t^2-r^2}\right) = \frac{m(+\cancel{2}r)}{\cancel{2}\sqrt{t^2-r^2}}\left(+ J_1\left(m\sqrt{t^2-r^2}\right)\right)$$

$$= \delta_{ab} \frac{i}{4\pi} \frac{m}{\sqrt{t^2-r^2}} \frac{1}{\cancel{r}} J_1\left(m\sqrt{t^2-r^2}\right) sg(t-t')$$

$$\boxed{[\phi_a(\vec{x},t), \phi_b^{\dagger}(\vec{y},t')] = \delta_{ab} \frac{i}{4\pi} sg(t-t') \frac{m}{\sqrt{(t-t')^2-|\vec{x}-\vec{y}|^2}} J_1\left(m\sqrt{(t-t')^2-|\vec{x}-\vec{y}|^2}\right)}$$

inside light-cone.

$$s = \sqrt{t^2-r^2}$$

$$\frac{1}{r}\frac{\partial}{\partial r} J_0(ms) = \frac{1}{r}\frac{\partial s}{\partial r}\frac{\partial}{\partial s} J_0(ms) = \frac{1}{r}\frac{-2r}{\sqrt{t^2-r^2}} = -\frac{2m}{\sqrt{t^2-r^2}}\frac{\partial}{\partial(ms)} J_0(ms)$$

$$-\frac{2m}{\sqrt{t^2-m^2}}\delta(ms) \qquad J_0(0)=1$$

$$= -\frac{2}{s}\delta(s)$$

$$[\ ,\ ] = \frac{i}{4\pi} sg(t-t')\,\delta_{ab}\,\frac{m}{s} J_1(ms) + \delta_{ab}\frac{i}{4\pi} sg(t-t')\left(-\frac{2}{s}\right)\delta(s)$$

$$\delta(s) = \delta\left(\sqrt{t^2-m^2}\right) = \delta\left(\sqrt{\xi}\right) = \frac{\delta(\xi)}{1/2\sqrt{\xi}} = 2s\,\delta(\xi) \qquad -\frac{2}{\cancel{s}}\,2\cancel{s}\,\delta(t^2-r^2)$$

$$\boxed{[\ ,\ ] = \frac{i}{4\pi} sg(t-t')\,\delta_{ab}\,\frac{m}{s} J_1(ms) \mp \frac{i}{\pi}\delta_{ab}\, sg(t-t')\,\delta(t^2-r^2)} \qquad t\geq r$$

Important to note that, in expressions such as

$$\int \frac{d^3k}{2\omega_k} e^{ikx}$$

$\frac{d^3k}{2\omega_k}$ is Lorentz invariant.

Indeed, obvious invariance under rotations. Consider a boost:

$$\omega' = ch\beta\, \omega - sh\beta\, k_1 \qquad \Big\|$$
$$k'_1 = ch\beta\, k_1 - sh\beta\, \omega$$

Change of variables is $\quad k'_1 = ch\beta\, k_1 - sh\beta \sqrt{k_1^2 + k_\perp^2 + m^2}$

$$dk'_1 = ch\beta\, dk_1 - \frac{sh\beta}{\cancel{2}\omega} \cancel{2} k_1\, dk_1 = \frac{ch\beta\,\omega - sh\beta\, k_1}{\omega}\, dk_1$$

$$ch\beta\,\omega - sh\beta\, k_1 = \omega' \quad \Rightarrow \quad \frac{dk'_1}{\omega'} = \frac{dk_1}{\omega} \quad \rightarrow \quad \frac{d^3k}{\omega} = \frac{d^3k'}{\omega'}$$

we can check this.

$$\omega' = (k_1'^2 + \underbrace{k_\perp^2 + m^2}_{\omega^2 - k_1^2})^{1/2} = (\underbrace{ch^2\beta\, k_1^2 + sh^2\beta\, \omega^2 - 2 sh\beta\, ch\beta\, k_1 \omega} + \omega^2 - k_1^2)^{1/2}$$

$$= (sh^2\beta\, k_1^2 + ch^2\beta\, \omega^2 - 2 sh\beta\, ch\beta\, k_1\omega)^{1/2}$$

$$= ch\beta\, \omega - sh\beta\, k_1 \quad \checkmark$$

$$\int_0^\infty \frac{\sin\left(p\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}} \cos bx \, dx =$$

$x = a\,\mathrm{sh}\,\xi \longrightarrow \sqrt{x^2+a^2} = a\,\mathrm{ch}\,\xi$

$$= \int_0^\infty \frac{\cancel{a\,\mathrm{ch}\,\xi}\, d\xi}{\cancel{a\,\mathrm{ch}\,\xi}} \sin(pa\,\mathrm{ch}\,\xi) \cos(ba\,\mathrm{sh}\,\xi).$$

$$= \int_0^\infty \sin\left(\frac{pa}{2}(e^\xi + e^{-\xi})\right) \cos\left(\frac{ba}{2}(e^\xi - e^{-\xi})\right) d\xi$$

$s(\alpha+\beta) = s\alpha\, c\beta + s\beta\, c\alpha$

$s(\alpha-\beta) = s\alpha\, c\beta - s\beta\, c\alpha$

$s\alpha\, c\beta = \frac{1}{2}\left(s(\alpha+\beta) + s(\alpha-\beta)\right)$

$$s\left(\frac{pa}{2}e^\xi + \frac{pa}{2}e^{-\xi}\right) \cos\left(\frac{ba}{2}(e^\xi - \frac{ba}{2}e^{-\xi}\right) =$$

$$= \frac{1}{2}\left(s\left(\left(\frac{pa}{2}+\frac{ba}{2}\right)e^\xi + \left(\frac{pa}{2}-\frac{ba}{2}\right)e^{-\xi}\right) + s\left(\left(\frac{pa}{2}-\frac{ba}{2}\right)e^\xi + \left(\frac{pa+ba}{2}\right)e^{-\xi}\right)\right)$$

Define $P_+ = (p+b)\,a \qquad P_- = (p-b)\,a$

$\xi \to -\xi$

$$= \frac{1}{2}\left(s\left(\frac{P_+}{2}e^\xi + \frac{P_-}{2}e^{-\xi}\right) + s\left(\frac{P_-}{2}e^\xi + \frac{P_+}{2}e^{-\xi}\right)\right)$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} d\xi\, \sin\left(\frac{P_+}{2}e^\xi + \frac{P_-}{2}e^{-\xi}\right) = \frac{1}{2}\int_{-\infty}^{\infty} d\xi\, \sin\left(\frac{P_+}{2}e^\xi\right) \cos\left(\frac{P_-}{2}e^{-\xi}\right) +$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} d\xi\, \cos\left(\frac{P_+}{2}e^\xi\right) \sin\left(\frac{P_-}{2}e^{-\xi}\right)$$

$u = e^{\xi} \qquad u: 0 \to \infty \qquad du = e^{\xi} d\xi$ (27)

$$\int_0^\infty \frac{\sin(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}} \cos bx \, dx = \frac{1}{2}\int_0^\infty \frac{du}{u} \sin\left(\frac{p_+}{2}u\right)\cos\left(\frac{p_-}{2u}\right) +$$

$$+ \frac{1}{2}\int_0^\infty \frac{du}{u} \cos\left(\frac{p_+}{2}u\right)\sin\left(\frac{p_-}{2u}\right)$$

$$\left(p_- = sg(p_-)\,|p_-| \right)$$

$$= \frac{1}{2}\int_0^\infty \frac{du}{u}\sin\left(\frac{p_+}{2}u\right)\cos\left(\frac{|p_-|}{2u}\right) + \frac{1}{2}\int_0^\infty \frac{du}{u}\cos\left(\frac{p_+}{2}u\right) sg(p_-) \sin\left(\frac{|p_-|}{2u}\right)$$

$$V = \frac{p_+ u}{2} \qquad u = 2V/p_+ \qquad\qquad V = \frac{2u}{|p_-|}$$

$$= \frac{1}{2}\int_0^\infty \frac{dV}{V}\sin V \cos\left(\frac{p_+|p_-|}{4V}\right) + \frac{1}{2} sg(p_-)\int_0^\infty \frac{dV}{V}\sin V\cos\left(\frac{p_+|p_-|}{4V}\right)$$

if $sg(p_-) = -1 \Rightarrow$ <u>gives 0</u> $\qquad p-b<0 \Rightarrow 0$

if $sg(p_+) = +1 \Rightarrow$ they add up.

$$\int_0^\infty \frac{\sin(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}}\cos bx\,dx = \begin{cases} 0 & \text{if } p<b \\ \displaystyle\int_0^\infty \frac{dV}{V}\sin V\cos\left(\frac{|p_+p_-|}{4V}\right) = \frac{\pi}{2} J_0\left(\sqrt{\underbrace{p_+p_-}_{a^2(p^2-b^2)}}\right) & \text{if } p>b \end{cases}$$

standard formula for J_0

Same property for retarded & advanced prop.

$$(\underset{\uparrow_x}{\partial_\mu}\partial^\mu + m^2)\Big(\theta(t_x-t_y)\,[\phi_a(\vec{x},t_x),\phi_b^\dagger(\vec{y},t_y)]\Big) =$$

$$= \partial_{t_x}\delta(t_x-t_y)\,[\phi_a(\vec{x},t_x),\phi_b^\dagger(\vec{y},t_y)] +$$

$$+ 2\delta(t_x-t_y)\,[\partial_{t_x}\phi_a(\vec{x},t_x),\phi_b^\dagger(\vec{y},t_y)] +$$

$$+ \theta(t_x-t_y)\,[\underbrace{\partial_{t_x}^2\phi_a(\vec{x},t_x) - \partial_{\vec{x}}^2\phi_a(\vec{x},t_x) + m^2\phi_a(\vec{x},t_x)}_{0},\phi_b^\dagger(\vec{y},t_y)]$$

Let's check what this is by integrating against a function $F(t_x)$

$$\int dt_x\,\Big(\underset{\uparrow \text{by parts}}{\partial_{t_x}}\delta(t_x-t_y)\,[\phi_a(\vec{x},t_x),\phi_b^\dagger(\vec{y},t_y)] + 2\delta(t_x-t_y)\,[\partial_{t_x}\phi_a,\phi_b^\dagger(\vec{y},t_y)]\Big)F(t_x)$$

$$= -\partial_{t_x}\Big([\phi_a(\vec{x},t_x),\phi_b^\dagger(\vec{y},t_y)]\,F(t_x)\Big)\Big|_{t_x=t_y} + 2\,[\partial_{t_x}\phi_a,\phi_b^\dagger(t_y)]\,F(t_y)\Big|_{\uparrow t_x=t_y}$$

$$= -\partial_{t_x}F(t_x)\,\underbrace{[\phi_a(\vec{x},t_x),\phi_b^\dagger(\vec{y},t_y)]\Big|_{t_x=t_y}}_{0 \text{ if } t_x=t_y} + \underbrace{[\underbrace{\partial_{t_x}\phi_a}_{\Pi_a},\phi_b^\dagger(t_y)]}_{-i\delta^{(3)}(\vec{x}-\vec{y}) \text{ if } t_x=t_y}\,F(t_y)\Big|_{\uparrow t_x=t_y}$$

$$= -i\delta^{(3)}(\vec{x}-\vec{y})F(t_y) \leftarrow \text{same as } \int dt_x\,\delta(t_x-t_y)F(t_x)$$

$\Rightarrow$

$$(\partial^2+m^2)D_R(x-y) = (\partial^2+m^2)\Big(\theta(t_x-t_y)\,[\phi_a(x),\phi_b^\dagger(y)]\Big) =$$

$$= -i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta(t_x-t_y) = -i\,\delta^{(4)}(x-y)$$