∘) High energy → low energy.

like EFT → less parameters.
finite #
ρ, etc should appear.

U.V. cut-off.

→) map out space of allowed S-matrices
→ find points distinguish vertices.

Paulos Penedones
Toledo van Rees
Vieira

$\pi^0 + \pi^0 \to \pi^0 + \pi^0$

$$F(s,t,u) = f_0 + \int_4^\infty dx \, K(s,t,u;x)\, \sigma(x) + \iint dx\,dy\, K(s,t,u;x,y)\, \rho(x,y)$$

$$K(s,t,u;x) = \frac{1}{\pi}\left(\frac{1}{x-s} + \frac{1}{x-t} + \frac{1}{x-u}\right)$$

$$K(s,t,u;x,y) = \frac{1}{\pi^2}\left(\frac{1}{(x-s)(y-t)} + \frac{1}{(x-t)(y-u)} + \frac{1}{(x-t)(y-u)}\right)$$

$$f_\ell(s) = \frac{1}{4}\int_{-1}^{1} d\mu \, P_\ell(\mu)\, F(s^+, t, u)$$

$$t, u = -\frac{(s-4)}{2}(1 \pm \mu)$$

$$h_\ell(s) = \pi\sqrt{\frac{s-4}{s}}\, f_\ell(s)$$

$$S_\ell = 1 + i h_\ell(s) \qquad |S_\ell| \leq 1 \quad ; \quad |h_\ell(s)|^2 \leq 2\,\mathrm{Im}\, h_\ell(s)$$

$$F(s^+, t) = \sum_{\ell\ \mathrm{even}} \int_4^\infty ds \, \frac{2}{\pi}\sqrt{\frac{s}{4-s}}\,(2\ell+1)\, h_\ell(s)\, P_\ell\left(1 + \frac{2t}{s-4}\right)$$

②

$$\sum_{n=0}^{\infty} t^n P_n(z) = \frac{1}{\sqrt{1-2zt+t^2}} \qquad |z| \leq \frac{1+t^2}{2t}$$

$$\sum_{l=0}^{\infty} a_l P_l(\mathrm{ch}\,\mu) < \infty \qquad \lim_{l\to\infty} |a_l|^{1/l} = e^{-\alpha} \qquad |\mu| < \alpha$$

$$\sum_{l=0}^{\infty} (2l+1)\, h_l(s)\, P_l(\mathrm{ch}\,\mu) \qquad \mathrm{ch}\,\mu = 1 + \frac{2t}{s-4}$$

$$\mathrm{ch}\,\mu < \frac{s+4}{s-4} = \mathrm{ch}\,\alpha_1(s)$$

$$\lim_{l\to\infty} |h_l(s)|^{1/l} \leq e^{-\alpha_1(s)} \qquad |h_l(s)| \sim \left(\frac{\sqrt{s}-2}{\sqrt{s}+2}\right)^{l} - \vec{\Lambda}_l(s)$$

$$\sum_{l=0}^{\infty} (2l+1)\, \mathrm{Im}\, h_l(s)\, P_l(\mathrm{ch}\,\mu) = \sum_{l=0} (2l+1) \frac{1}{2} |h_l(s)|^2 P_l(\mathrm{ch}\,\mu)$$

$$|\mu| < 2\alpha_1(s)$$

$$= \sigma(s) + \int_4^{\infty} dy\, \rho(s,y) \left(\frac{1}{y-t} + \frac{1}{y-u} \right)$$

$$\rho(s,y) = 0 \qquad y < t_2^+(s) = 16 + \frac{64}{s-4}$$

No particle production

$$\rho(s,y) = 0 \quad s > 4 \quad ; \quad 4 < y < 16$$

crossing $\rho(s,y) = 0 \quad \forall y > 4 \quad 4 \leq s < 16 \Rightarrow \mathrm{Im}\, h_l(s) = 0$

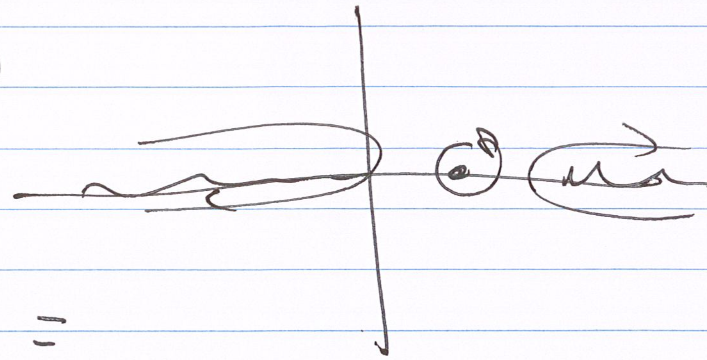
$$2\, \mathrm{Im}\, h_l(s) \geq |h_l(s)|^2 \Rightarrow h_l(s) = 0 \quad l \geq 2.$$

free theory

2d Dual

From generalized disp. relations.

$$F = \sum_{a=1,\pm} n_a S_a(s_0)$$



$$S_a(s_0) = \frac{1}{2\pi i}\oint_C \frac{S_a(s)}{s-s_0} =$$

$$= \frac{1}{2\pi i}\left[\int_{4m^2}^{\infty} + \int_{-\infty}^{0}\right] \Delta S_a \, ds$$

$$F = \frac{1}{2\pi i}\oint_C K_a(s)\, S_a(s) = \frac{1}{2\pi i}\left[\int_4^{\infty} + \int_{-\infty}^{0}\right] \Delta(K_a S_a)\, ds$$

$$= \frac{2}{\pi}\int_4^{\infty} \mathrm{Im}\left(K_a(s^+)\, S_a(s^+)\right) ds \le \frac{2}{\pi}\int_4^{\infty} |K_a S_a| \le \frac{2}{\pi}\int |K_a|$$

$$\max_{\{S_a\}} \sum n_a S_a(s_0) \le \min_{\{K_a\}} \frac{2}{\pi}\int_4^{\infty} \sum |K_a|\, ds.$$

$$\boxed{K_a = \frac{2 i n_a}{(s-2)\sqrt{s}\sqrt{4-s}}}$$

simplest $K_a = \frac{i n_a}{d\theta}$ $K_a = \frac{n_a}{s-2}\,\frac{d\theta}{ds}$ $\frac{i n_a}{d\theta}$,

for $\boxed{s_0 = 2}$ $s = 4\,\mathrm{ch}^2\theta/2$

Duality gap closes $\Rightarrow$ $S_a = e^{i\varphi_a} \Rightarrow K_a = |K_a|\, e^{-i\varphi_a}$

$$\Rightarrow \boxed{S_a = \frac{\bar{K}_a}{|K_a|}}$$

(4d) use 2d?

$$F(s) = f(s, t=0, u=4-s) \qquad \sum |u_i v_i| \le \sqrt{\sum u_i^2}\sqrt{\sum v_i^2}$$

$F(s) = f(4-s)$ like 2d!

Chung - Mau '76

$$F(s_0) = \frac{2}{\pi}\int_4^\infty \mathrm{Im}\left[F(s^+) K(s^+, s_0)\right]$$

$$|F(s_0)| \le \sum_\ell \int_4^\infty ds \underbrace{\frac{2}{\pi}\sqrt{\frac{s}{4-s}}(2\ell+1)\sqrt{2\,\mathrm{Im}\,h_\ell}\,|K(s', s_0)|}_{u_i v_i}$$

$$a_2 = \sum_\ell \int_4^\infty \underbrace{\frac{4}{15\pi s^3}\sqrt{\frac{s}{4-s}}(2\ell+1)\,2\,\mathrm{Im}\,h_\ell(s) P_\ell\left(1+\frac{2x^4}{s-4}\right)}$$

$$|F(s_0)| \le \sqrt{a_2}\min_{\{K\}}\overbrace{\sqrt{\sum_\ell \int_4^\infty ds \frac{15}{\pi}\sqrt{\frac{s}{4-s}}\frac{(2\ell+1)}{P_\ell\left(1+\frac{2}{s-4}\right)} s^3 |K(s', s_0)|^2}}^{u_i^2}$$

$$F(s,t) = F(s_0, t) + \frac{(s-s_0)(s-u_0)}{\pi}\int_4^\infty ds' A(s', t)\frac{2s'-4+t}{(s'-s)(s'-u)(s'-s_0)(s'-u_0)}$$

$$|F(s,t)| \le \underbrace{|F(0,t)|}_{\sim\sqrt{a_2}} + \frac{su}{\pi}\int_4^\infty ds' \frac{A(s', t)}{s'^3}\max_{s'}\frac{(2s'-4+t)\,s'^3}{(s'-s)(s'-u)s'(s'-4+t)}$$

$$|F(s,t)| \le |F(t,0)| + \frac{16\,su\,(4+t)}{\pi(4-s)(4-u)t}\frac{15}{4}a_2 \;;\; a_2 = \frac{4}{15}\int_4^\infty \frac{A(s', t)}{s'^3}ds'$$

Bound on $F(s_0, 0)$ Łukaszuk-Martin, 1967 (5)

$$\sum_i u_i^2 \leq \sum_i u_i v_i \leq \sqrt{\sum_i u_i^2}\sqrt{\sum_i v_i^2} \leq \sum_i v_i^2$$

no sub.

$$F(s_0, 4-s_0) = \frac{1}{\pi}\int_4^\infty ds\, A(s, 4-s_0)\left[\frac{1}{s-s_0}+\frac{1}{s}\right]$$

$$= \int_4^\infty \sum_\ell \frac{2}{\pi^2}\sqrt{\frac{s}{s-4}}\,(2\ell+1)\,\mathrm{Im}\,h_\ell(s)\,P_\ell\left(1+\frac{2(4-s_0)}{s-4}\right)\cdot\left[\frac{1}{s-s_0}+\frac{1}{s}\right]$$

u_i^2

$$|F(s_0)| \leq \sum_\ell \int_4^\infty ds\,\frac{2}{\pi}\sqrt{\frac{s}{s-4}}\,(2\ell+1)\sqrt{2\,\mathrm{Im}\,h_\ell(s)}\,|K(s, s_0)|$$

$u_i v_i$

$$0 \leq F(s_0, 0) \leq \int_4^\infty ds \sum_\ell \frac{2}{\pi}\sqrt{\frac{s}{s-4}}\,\frac{2\ell+1}{P_\ell\left(1+\frac{2(4-s_0)}{s-4}\right)}\,|K(s)|^2\,\frac{s(s-s_0)}{2s-s_0}$$

v_i^2

Dual from conic.

$$F = \sum_n a_n \alpha_n \qquad a_2 = \sum_n a_{2n} \alpha_n \leq a_2^{max}$$

$$h_e(s_j) = \sum_n h_{ej,n} a_n \quad ; \quad h_e^A = \frac{h_e}{\alpha_e} \qquad h_e^B = \frac{h_e}{\alpha_e^2}$$

$$(\mathrm{Re}\, h_e^A)^2 + (\mathrm{Im}\, h_e^A)^2 \leq 2\, \mathrm{Im}\, h_e^B (s_j)$$

$$\vec{x}^2 \leq 2\alpha\beta \ ; \ \vec{y}^2 \leq 2\xi\eta \Rightarrow \vec{x}\cdot\vec{y} + \alpha\xi + \beta\eta \geq 0$$

$$\alpha, \beta, \xi, \eta \geq 0$$

$$\vec{x}\cdot\vec{y} + \alpha\beta + \beta\eta \geq 0 \ , \ \forall\, \vec{x}^2 \leq 2\alpha\beta \Rightarrow \vec{y}^2 \leq 2\xi\eta$$

$$L = \sum_n \alpha_n a_n + \underbrace{\lambda}_{\geq 0}\underbrace{(a_2^{max} - a_{2n}\alpha_n)}_{\geq 0} +$$

$$+ \sum_{ej} \mathrm{Re}\left[K_{ej}^A \sum_n h_{ej,n}^A \alpha_n\right] + K_{ej}^B \sum_n h_{ej}^B \alpha_n + \xi_{ej} \geq 0$$

$$\lambda \geq 0 \qquad |K_{ej}^A|^2 \leq 2 K_{ej}^B \xi_{ej}$$

$$\Theta_n = \alpha_n - \lambda a_{2n} + \sum_{ej} \mathrm{Re}\left[K_{ej}^A h_{ej,n}^A\right] + K_{ej}^B h_{ej,n}^B$$

$$L = \sum_n \Theta_n \alpha_n + \lambda a_2^{max} + \xi_{ej} \leq \|\Theta_n\| \, \|\alpha_n\| +$$

$$+ \lambda a_2^{max} + \xi_{ej}$$

$$G(\omega_1,\omega_2) = \frac{1}{\pi^2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{g(x,y)}{(x-\omega_1)(y-\omega_2)}$$

$$\Delta_{12} G(x,y) = G(x^+,y^+) - G(x^-,y^+) - G(x^+,y^-) + G(x^-,y^-)$$

$$= -4\, g(x,y)$$

$$\int_{-\infty}^{\infty} dx\; G(x^\pm,\omega_2) = 0 \quad ; \quad \int_{-\infty}^{\infty} dx\; G(\omega_1, y^\pm) = 0$$

$$G(\omega_1,\omega_2) = H(\omega_1,\omega_2)\, K(\omega_1,\omega_2)$$

$$\int_{-\infty}^{\infty} dx\,dy \left[\Delta_{12} H(x,y)\, K(x^-,y^+) - H(x^+,y^-)\, \Delta_{12} K(x,y)\right] = 0$$

$$\int_{-\infty}^{\infty} dx\,dy \Big(H(x^+,y^+) K(x^-,y^+) - H(x^-,y^+) K(x^-,y^+) - H(x^+,y^-) K(x^-,y^+) + H(x^-,y^-) K(x^-,y^+) - H(x^+,y^-) K(x^+,y^+) + H(x^+,y^-) K(x^-,y^+) + H(x^+,y^-) K(x^+,y^-) - H(x^+,y^-) K(x^-,y^-) \Big) = 0$$

$$H(s,t) = \frac{1}{\pi^2}\int_a dx \int_a dy \frac{\rho(x,y)}{(x-s)(y-t)}$$

$$K(s,t) = -\frac{1}{(s-s_0)(t-t_0)} + \frac{i}{\pi^2}\int_a^{\infty} dx \int_{a-x}^{\infty} dy \frac{\bar{k}(x,y)}{(s-x)(t-y)}$$

$$\int_{-\infty}^{\infty} dx \int_{4}^{\infty} dy \left[\left(-\frac{4}{\pi^2} \rho(x,y)\right) K(x; y^+) \right] -$$

$$- \int_{4}^{\infty} dx \int_{4-x}^{0} dy \left(-\frac{4i}{\pi^2} H(x^+, y^-) \bar{K}(x,y) \right) - \frac{\pi^2}{4} H(s_0, t_0) = 0$$

$$H(s_0, t_0) = \frac{1}{\pi^2} \int \rho(x,y) K(x; y^+) + \frac{i}{\pi^2} \int_{4}^{\infty} \left(\int_{4-x}^{0} H(x^+, y^-) \bar{K}(x,y) \right)$$

$$\| \rho \| \, \| K(x; y^+) \| \leq M$$

$$\sum (2\ell+1) P_\ell \left(1 + \frac{2y}{4-x}\right) h_\ell(x)$$

$$\sum_\ell k_\ell h_\ell$$

$$(-i S_\ell + i) k_\ell$$

$$S_\ell = 1 + i h_\ell$$

$$i (S_\ell - 1) h_\ell$$

$$\frac{1}{\pi^2} (S_0 h_\ell - k_\ell)$$

$$|h_\ell| - k_\ell$$

$$H(s, t) = \frac{2}{\pi} \sqrt{\frac{s}{s-4}} \sum_\ell (2\ell+1) h_\ell(s) P_\ell \left(1 + \frac{2t}{s-4}\right)$$

$$\int_{4}^{\infty} ds \int_{4-s}^{0} dt \, \frac{2}{\pi} \sqrt{\frac{s}{s-4}} \sum_\ell (2\ell+1) h_\ell(s) P_\ell \left(1 + \frac{2t}{s-4}\right) \bar{k}(s,t)$$

$$+ \sum_{\substack{\ell=0 \\ \ell \text{ even}}}^{\infty} h_\ell(s) k_\ell(s)$$