Infinitesimal gauge transf.

$$U = e^{i\theta_a t^a}$$

t^a basis for hermitian traceless matrices

$[t^a, t^b]$ is anti-hermitian an traceless

$$\Rightarrow \boxed{[t^a, t^b] = i f^{abc} t^c} \qquad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$

structure constants. $\quad f^{abc}$ totally antisymmetric

$$A_\mu = A_\mu^a t^a$$

$$D_\mu = \partial_\mu - i g A_\mu^a t^a$$

coupling constant. by rescaling A_μ.

$$\delta\psi = i\theta_a t^a \psi$$

$$\delta A_\mu^a \; ; \qquad A \to U A U^{-1} - \frac{i}{g}\partial_\mu U U^{-1}$$

$$\tilde{A} \simeq (1 + i\theta_a t^a) A (1 - i\theta_a t^a) - i\, i\, \partial_\mu \theta_a t^a + \dots$$

$$\simeq A + i\theta_a [t^a, A] + \frac{1}{g}\partial_\mu \theta_a t^a$$

$$\tilde{A}_\mu^a t^a \simeq A_\mu^a t^a + i\theta_a [t^a, t^b] A_\mu^b + \frac{1}{g}\partial_\mu \theta_a t^a = \left(A_\mu^a + \theta_c f^{cba} A_\mu^b + \frac{\partial_\mu}{g}\theta^a\right) t^a$$

$$\delta A_\mu^a = \frac{1}{g}\partial_\mu \theta^a \cancel{+} f^{abc}\theta^b A_\mu^c$$

$$F_{\mu\nu}^a t^a = \partial_\mu A_\nu^a t^a - \partial_\nu A_\mu^a - ig\underbrace{[t^a, t^b]}_{if^{abc} A^c} A_\mu^a A_\nu^b$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c$$

$\delta F_{\mu\nu}^a$.

$$F \Rightarrow U F U^{-1}$$

$$\delta F_{\mu\nu}^a = -f^{abc}\theta^b F_{\mu\nu}^c$$

Jacobi id $\quad [[t^a, t^b] t^c] + [[t^c t^a] t^b] + [[t^b t^c] t^a] = 0$

$$f^{abd} f^{dce} + f^{cad} f^{dbe} + f^{bcd} f^{dae} = 0.$$

<u>Summary</u>.

$$\delta \psi = i\theta_a t^a \psi$$

$$\delta A_\mu^a = \frac{1}{g}\partial_\mu \theta^a - f^{abc}\theta^b A_\mu^c$$

$$\delta F_{\mu\nu}^a = -f^{abc}\theta^b F_{\mu\nu}^c$$

$$D_\mu \psi = \partial_\mu \psi - ig A_\mu^a t^a \psi$$

Quantization of gauge theories

Objective:

.) Unitarity: positive definite norm states → probability interpretation

unitary S-matrix.

If we add extra polarizations we expect

$$[a_{\sigma_1}, a^{\dagger}_{\sigma_2}] = -\epsilon^{\mu}_{(\sigma_1)}\,\epsilon_{\mu(\sigma_2)}$$

for 3 space-like pol. this gives $\delta_{\sigma_1\sigma_2}$

But for time-like pol. we get

$$[a_0, a_0^{\dagger}] = -1$$

But $|1\rangle = a_0^{\dagger}|0\rangle$; $\langle 1|1\rangle = \langle 0|a_0\, a_0^{\dagger}|0\rangle =$

$$= \langle 0|[a_0, a_0^{\dagger}] + \underbrace{a_0^{\dagger} a_0|0\rangle}_{0} =$$

$$= -\langle 0|0\rangle = -1$$

$|1\rangle$ is a negative norm state!

Since we need to eliminate these states, things are ok but we need to be sure.

Once we eliminate some states unitarity means (2)
physically that those states cannot be produced
in collisions of physical states.

i.e. if we scatter physical polarizations, unphysical
ones are not produced.

∘) global symmetries are preserved, in particular
Lorentz symmetry.

We argued that this requires gauge inv.

Gauge invariance is not a physical symmetry so we
do not need to preserve it. However it is a useful
tool to have something analogous to gauge freedom.

∘) Renormalizability

$$S = \frac{1}{4g^2}\int d^4x \; (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) = \frac{1}{2g^2}\int d^4x \; \partial_\mu A_\nu \partial_0^\mu A_0^\nu - \partial_\mu A_\nu \partial^\nu A^\mu$$

there is no $\partial_0 A_0$ $\Rightarrow$ $\boxed{\Pi_0 = 0}$ constraint.

no time derivatives

eqn. for A_0 is a constraint: $D_\mu F^{\mu 0} = J^0 \Rightarrow \boxed{D_i F^{i0} = J^0}$

One way to implement the analogous of gauge inv. at the quantum level is the BRST quantization procedure

<u>Kinematics</u>

·) large space of states ~~with~~ (including negative norm states)

··) Q : fermionic operator satisfying:

$$Q^2=0 \quad ; \quad Q^\dagger = Q . \qquad Q_{BRST} : \text{BRST charge.}$$

···) Define physical states as those satisfying

$$Q|\text{phys}\rangle = 0$$

given any state $|\chi\rangle$; $Q|\chi\rangle$ is physical

since $$Q\, Q|\chi\rangle = Q^2|\chi\rangle = 0$$

they have zero norm:

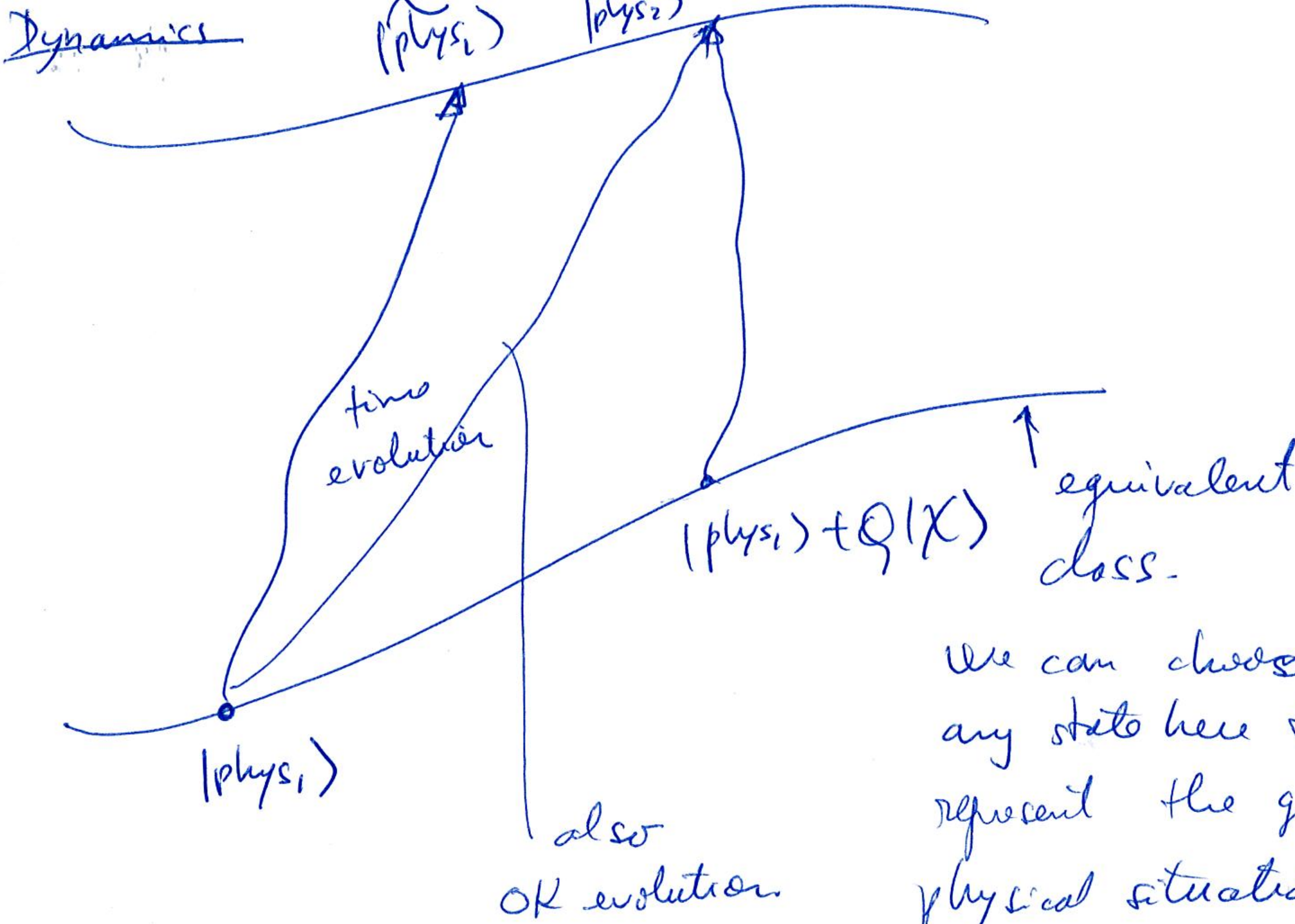
$$\langle\chi| Q^\dagger Q|\chi\rangle = \langle\chi|Q^2|\chi\rangle = 0 .$$

we define classes of equivalence

$$|\text{phys}_1\rangle \equiv |\text{phys}_2\rangle \iff |\text{phys}_2\rangle = |\text{phys}_1\rangle + Q|\chi\rangle$$

for some (generically unphysical) $|\chi\rangle$.

Dynamics



we can choose any state here to represent the given physical situation

$\partial_t |\psi\rangle = -i H |\psi\rangle$ $\boxed{[Q,H]=0}$ $Q|\psi\rangle = 0$ preserved.

Take $\tilde{H} = H + [Q,V]$ for any V.

$\partial_t |\tilde{\psi}\rangle = -i \tilde{H} |\psi\rangle$

$$|\psi(t+\delta t)\rangle = |\psi(t)\rangle - i H |\psi(t)\rangle \underbrace{-i\, QV|\psi\rangle}_{} + i \underbrace{VQ|\psi\rangle}_{0}$$

differs by a trivial state so it is an OK evolution

since $|\psi\rangle$ is physical

$\underline{\tilde{H} \equiv H}$ we can use both.

Norm should be positive definite

$$\langle \text{phys}_1 | \text{phys}_2 \rangle > 0 \qquad \text{unless} \quad |\text{phys}\rangle = Q|\chi\rangle$$

also

$$\langle \text{phys}_1 | \left(|\text{phys}_2\rangle + Q|\chi\rangle \right) =$$

$$= \langle \text{phys}_1 | \text{phys}_2 \rangle + \langle \text{phys}_1 | \overset{0}{\overleftarrow{Q}} | \chi \rangle$$

$$= \langle \text{phys}_1 | \text{phys}_2 \rangle$$ norm is BRST invariant.

Any operator $\mathcal{O} \to \mathcal{O} + [Q, \underset{\uparrow \text{ arbitrary}}{A}]$

$$\langle \psi_1 | \mathcal{O} | \psi_2 \rangle \; ; \quad \langle \psi_1 | \mathcal{O} + QA - AQ | \psi_2 \rangle =$$

$$= \langle \psi_1 | \mathcal{O} | \psi_2 \rangle$$

if both $|\psi_{1,2}\rangle$ are physical.

Also ghost number $= 0$.

BRST quantization provides us with the freedom to choose a representative state $|\psi\rangle + Q|\chi\rangle$ (kinematics) among an equivalence class

arbitrary

and also to choose a Hamiltonian $H + [Q, V]$ among an equivalence class.

This freedom can be use to great advantage since different choices make different properties manifest (unitarity, Lorentz inv. and renormalizability).

Q_{BRST} is a canonical transf.

$$\mathcal{L} = p\dot{q} - H$$

$$\mathcal{L} \rightarrow \mathcal{L} + \delta_Q I$$

BRST symmetry for non-abelian gauge theories.

A_μ ; matter fields: ψ, ϕ, --

c, $\bar{c}$ ghosts fermionic scalars in

$N_{gh}=1$, -1 the adjoint representation, hermitian.

B auxiliary bosonic field in the adjoint rep.

Intuition: fermionic ghosts cancel contributions of unphysical polarizations in loops.

$$\delta_{BRST}\phi_A = i[Q_{BRST}, \phi_A]_\pm \doteq i s\phi_A$$

notation.

$sA = D_\mu c = \partial_\mu c - i(A_\mu c - cA_\mu)$; $\delta A^a_\mu = \partial_\mu c^a + g f^{abc} A^b_\mu c^c$

$s\psi = i c \psi$ (gauge transf. with fermionic parameter)

$sc = i cc$	$\delta c^a = -\frac{1}{2} g f^{abc} c^b c^c$
$s\bar{c} = B$	$\delta \bar{c}^a = B^a$
$sB = 0$	$\delta B^a = 0.$

$\underline{N_{gh} = 0}$ for physical states.

$s^2 = 0 \qquad Q^2 = 0$

$$s(sA_\mu) = \partial_\mu(icc) - i D_\mu c c + i A_\mu icc + i icc A_\mu - $$
$$- i c D_\mu c$$

$$= i\partial_\mu c c + \underbrace{i c \partial_\mu c} - i D_\mu c c + i(iA_\mu c - icA_\mu)c$$
$$\underbrace{+ i i c A_\mu c + i ic cA_\mu} - i c D_\mu c$$
$$= \cancel{iD_\mu c c} + \cancel{i c D_\mu c} - \cancel{iD_\mu cc} - \cancel{icD_\mu c} = 0 \checkmark$$

$s(s\phi) = i icc\phi - ic\, ic\phi = 0 \checkmark$

$s(sc) = i(icc)c - ic\, icc = 0 \checkmark$

$s(s\bar{c}) = sB = 0 \checkmark$

$ssB = 0 \checkmark$

Take $\phi_{1,2}$ bosonic

$ss(\phi_1\phi_2) = s(s\phi_1\, \phi_2 + \phi_1 s\phi_2) =$

$= \underbrace{s^2\phi_1}_{0} \underbrace{- s\phi_1 s\phi_2 + s\phi_1 s\phi_2}_{0} + \underbrace{\phi_1 s^2\phi_2}_{0}$

Lagrangian: $\mathcal{L}[A_\mu, \partial_\nu A_\mu, \phi, \partial_\mu\phi]$ is

BRST inv. iff it is gauge invariant.

We can add $s\Psi$ for some local fermionic Ψ.

Choose a function $\mathcal{F}(A,\phi \dots)$, e.g. $\partial_\mu A^\mu$; or $n_\mu A^\mu$.

and take $\Psi = \int d^4x \, \mathrm{Tr}(\bar{c}\mathcal{F} + \frac{1}{2}\xi \bar{c} B)$

$$\delta\Psi = \int d^4x \; \mathrm{Tr}\left[B\mathcal{F} - \bar{c}\,\frac{\delta\mathcal{F}}{\delta A_\mu} D_\mu c + \frac{1}{2}\xi BB\right]$$

•) $\mathcal{F}_1 = \partial^\mu A_\mu$

$$\delta\Psi_1 = \int d^4x \; \mathrm{Tr}\left[B\,\partial^\mu A_\mu - \bar{c}\,\partial^\mu D_\mu c + \frac{1}{2}\xi BB\right]$$

$$= \int d^4x \; \mathrm{Tr}\left[B\,\partial^\mu A_\mu + \partial^\mu\bar{c}\, D_\mu c + \frac{1}{2}\xi BB\right]$$

••) $\mathcal{F}_2 = n^\mu A_\mu$

$$\delta\Psi_2 = \int d^4x \; \mathrm{Tr}\left[B\, n^\mu A_\mu - \bar{c}\, n^\mu D_\mu c + \frac{1}{2}\xi BB\right]$$

$$\mathrm{Tr}\left(B\mathcal{F} + \frac{1}{2}\xi BB\right) = \mathrm{Tr}\,\frac{\xi}{2}\Big(\underbrace{B + \frac{1}{\xi}\mathcal{F}}_{\tilde{B}}\Big)^2 - \frac{1}{2\xi}\mathrm{Tr}\,\mathcal{F}\mathcal{F}.$$

$$\mathcal{L}_1 = -\frac{1}{4g^2}\mathrm{Tr}F^2 - \frac{1}{2\xi}\mathrm{Tr}(\partial^\mu A_\mu)^2 + \mathrm{Tr}\,\partial^\mu\bar{c}\, D_\mu c + \frac{1}{2}\xi\,\mathrm{Tr}\tilde{B}^2$$

$$\mathcal{L}_2 = -\frac{1}{4g^2}\mathrm{Tr}F^2 - \frac{1}{2\xi}\mathrm{Tr}(n^\mu A_\mu)^2 + \mathrm{Tr}\,\underbrace{\bar{c}\, n^\mu D_\mu c}_{\bar{c}[n^\mu A_\mu,\, c]} + \frac{1}{2}\xi\,\mathrm{Tr}\tilde{B}^2$$

$\xi \to 0 \quad m^2_{(nA)} \to \infty$

goes away

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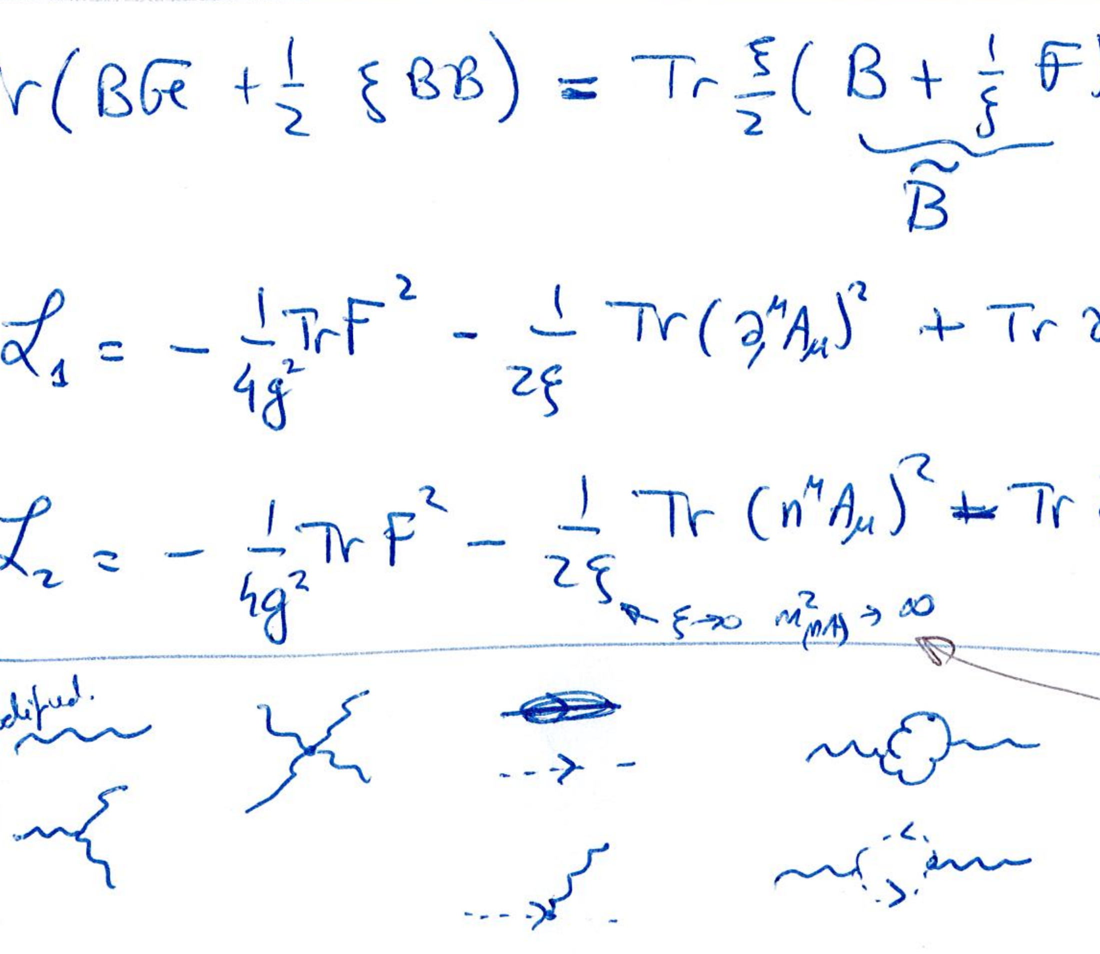
modified.



Also A_0 can be eliminated from the constraints, ⇒ physical gauge.

BRST current

$$j^\mu_{BRST} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} \delta \phi_A$$

all fields

Fermions?

$$\mathcal{L} = \psi_1 A \psi_2 \; - -$$

$$\frac{\delta \mathcal{L}}{\delta \psi_1} = A \psi_2 = 0 \qquad \frac{\delta \mathcal{L}}{\delta \psi_2} = - \psi_1 A = 0.$$

e.o.m. $$\partial_\mu \frac{\vec{\delta} \mathcal{L}}{\delta \partial_\mu \psi} - \frac{\vec{\delta} \mathcal{L}}{\delta \psi} = 0$$

$$\partial_\mu j^\mu = \partial_\mu \frac{\vec{\delta} \mathcal{L}}{\vec{\delta} \partial_\mu \phi_A} \delta \phi_A + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} \partial_\mu \delta \phi_A$$

$$= \frac{\vec{\delta} \mathcal{L}}{\vec{\delta} \phi_A} \delta \phi_A + \frac{\vec{\delta} \mathcal{L}}{\vec{\delta} \partial_\mu \phi_A} \partial_\mu \delta \phi_A = 0.$$

$$j^\mu_{BRST} = \delta \phi_A \frac{\vec{\delta} \mathcal{L}}{\vec{\delta} \partial_\mu \phi_A}$$

$$j^\mu_{BRST} = D_\nu c \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} + i c c \frac{\delta \mathcal{L}}{\delta \partial_\mu c} + B \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{c}}$$

$$= -D_\nu c \frac{1}{g^2} F^{\mu\nu} - i c c \partial^\mu \bar{c} + B D_\mu c$$

$$Q_{BRST} = \int d^3x \left[-\frac{1}{g^2} F^{0\nu} D_\nu c - i \dot{\bar{c}} c c + B D_0 c \right]$$

For Q_{BRST} to be hermitian c, $\bar{c}$ should be hermitian.

$$\boxed{Q_{BRST} = \int d^3x \frac{1}{g^2} c \underbrace{D_j F^{0j}}_{\text{constraints}} - i \dot{\bar{c}} c c + B D_0 c}$$

Free Lagrangian (quadratic in the fields).

We rescale $A_\mu \to g A_\mu$ to eliminate $\frac{1}{g^2}$.

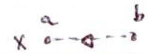
$$\mathcal{L}_1 = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} + \partial^\mu \bar{c}^a \partial_\mu c^a$$

$$= -\frac{1}{2} \partial_\mu A^a_\nu (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) - \frac{1}{2\xi} \partial_\mu A^\mu \partial_\nu A^\nu + \partial^\mu \bar{c} \partial_\mu c$$

$$S = -\frac{1}{2} \int \partial_\mu A^a_\nu \partial^\mu A^{\nu a} + \frac{1-1/\xi}{2} \int \partial_\mu A^{\mu a} \partial_\nu A^\nu + \int \partial^\mu \bar{c}^a \partial_\mu c^a.$$

$\xi = 1$ particularly simple. $\quad -\frac{1}{2}\int (\partial_\mu A_0^a)^2 + \int \underbrace{\frac{1}{2} (\partial_\mu A_i^a)^2}_{\text{like scalar field}}$

wrong sign.

 $= \frac{i\delta^{ab}}{p^2}$ $\quad\Big\|\quad$ $\langle c^a(x)\, \bar{c}^b(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}$ 

e.o.m.

$$+\partial_\mu \partial^\mu A^{a\nu} - (1-1/\xi)\, \partial_{\mu\nu} A^{a\mu} = 0$$

solutions $\epsilon_\nu e^{ikx}$ $\to$ $-k^2 \epsilon_\nu + (1-1/\xi)\, k_\nu k^\mu \epsilon_\mu = 0$

k^ν: $\quad -k^2 (k\epsilon) + (1-1/\xi)\, k^2 (k\epsilon) = 0 \Rightarrow \underline{k^2 (k\epsilon) = 0}$

a) $k^2 = 0$ or $(k\epsilon) = 0$ $\quad\Big|\quad$ if $k^2 = 0 \Rightarrow k_\nu (k\epsilon) = 0 \Rightarrow (k\epsilon) = 0$

if $(k\epsilon) = 0 \Rightarrow k^2 \epsilon_\nu = 0 \Rightarrow \boxed{k^2 = 0}$

So, classical solutions are such that.

$k^2 = 0$ and $(k\epsilon) = 0$. $\to$ not solutions

polarizations $\epsilon_\mu^{(1)}$; $\epsilon_\mu^{(2)}$; k_μ , $\tilde{k}_\mu$

$k_\mu \to (\omega, \vec{k})$ $\quad$ $\tilde{k}_\mu \to (\omega, -\vec{k})$ $\quad$ $\tilde{k} \cdot k \neq 0$

$\epsilon_\mu^{(1)}$; $\epsilon_\mu^{(2)}$; k_μ $\to$ solutions

$k_\mu \to$ not physical

We can use canonical quantization or use $\langle T A_\mu(x) A_\nu(x) \rangle$ satisfies Green's function equation:

$$\boxed{\partial_\mu \partial^\mu G^{\nu\alpha}_{ab}(x-y) - (1 - 1/\xi)\, \partial_\mu\, G^{\mu\alpha}_{ab}(x-y) = i\delta^{\nu\alpha}(x-y)\delta_{ab}}$$

in momentum space

$$-k^2 G^{\nu\alpha}(k) + (1 - 1/\xi)\, k_\mu k_\nu G^{\mu\alpha} = i\eta^{\nu\alpha}$$

$$k_\nu \quad \cancel{-k^2 k_\nu G^{\nu\alpha} + (1-1/\xi) k^2 k_\mu G^{\mu\alpha}} = i k^\alpha$$

$$-1/\xi\, k_\mu G^{\mu\alpha} = \frac{i k^\alpha}{k^2} \qquad k_\mu G^{\mu\alpha} = -i\xi \frac{k^\alpha}{k^2}$$

$$G^{\mu\alpha} = A\,\eta^{\mu\alpha} + B\, k^\mu k^\alpha$$

$$k_\mu G^{\mu\alpha} = A k^\alpha + B k^2 k^\alpha$$

$$A + Bk^2 = -i\xi/k^2$$

$$B = -\frac{i\xi}{k^4} - \frac{A}{k^2}$$

$$G^{\mu\alpha} = -i\xi \frac{k^\mu k^\alpha}{k^4} + A\left(\eta^{\mu\alpha} - \frac{k^\mu k^\alpha}{k^2}\right)$$

(14)

$$\cancel{+i\xi \frac{k^\nu k^\alpha}{k^2}} - Ak^2\eta^{\mu\alpha} + A k^\mu k^\alpha + (1-1/\xi)k^\nu\left(-i\xi \frac{k^\alpha}{k^2}\right)$$

$$= -A k^2 \eta^{\mu\alpha} + A k^\nu k^\alpha + i \frac{k^\nu k^\alpha}{k^2} = i\eta^{\nu\alpha}$$

$$-i/k^2$$

$$G^{\mu\alpha} = -\frac{i}{k^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\alpha}{k^2}\right) - i\xi \frac{k^\mu k^\alpha}{k^4}$$

$$\boxed{G^{\mu\alpha}_{ab} = \left(-\frac{i}{k^2}\eta^{\mu\nu} + i(1-\xi)\frac{k^\mu k^\alpha}{k^4}\right)\delta^{ab}}$$

$$= -\frac{i}{k^2+i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)\delta^{ab}$$

ξ-dependent $\Rightarrow$ should not contribute

$\Rightarrow$ prescription does not matter.

<u>Take $\xi = 1$</u>

$$A^a_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left[a^{\sigma a}_k \epsilon_{\sigma\mu} e^{ikx} + a^{\dagger\sigma a}_k \epsilon^*_{\sigma\mu} e^{-ikx} \right]$$

$$c^a(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left[c^a_k e^{ikx} + c^{\dagger a}_k e^{-ikx} \right]$$

$$\bar{c}^a(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega}} \left[\bar{c}^a_k e^{ikx} + \bar{c}^{\dagger a}_k e^{-ikx} \right]$$

$c, \bar{c}$ hermitian.
$\Rightarrow c^a, \bar{c}^a$ real.

notice also $\Pi_c = \dot{\bar{c}}$

so anti commutations

$$\{c^\dagger, \bar{c}\} = \delta_{k-k'}$$
$$\{c, \bar{c}^\dagger\} = \delta_{k-k'}.$$

$\delta A_\mu = \partial_\mu c$ free theory $\delta \bar{c} = B = \partial_\mu A^\mu$.

$[Q, a^{\dagger\sigma}_k \epsilon^*_{\sigma\mu}] = k_\mu c^\dagger_k$ $\quad [Q, \bar{c}^\dagger_k] = (k\epsilon^*) a^\dagger_k$

~~$Q a^{(1)\dagger}_k \epsilon_\mu |0\rangle = ((k\epsilon_1) c^\dagger |0\rangle = 0$~~ $\quad Q a^{\dagger\sigma}_k \epsilon_{\sigma\mu} |0\rangle = k_\mu c^\dagger_k |0\rangle$

same (?) on $\epsilon^{(1)\mu}$. $\quad Q a^{\dagger\sigma}_k |0\rangle = (k\epsilon_1) c^\dagger_k |0\rangle = 0$

$\epsilon^{(1)}, \epsilon^{(2)}$ are physical.

$$Q\, a_k^{\dagger(3)} k_\mu |0\rangle = k_\mu c_k^\dagger |0\rangle$$

$\tilde{k}_\mu \cdot k_\mu \neq 0$ $\quad Q\, a_k^{\dagger(3)} |0\rangle = c_k^\dagger |0\rangle \neq 0$

$\boxed{k_\mu \rightarrow \text{not physical}}$

good! use it for gauge transf.!

also $c_k^\dagger |0\rangle$ trivial

$$Q\, a_k^{\dagger(4)} \tilde{k}_\mu |0\rangle = k_\mu c_k |0\rangle$$

$\tilde{\epsilon}^\mu \ldots$ $\quad Q\, a_k^{\dagger(4)} |0\rangle = 0\,,$ $\quad \tilde{k}_\mu \rightarrow$ physical.

but trivial

$$Q\left(\bar{c}_k^\dagger |0\rangle\right) = \sum_\sigma (K^\mu \epsilon_\mu^\sigma)\, a_k^{\dagger\sigma} |0\rangle$$

$$= (K^\mu \tilde{k}_\mu)\, a_k^{\dagger(4)} |0\rangle$$

$a_k^{\dagger(4)} |0\rangle = Q|X\rangle \rightarrow$ trivial

adding this polarization does not change anything.

also $\bar{c}_k^\dagger |0\rangle$ not physical

So $\epsilon^{(1)}, \epsilon^{(2)}$ physical & not trivial !!

$\bar{c}_k^\dagger |0\rangle\,;\ k_\mu \leftarrow$ not physical.

$c_k^\dagger |0\rangle\,;\ \tilde{k}_\mu \leftarrow$ trivial. ✓

(17) (d)

$$\mathcal{L} = -\frac{1}{4g^2}\,\mathrm{Tr}\,F^2 + \mathrm{Tr}\,B\,\partial^\mu A_\mu + \mathrm{Tr}\,\partial^\mu \bar{c}\,D_\mu c + \frac{1}{2}\xi\,\mathrm{Tr}\,BB$$

$$-\frac{1}{2g^2} F_{0j}F^{0j} - \frac{1}{4g^2} F_{ij}F^{ij}$$

$$-\frac{1}{2g^2}\left(\partial_0 A_j - \partial_j A_0 - i[A_0, A_j]\right)F^{0j} - \frac{1}{4g^2}F_{ij}F^{ij}$$

$$\frac{\delta \mathcal{L}}{\delta \partial_0 A_j} = -\frac{1}{g^2}F^{0j} = \Pi_j\,; \quad \frac{\delta \mathcal{L}}{\delta \partial_0 A_0} = +B\,; \quad \frac{\vec{\delta}\mathcal{L}}{\delta \dot{\bar{c}}} = +D_0 c$$

By the way in gauge $n^\mu A_\mu = 0$; $n^\mu = \hat{z}$
$D_i F^{0i} = \rho \Rightarrow -\partial_3^2 A_0 = \rho - \partial_\ell F^{0\ell} + i[A_\ell, F^{0\ell}]$
$\ell = 1,2$
$\Rightarrow$ solve for A_0.

$$\frac{\vec{\delta}\mathcal{L}}{\delta \partial_0 c} = -\partial_0 \bar{c} \qquad \frac{\delta \mathcal{L}}{\delta B} = 0$$

$$\Pi_j = \frac{1}{g^2}F_{0j} \qquad \Pi_0 = +B \qquad \Pi_{\bar{c}} = +D_0 c \qquad \Pi_c = -\partial_0 \bar{c} \qquad \Pi_B = 0$$

eliminate B

$$\cancel{H = \frac{1}{g^2}F_{0j}} \quad H = \Pi_j\,\partial_0 A_j + \cancel{\Pi_0\,\partial_0 A_0} + \partial_0\bar{c}\,\Pi_{\bar{c}} + \partial_0 c\,\Pi_c$$

$B = \Pi_0$

$$+\frac{1}{2g^2}F_{0j}F^{0j} + \frac{1}{4g^2}F_{ij}F^{ij} \cancel{+ B\,\partial_0 A_0} + B\,\partial^j A_j +$$

$$+\partial_0\bar{c}\,D_0 c - \partial^j\bar{c}\,D_j c - \frac{1}{2}\xi\,BB$$

Recall constraint $D_i\Pi^i = 0$ (pure YM)

(18) (b)

$$\Pi_j = \frac{1}{g^2}\left(\partial_0 A_j - \partial_j A_0 - i[A_0, A_j]\right)$$

$$\partial_0 A_j = g^2 \Pi_j + \partial_j A_0 + i[A_0, A_j]$$

$$= g^2 \Pi_j + \partial_j A_0 - i[A_j, A_0] = g^2 \Pi_j + D_j A_0$$

$$\Pi_{\bar c} = +\partial_0 c - i[A_0, c] \Rightarrow \partial_0 c = +\Pi_{\bar c} + i[A_0, c]$$

$$H = \underbrace{g^2 \Pi_j \Pi_j} + \underbrace{\Pi_j D_j A_0} + \cancel{\Pi_c \Pi_{\bar c}} + \underbrace{\Pi_{\bar c} \Pi_c} + \underbrace{i[A_0, c]\Pi_c}$$

$$-\frac{1}{2} g^2 \Pi_j \Pi_j + \frac{1}{4g^2} F_{ij} F^{ij} \underbrace{- \Pi_0 \partial^j A_j} - \cancel{\Pi_c \Pi_{\bar c}} +$$

$$\underbrace{+ \partial^j \bar c D_j c} + \frac{1}{2} \xi \Pi_0^2$$

$$\boxed{\begin{aligned} H = {} & \frac{1}{2} g^2 \Pi_j \Pi_j + \frac{1}{4g^2} F_{ij} F^{ij} \\ & + \Pi_{\bar c} \Pi_c + \Pi_j D_j A_0 + i[A_0, c]\Pi_c \\ & - \Pi_0 \partial^j A_j + \partial^j \bar c D_j c + \frac{1}{2} \xi \Pi_0^2 \end{aligned}}$$

$$Q = \int -\frac{1}{g^2} F^{0j} D_j c - i\, \bar{c}\, c c + B D_0 c$$

$$Q = \int \Pi_j D_j c + i\, \Pi_c\, c c + \Pi_0\, \Pi_{\bar{c}}$$

$$= \cancel{-\int c D_j \Pi_j} = + \int c D_j \Pi^j + i \Pi_c c c + \Pi_0 \Pi_{\bar{c}} \; ; \; D_j \Pi^j \text{ are the constraints}$$

universal form

indep. of Ψ

gauge fixing

$$i \{ Q, -\tfrac{1}{2} \xi \Pi_0 \bar{c} \} = -\tfrac{1}{2} \xi \Pi_0^{\,2}$$

$$\cancel{[AB, C] =}$$
$$\{A, BC\} = [A, B] C + B \{A, C\}$$

$$i \{ Q, A_0 \Pi_c \} = i [Q, A_0] \Pi_c + i A_0 \{ Q, \Pi_c \}$$

$$= i \, \Pi_{\bar{c}} \Pi_c + i A_0\, i D_j \Pi_j - A_0 \{ \Pi_c c c, \Pi_c \}$$

$$= \Pi_{\bar{c}} \Pi_c - A_0 D_j \Pi_j + i\, \Pi_c c A_0 - i\, \Pi_c A_0 c$$

$$= \Pi_{\bar{c}} \Pi_c + \overset{\text{by parts}}{\Pi_j D_j A_0} + \underbrace{i\, \Pi_c [c, A_0]}$$

$$-i [c, A] \Pi_c$$

$$+ i [A_0, c] \Pi_c$$

$$= \Pi_{\bar{c}} \Pi_c + \Pi_j D_j A_0 + i [A_0, c] \Pi_c \quad \checkmark$$

$$i \{ Q, A_j \partial^j \bar{c} \} = D_j c\, \partial^j \bar{c} + i \{ \Pi_0 \Pi_{\bar{c}}, A_j \partial^j \bar{c} \}$$

$$= \cancel{\;} - \partial^j \bar{c} D_j c - \Pi_0 \partial^j A_j \quad \checkmark$$

20 d

<u>We find:</u>

$$H = \int d^3x \left\{ \frac{1}{2} g^2 \, \Pi_j \Pi_j + \frac{1}{4g^2} F_{ij} F^{ij} \right\}$$

$$+ i \int d^3x \, \{Q, K\}$$

where

$$K = \int d^3x \left\{ -\frac{1}{2} \xi \, \Pi_0 \, \bar{c} + A_0 \, \Pi_c + A_j \, \partial^j \bar{c} \right\}$$

Since $\Pi_j = \frac{1}{g^2} F_{0j}$ $\qquad F_{12} = B_3$ etc.

$$H = \frac{1}{2g^2} \underbrace{\int d^3x \, (E_j E_j + B_j B_j)}_{\checkmark} + i \int d^3x \, \{Q, K\}$$

<u>Recall:</u>

$$T_{00} = \frac{1}{g^2} \mathrm{Tr}\left(F_{0j} F_0^{\ j} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right)$$

$$- F_{0j} F_{0j} + \frac{1}{2} F_{0j} F_{0j} - \frac{1}{4} F_{ij} F_{ij}$$

$$= \frac{1}{2g^2} F_{0j} F_{0j} + \frac{1}{4g^2} F_{ij} F_{ij} = \frac{1}{2g^2} (E_i^2 + B_i^2)$$

BRST identities

(Also Ward id, Slavnov Taylor id.)
Ward Takahashi id, BRS id.

More primitive id.

$$\frac{\delta S}{\delta \phi_A} = 0$$ e.o.m. is satisfied in Q.M. by the Heisenber op.

Indeed $\partial_t \phi_A = -i\,[H, \phi_A]$

but this is also classical eqn. (using Poisson bracket).

$$\langle 0 | T \frac{\delta S}{\delta \phi_A} B(\phi_A) \rangle = 0$$

But

$$\frac{\delta S}{\delta \phi_A} = \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} - \frac{\delta \mathcal{L}}{\delta \phi_A} = 0$$

$$\partial_\mu \langle 0 | T \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} B(\phi_A) | 0 \rangle - \langle 0 | T \frac{\delta \mathcal{L}}{\delta \phi_A} B(\phi_A) | 0 \rangle \overset{?}{=} 0$$

↑
put derivative outside ⇒ good for Fourier transf.

Only issue is ∂_0. because of time order.

$$\partial_0^x \langle 0 | T \phi_A(x) \phi_B(y) | 0 \rangle = \langle 0 | T \partial_0^x \phi_A(x) \phi_B(y) | 0 \rangle + \delta(t_x - t_y) \cdot \text{jump}$$

$t_x > t_y \qquad \phi_A(x)\ \phi_B(y)$

$t_x < t_y \qquad \phi_B(y)\ \phi_A(x)$

$$\text{jump} = \phi_A(x)\,\phi_B(y) - \phi_B(y)\,\phi_A(x) = [\phi_A(x), \phi_B(y)]$$

$$\partial_0^x \langle 0|T\,\phi_A(x)\,\phi_B(y)|0\rangle = \langle 0|T\,\partial_0^x\phi_A(x)\,\phi_B(y)|0\rangle +$$

$$+\ \delta(t_x - t_y)\,\langle 0|[\phi_A(x), \phi_B(y)]_{E.T.}|0\rangle$$

$$\partial_\mu \langle 0|T\frac{\delta h}{\delta\partial_\mu\phi_A}\,B(\phi_A)|0\rangle - \langle 0|T\frac{\delta h}{\delta\phi_A}\,B(\phi_A)|0\rangle =$$

$$= \delta(t_x - t_y)\,\langle 0|\left[\frac{\delta h}{\delta\partial_0\phi_A}, B(\phi_A)\right]|0\rangle + \underbrace{\langle 0|T\frac{\delta S}{\delta\phi_A}B(\phi)|0\rangle}_{0}$$

$$= \delta(t_x - t_y)\,\langle 0|[\Pi_A, B(\phi_A)]_{E.T.}|0\rangle$$

$$= -i\,\delta(t_x - t_y)\,\delta(\vec{x} - \vec{y})\,\frac{\delta B}{\delta\phi_A}$$

$$\langle 0|\hat{T}\frac{\delta S}{\delta\phi_{A(x)}}B(\phi_A)|0\rangle = -i\,\delta^{(4)}(x-y)\,\frac{\delta B}{\delta\phi_A}$$

w/ convention derivatives outside.

Use $\phi_A = \bar{c}$

$\partial^\mu D_\mu c = 0$ e.o.m. for $\bar{c}$ with sign

$$\partial^\mu \partial_\mu c - i\,\partial^\mu [A_\mu, c] = 0 \qquad \frac{\vec{\delta} \mathcal{L}}{\delta \partial_\mu \bar{c}} = D_\mu c$$

$$\Box_x \langle 0 | \hat{T}\, c^a(x)\, \bar{c}^b(y) | 0 \rangle - i\,\partial^\mu \langle 0 | \hat{T} \big(\underbrace{[A_\mu, c]}_{x}\big)^a \bar{c}^b(y) | 0 \rangle =$$

$$= -i\,\delta^{(4)}(x-y)\,\delta^{ab}$$

gauge group.

$$\delta_{BRST} \langle 0 | \hat{T}\, \phi_{A_1} \cdots \phi_{A_n} | 0 \rangle = 0$$

$$\delta \langle 0 | \hat{T}\, \partial^\mu_x A^a_\mu\, \bar{c}^b_y | 0 \rangle =$$

$$= \langle 0 | \hat{T} \big(\underbrace{\partial^\mu D_\mu c}_{x}\big)^a \bar{c}^b_y | 0 \rangle + \langle 0 | \hat{T} (\partial^\mu A_\mu)^a B^b | 0 \rangle$$

$$= -i\,\delta^{(4)}(x-y)\,\delta^{ab} + \frac{1}{\xi} \langle 0 | \hat{T}\, \partial^\mu A^a_\mu\, \partial^\nu A^b_\nu | 0 \rangle$$

$$\partial^\mu_x \partial^\nu_y \langle 0 | \hat{T}\, A^a_\mu(x)\, A^b_\nu(y) | 0 \rangle = -i\,\xi\,\delta^{(4)}(x-y)\,\delta^{ab}$$

$$-i\, q^\mu q^\nu K^{ab}_{\mu\nu} = -i\,\xi\,\delta^{ab}$$

Recall

$$G^{\mu\nu} = -\frac{i}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) - i\xi \frac{k^\mu k^\nu}{k^4}$$

This should be valid at all loops.

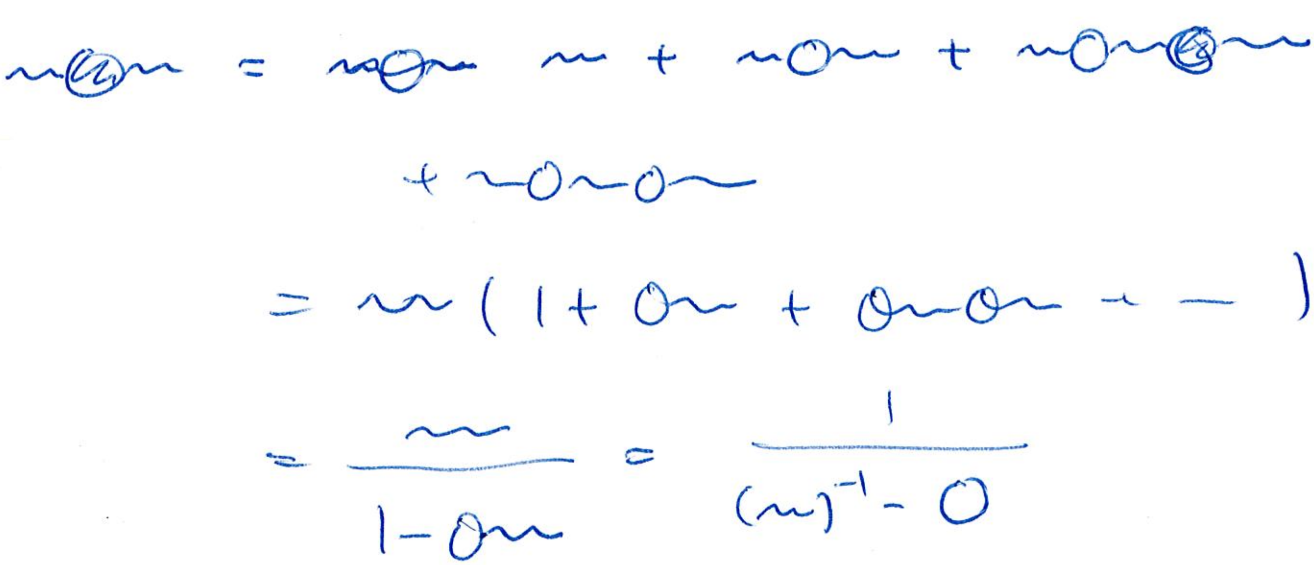
$$G^{\mu\alpha} = A(q^2)\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + B(q^2)\, q^\mu q^\alpha$$

BRST id. $\Rightarrow$ $B(q^2) = -\dfrac{i\xi}{q^4}$

$$A(q^2) = -\frac{i}{q^2+i\varepsilon}\,\frac{1}{1+\Pi(q^2)}$$

by definition.

$$G^{\mu\alpha}_{full} = -\frac{i}{k^2+i\varepsilon}\,\frac{(\eta^{\mu\nu} - k^\mu k^\nu/k^2)}{(1+\Pi(q^2))} - \frac{i\xi}{k^4}k^\mu k^\alpha$$



$$(\ )^{-1} - O = G^{-1}$$

$$O = (\ )^{-1} - G^{-1} = G_0^{-1} - G^{-1}$$

$$G^{\mu\alpha} = A\left(\eta^{\mu\alpha} - \frac{q^\mu q^\alpha}{q^2}\right) + B q^\mu q^\alpha$$

$$(G^{-1})_{\alpha\beta} = \tilde{A}\left(\eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right) + \tilde{B} q_\alpha q_\beta$$

$$G\, G^{-1} = A\tilde{A}\left(\eta^{\mu\alpha} - \frac{q^\mu q^\alpha}{q^2}\right) + B\tilde{B} q^2 q^\mu q^\alpha$$

$$\tilde{A} = 1/A \qquad -\frac{q^\mu q^\nu}{q^2} + B\tilde{B} q^2 q^\mu q^\nu = 0$$

$$\tilde{B} = \frac{1}{B q^4} \qquad B\tilde{B} = 1/q^4$$

$$(G^{-1}_{full}) = \underset{\sim}{ik^2}(1+\Pi^0)\, \mathbb{P}_{\mu\nu} + \frac{ik^4}{\xi k^4} q_\mu q_\nu$$

$$(G_0^{-1}) = ik^2\ \mathbb{P}_{\mu\nu} + \frac{i}{\xi} q_\mu q_\nu$$

(1-PI diags.)

$$O = \Pi_{\mu\nu} = -ik^2\,\Pi^0(q^2)\,\mathbb{P}_{\mu\nu}$$

self energy is transverse.

Check that BRST current is conserved

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + B \partial^\mu A_\mu + \partial^\mu \bar{c} D_\mu c + \frac{1}{2} \xi BB$$

trace assumed.

e.o.m.)

A_μ)

$$-\frac{1}{2g^2} (\partial_\mu A_\nu - i A_\mu A_\nu) F^{\mu\nu} - A_\mu \partial^\mu B + \partial^\mu \bar{c} (\partial_\mu c - i A_\mu c + i c A_\mu)$$

only terms with A_μ

$$+ \frac{1}{2g^2} A_\nu \partial_\mu F^{\mu\nu} + \frac{i}{2g^2} A_\mu A_\nu F^{\mu\nu} - A_\mu \partial^\mu B + i A_\mu c \partial^\mu \bar{c} + i A_\mu \partial^\mu \bar{c} c$$

$\delta/\delta A_\nu$ also from here $\to$ factor of 2.

$$\frac{1}{g^2} \partial_\mu F^{\mu\nu} + \frac{i}{g^2} \underbrace{A_\mu F^{\nu\mu}}_{-A_\mu F^{\mu\nu}} + \frac{i}{g^2} F^{\mu\nu} A_\mu - \partial^\nu B + i (c \partial^\nu \bar{c} + \partial^\nu \bar{c} c)$$

$$\frac{1}{g^2} D_\mu F^{\mu\nu} - \partial^\nu B + i (c \partial^\nu \bar{c} + \partial^\nu \bar{c} c) = 0$$

$$\boxed{\frac{1}{g^2} D_\mu F^{\mu\nu} = \partial^\nu B - i (c \partial^\nu \bar{c} + \partial^\nu \bar{c} c)}$$ e.o.m. for A_ν.

B) $\partial^\mu A_\mu + \xi B = 0 \Rightarrow \boxed{B = -\frac{1}{\xi} \partial_\mu A^\mu}$

$\bar{c}$) $-\bar{c} \partial^\mu D_\mu c \to \boxed{\partial^\mu D_\mu c = 0}$

c) $\boxed{-D_\mu \partial^\mu \bar{c} = 0}$ $D_\mu \partial^\mu \bar{c} \neq \partial^\mu D_\mu \bar{c}$

$$j^\mu = -\frac{1}{g^2} F^{\mu\nu} D_\nu c - i\,\partial^\mu \bar{c}\, c c + B D^\mu c$$

trace assumed

$$\partial_\mu j^\mu \stackrel{?}{=} 0$$

$$\partial_\mu j^\mu = -\frac{1}{g^2} \partial_\mu F^{\mu\nu} D_\nu c - \frac{1}{g^2} F^{\mu\nu} \partial_\mu D_\nu c - i\,\partial_\mu(\partial^\mu \bar{c}\, c c)$$

$$+ \partial_\mu B\, D^\mu c + B \underbrace{\partial_\mu D^\mu c}_{0 \text{ by } \bar{c} \text{ e.o.m.}}$$

Terms with $F^{\mu\nu}$: $\frac{1}{g^2}\partial_\mu F^{\mu\nu} = \frac{1}{g^2} D_\mu F^{\mu\nu} + \frac{i}{g^2}[A_\mu, F^{\mu\nu}] =$

$$= \partial^\nu B - i\,(c\,\partial^\nu \bar{c} + \partial^\nu \bar{c}\, c) + \frac{i}{g^2}[A_\mu, F^{\mu\nu}]$$

$$\partial_\mu j^\mu \stackrel{\text{trace assumed}}{=} \cancel{-\partial^\nu B D_\nu c} + i\,(c\,\partial^\nu \bar{c} + \partial^\nu \bar{c}\, c) D_\nu c - \frac{i}{g^2} A_\mu F^{\mu\nu} D_\nu c +$$

$$+ \frac{i}{g^2} F^{\mu\nu} A_\mu D_\nu c - \frac{F^{\mu\nu}}{g^2} \partial_\mu D_\nu c - i\,\partial_\mu(\partial^\mu \bar{c}\, c c) \cancel{+ \partial_\mu B D^\mu c}$$

$$= i\,\partial^\nu \bar{c}\,(D_\nu c\, c + c\, D_\nu c) + \frac{1}{g^2} F^{\mu\nu}(-i D_\nu c\, A_\mu + i A_\mu D_\nu c -$$

$$- \partial_\mu D_\nu c) - i\,\partial_\mu(\partial^\mu \bar{c}\, c c)$$

$$= i\,\partial^\nu \bar{c}\, D_\nu(cc) + \frac{1}{g^2} F^{\mu\nu}(-D_\mu D_\nu c) - i\,\partial_\nu(\partial^\nu \bar{c}\, c c)$$

c

$$= i\,\partial^\nu \bar{c}\, D_\nu(cc) - \frac{1}{2g^2} F^{\mu\nu} \underbrace{[D_\mu, D_\nu]}_{-i[F_{\mu\nu},\, c]} c - i\partial_\nu(\partial^\nu \bar{c}\, cc)$$

$F^{\mu\nu} F_{\mu\nu} c - F^{\mu\nu} c F_{\mu\nu} = 0$ by trace.

$$= i\,\partial^\nu \bar{c}\, D_\nu(cc) - i\,\partial_\nu \partial^\nu \bar{c}\, cc - i\,\partial^\nu \bar{c}\, \partial_\nu(cc)$$

But $D_\mu \partial^\mu \bar{c} = 0 \Rightarrow \partial_\mu \partial^\mu \bar{c} - i\,[A_\mu, \partial^\mu \bar{c}] = 0$

$$= i\,\partial^\nu \bar{c}\, D_\nu(cc) - i\left(i\,[A_\mu, \partial^\mu \bar{c}]\, cc + \partial^\nu \bar{c}\, \partial_\nu(cc)\right)$$

$i\,A_\mu \partial^\mu \bar{c}\, cc - i\,\partial^\mu \bar{c}\, A_\mu cc$

$i\,\partial^\mu \bar{c}\, cc A_\mu - i\,\partial^\mu \bar{c}\, A_\mu cc \leftarrow$ cyclicity.

$- i\,\partial^\mu \bar{c}\, [A_\mu, cc]$

$$= i\,\partial^\nu \bar{c}\, D_\nu(cc) - i\,\partial^\nu \bar{c}\, (\partial_\nu(cc) - i\,[A_\mu, cc])$$

$$= i\,\partial^\nu \bar{c}\, D_\nu(cc) - i\,\partial^\nu \bar{c}\, D_\nu(cc) = 0$$

$\partial_\mu j^\mu = 0$ by <u>e.o.m.</u>

Derivation of j^μ by using space time dependent parameter. (d)

$$\delta A_\mu = \epsilon(x) D_\mu c \quad ; \quad \delta c = i\epsilon(x) c c \quad ; \quad \delta \bar{c} = \epsilon(x) B \, , \quad \delta B = 0$$

$\epsilon(x)$: fermion ; depends on x ; it is invariant under $SU(N)$
($\Rightarrow$ not a matrix.).

$\Rightarrow$ variation action should be zero for $\epsilon(x)$ indep. of x..

$\Rightarrow \delta S = \int \partial_\mu \epsilon \, j^\mu$. for minimum $\delta S = 0 \Rightarrow \underline{\underline{\partial_\mu j^\mu = 0}}$

$$\delta F_{\mu\nu} = \partial_\mu (\epsilon D_\nu c) - \partial_\nu (\epsilon D_\mu c) - i \left[\epsilon D_\mu c, A_\nu\right] - i\left[A_\mu, \epsilon D_\nu c\right]$$

$$= \partial_\mu \epsilon D_\nu c - \partial_\nu \epsilon D_\mu c + \epsilon \partial_\mu D_\nu c - \epsilon \partial_\nu D_\mu c -$$

$$- i\epsilon D_\mu c A_\nu + i A_\nu \epsilon D_\mu c - i A_\mu \epsilon D_\nu c + i \epsilon D_\nu c A_\mu$$

$$= \partial_\mu \epsilon D_\nu c - \partial_\nu \epsilon D_\mu c + \epsilon\left(\partial_\mu D_\nu c - i A_\mu D_\nu c + i D_\nu c A_\mu - (\mu \leftrightarrow \nu)\right)$$

$$= \partial_\mu \epsilon D_\nu c + \epsilon D_\mu D_\nu c - (\mu \leftrightarrow \nu)$$

$$\delta F_{\mu\nu} F^{\mu\nu} = 2 \partial_\mu \epsilon D_\nu c F^{\mu\nu} + \frac{1}{2}\epsilon \underbrace{[D_\mu, D_\nu]}_{-i[F_{\mu\nu}, c]} c F^{\mu\nu}$$

$$= 2\partial_\mu \epsilon D_\nu c F^{\mu\nu} - \frac{i}{2}\epsilon [F_{\mu\nu}, c] F^{\mu\nu}$$

0 by trace.

$$-\frac{1}{4g^2}\delta \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2g^2}\mathrm{Tr}(\delta F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{g^2}\partial_\mu \epsilon D_\nu c F^{\mu\nu} + \frac{i}{g^2}\epsilon\left(F_{\mu\nu} c F^{\mu\nu} - c F_{\mu\nu} F^{\mu\nu}\right)$$

(e)

$$\delta\left(-\frac{1}{4g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\right) = -\frac{1}{g^2}\partial_\mu\epsilon\,\mathrm{Tr}\,(F^{\mu\nu}D_\nu c)$$

$$\delta(B\partial^\mu A_\mu) = B\,\partial^\mu(\epsilon D_\mu c)$$

$$\delta(\partial^\mu\bar{c}\,D_\mu c) = \partial^\mu(\epsilon B)D_\mu c + \partial^\mu\bar{c}\;\delta(\partial_\mu c - i[A_\mu, c])$$

$$= \partial^\mu(\epsilon B)D_\mu c + \partial^\mu\bar{c}\,D_\mu(\epsilon i cc) - i\partial^\mu\bar{c}\,\underbrace{[\epsilon D_\mu c, c]}_{\epsilon D_\mu c c - c\epsilon D_\mu c}$$

$D_\mu\epsilon = \partial_\mu\epsilon$

$= \epsilon D_\mu c c + cD_\mu c$

$= \epsilon D_\mu(cc)$

$$= \partial^\mu(\epsilon B)D_\mu c + i\partial^\mu\bar{c}\partial_\mu\epsilon cc + \cancel{i\partial^\mu\bar{c}\epsilon D_\mu(cc)} \cancel{- i\partial^\mu\bar{c}\epsilon D_\mu(cc)}$$

$$= \partial^\mu(\epsilon B)D_\mu c - i\partial_\mu\epsilon\partial^\mu\bar{c}cc$$

$$\delta\mathcal{L} = -\frac{1}{g^2}\partial_\mu\epsilon F^{\mu\nu}D_\nu c + B\partial^\mu(\epsilon D_\mu c) + \partial^\mu(\epsilon B)D_\mu c -$$

$$- i\partial_\mu\epsilon\partial^\mu\bar{c}cc$$

$$\delta S = \int d^4x\,\Big\{\epsilon\Big(\frac{1}{g^2}\partial_\mu(F^{\mu\nu}D_\nu c) - D_\mu c\,\partial^\mu B - B\partial^\mu D_\mu c$$

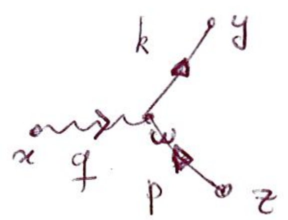
$$+ i\partial_\mu(\partial^\mu\bar{c}cc)\Big)$$

$$= \int d^4x\,\epsilon\,\partial_\mu\underbrace{\Big[\frac{1}{g^2}F^{\mu\nu}D_\nu c - BD^\mu c + i\partial^\mu\bar{c}cc\Big]}_{-j^\mu}$$

✓

WT identity for QED

Definition of vertex function:



$$\langle 0 | T \{ A_\mu(x) \psi_a(y) \bar{\psi}_b(z) \} | 0 \rangle$$

At lowest order.

$$\langle 0 | T \{ \underbrace{-ie\int d^4w\, \bar{\psi}_c \gamma^\nu_{cd} A_\nu \psi_d}_{w} \; A_\mu(x) \psi_a(y) \bar{\psi}_b(z) \} | 0 \rangle$$

$$= -ie\int d^4w \; S_{db}(w-z)\, \gamma^\nu_{cd}\, S_{ac}(y-w)\, G_{\nu\mu}(w-x)$$

$$= -ie\int d^4w \; \left[S(y-w)\, \gamma^\nu\, S(w-z) \right]_{ab} G_{\nu\mu}(w-x)$$

$$S(x) = \int \frac{d^4p}{(2\pi)^4} S(p)\, e^{-ipx} \quad ; \quad S(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} = \frac{i}{\not{p}-m}$$

$$\langle 0 | T \{ A_\mu(x) \psi_a(y) \bar{\psi}_b(z) \} | 0 \rangle = -ie \int d^4w \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \cdot$$

$$\cdot\, e^{-ik(y-w) - ip(w-z) - iq(w-x)} \qquad S(k)\, \gamma^\nu\, S(p)\, G_{\nu\mu}(q)$$

$\int d^4w \to (2\pi)^4 \delta^{(4)}(k-p-q)$

$$= -ie \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \; S_0(p+q)\, \gamma^\nu\, S_0(p)\; G^{(0)}_{\nu\mu}(q) \cdot e^{-ip(y-z)} \cdot e^{-iq(y-x)}$$

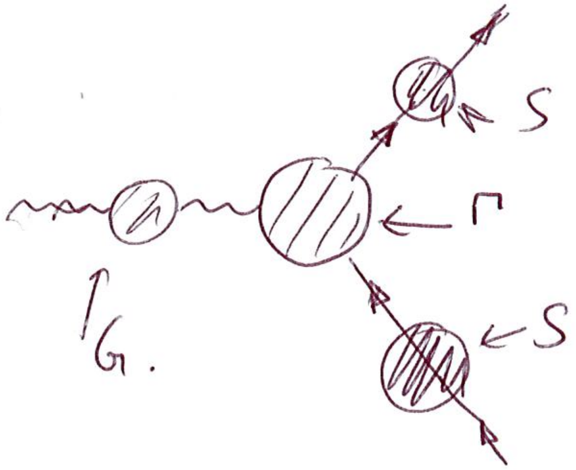
lowest order,

$-ipy - iqy + ipz + iqx$

$\boxed{\text{All orders}}$ Definition:

$$\langle 0 | T \{ A_\mu(x) \psi_a(y) \bar{\psi}_b(z) \} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} S(p+q)\, \Gamma^\nu(p, p+q)\, S(p) \cdot G_{\nu\mu}(q)\, e^{-ip(y-z) - iq(y-x)}$$

where $S(p+q)$, $S(p)$, $G_{\nu\mu}(q)$ $\boxed{\text{are exact}}$ prop.



Now we have

$$\partial^\mu_x \langle 0 | T \{ A_\mu(x)\, \psi_a(y)\, \bar{\psi}_b(z) \} | 0 \rangle =$$

$$= \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4}\, e^{-ip(y-z) - iq(y-x)} \quad S_0(p+q)\, \Gamma^\nu(p, p+q)\, S(p)$$

$$\underbrace{G_{\nu\mu}(q) \quad i q^\mu}$$

$$\xi \frac{q_\nu}{q^2}$$

$$= \xi \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4}\, e^{-ip(y-z) - iq(y-x)} \quad \frac{1}{q^2}\, S_0(p+q)\, q_\nu \Gamma^\nu(p, p+q) \cdot S(p)$$

(3)

$$\delta_{BRST} \langle 0 | T \{ \bar{c}(x) \psi_a(y) \bar{\psi}_b(z) \} | 0 \rangle =$$

$\delta\psi = ic\psi$
$\delta\psi^\dagger = +i\psi^\dagger c$
$\delta\bar{\psi} = +i\bar{\psi} c$

$$= \langle 0 | T \{ B(x) \psi_a(y) \bar{\psi}_b(z) \} | 0 \rangle +$$

for QED

$$+ e \langle 0 | T \{ \bar{c}(x) i (c\psi)_y \bar{\psi} \} | 0 \rangle +$$

$$+ e \langle 0 | T \{ \bar{c}(x) \psi(y) (+i) \bar{\psi} c \} | 0 \rangle$$

$$= -\frac{1}{\xi} \langle 0 | T \{ \partial^\mu A_\mu(x) \psi(y) \bar{\psi}(z) \} | 0 \rangle -$$

$$- i \langle 0 | T \{ \bar{c}(x) c(y) \psi_y \bar{\psi}_z \} | 0 \rangle +$$

$$+ i \langle 0 | T \{ \bar{c}(x) c(z) \psi_y \bar{\psi}_z \} | 0 \rangle$$

QED : ghost decouple. $\langle c(x) \bar{c}(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} e^{-ik(x-y)}$

$$\int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{-ip(y-z) - iq(y-x)} \frac{1}{q^2} S_0(p+q) q_\nu \Gamma^\nu(p, p+q) . S(p) =$$

$$= -ie \langle 0 | T \{ \bar{c}(x) c(y) \} | 0 \rangle S(y-z) + ie \langle 0 | T \{ \bar{c}(x) c(z) \} | 0 \rangle .$$

$$= ie \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2} e^{-iq(y-x)} \int \frac{d^4p}{(2\pi)^4} e^{-ip(y-z)} S(p) + ie \int \frac{d^4 \cancel{q}}{(2\pi)^4} \frac{i}{\cancel{q}^2} e^{-i\cancel{q}(z-x)} \overbrace{e^{-ik(y-z)} \int \frac{d^4k}{(2\pi)^4} S(k)}^{S(y-z)}$$

④

$$\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, e^{-ip(y-z)-iq(y-x)}\frac{1}{q^2}\, S(p+q)\, q_\nu \Gamma^\nu(p,p+q)\, S(p) =$$

$$= -\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{1}{q^2}\, e^{-iq(y-x)-ip(y-z)}\, S(p) +$$

$$+ \int \frac{d^4 q}{(2\pi)^4}\frac{1}{q^2}\, e^{-iq(z-x)-ik(y-z)}\, S(k)$$

$$e^{-iq(z-x)-ip(y-z)-iq(y-z)} = e^{-iq(y-x)}e^{-ip(y-z)}$$

$$-iq(z-x)-ik(y-z) \Longrightarrow -iq(y-x)-ip(y-z)$$

$$\to \phi = -iq(z-\cancel{x}+y+\cancel{x}) - ik(y-z) + ip(y-z) = 0$$

$$iq(\cancel{y-z}) - ik(\cancel{y-z}) + ip(\cancel{y-z}) = 0$$

$$k = p+q$$

$$= -\int \frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{1}{q^2}\, e^{-iq(y-x)-ip(y-z)}\left(S(p) - S(p+q)\right)$$

$$S(p+q)\, q_\nu \Gamma^\nu(p,p+q)\, S(p) = -S(p) + S(p+q)$$

$$\boxed{q_\nu \Gamma^\nu(p,p+q) = e\left(-S^{-1}(p+q) + S^{-1}(p)\right)}$$

lowest order.

$$+ie\gamma^\nu q_\nu = e\left(\cancel{-(+i)}\,(\not p+\not q-m) + \cancel{(-i)}\,(\not p-m)\right)$$

$$\not q = \cancel{-\not p} - \not q + m \quad \cancel{+\not p} - m \leftarrow \underline{\text{sign.}}$$

Instanton solution

γ matrices for $\underline{SO(4)}$ $\quad \{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$; take simply $\gamma_j \to i\gamma_j$

$$\boxed{\gamma^0 = \sigma_1 \otimes \mathbb{1} \qquad \gamma_j = \sigma_2 \otimes \sigma_j}$$

$\Sigma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ generators of $SO(4)$.

$$\Sigma^{0i} = -\frac{i}{4} 2i\, \sigma_3 \otimes \sigma_j = \frac{1}{2}\sigma_3 \otimes \sigma_j = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}$$

$$\Sigma^{ij} = -\frac{i}{4} \mathbb{1} \otimes 2i\, \epsilon_{ijk}\sigma_k = \frac{\epsilon_{ijk}}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

$$\boxed{A_\mu = \Sigma_{\mu\nu}\, \partial_\nu \ln h}$$ (upper left corner).

$$A_0 = \Sigma'_{0i}\, \partial_i \ln h = \frac{1}{2}\sigma_a\, \partial_a \ln h$$

$$\boxed{A_0^a = \cancel{1}\; \partial_a \ln h}$$

$$A_j = \Sigma'_{j0}\, \partial_0 \ln h + \Sigma_{jk}\, \partial_k \ln h$$

$$= -\frac{1}{2}\sigma_j\, \partial_0 \ln h + \frac{1}{2}\epsilon_{jka}\sigma_a\, \partial_k \ln h$$

$$\boxed{A_j^a = -\delta^a_j\, \partial_0 \ln h + \epsilon_{ajk}\, \partial_k \ln h}$$

$$\partial_\nu A_\mu = \Sigma_{\mu\alpha}\, \partial_{\alpha\nu} \ln h$$

$$F_{\mu\nu} = \Sigma_{\nu\alpha}\, \partial_{\alpha\mu} \ln h - \Sigma_{\mu\alpha}\, \partial_{\alpha\nu} \ln h - $$

$$- i \left[\Sigma_{\mu\alpha}, \Sigma_{\nu\beta} \right] \partial_\alpha \ln h \; \partial_\beta \ln h$$

$$\left[\Sigma^{\mu\alpha}, \Sigma^{\nu\beta} \right] = i \left(\Sigma^{\mu\nu} \delta^{\alpha\beta} - \Sigma^{\mu\beta} \delta^{\alpha\nu} - \Sigma^{\alpha\nu} \delta^{\mu\beta} + \Sigma^{\alpha\beta} \delta^{\mu\nu} \right)$$

$$F_{\mu\nu} = \Sigma_{\nu\alpha}\, \partial_\alpha \left(\frac{\partial_\mu h}{h} \right) - \Sigma_{\mu\alpha}\, \partial_\alpha \left(\frac{\partial_\nu h}{h} \right) + \Sigma^{\mu\nu} \frac{\partial h \cdot \partial h}{h^2} -$$

$$- \Sigma^{\mu\beta} \frac{\partial_\nu h}{h} \frac{\partial_\beta h}{h} - \Sigma_{\alpha\nu} \frac{\partial_\mu h}{h} \frac{\partial_\alpha h}{h} + \cancel{\Sigma^{\alpha\beta} \delta_{\mu\nu} \frac{\partial_\alpha h}{h} \frac{\partial_\beta h}{h}} \to 0.$$

$$= \Sigma_{\nu\alpha} \frac{\partial_{\alpha\mu} h}{h} - \cancel{\Sigma_{\nu\alpha} \frac{\partial_\alpha h\, \partial_\mu h}{h^2}} - \Sigma_{\mu\alpha} \frac{\partial_{\alpha\nu} h}{h} +$$

$$+ \cancel{\Sigma_{\mu\alpha} \frac{\partial_\nu h\, \partial_\alpha h}{h^2}} + \Sigma^{\mu\nu} \frac{\partial h \cdot \partial h}{h^2} - \cancel{\Sigma^{\mu\beta} \frac{\partial_\beta h\, \partial_\nu h}{h^2}} - \cancel{\Sigma_{\alpha\nu} \frac{\partial_\mu h\, \partial_\alpha h}{h^2}}$$

$$\boxed{F_{\mu\nu} = \Sigma_{\nu\alpha} \frac{\partial_{\alpha\mu} h}{h} - \Sigma_{\mu\alpha} \frac{\partial_{\alpha\nu} h}{h} + \Sigma_{\mu\nu} \frac{\partial_\alpha h\, \partial_\alpha h}{h^2}}$$

$$\partial_\mu F_{\mu\nu} - i\,[A_\mu, F_{\mu\nu}] = 0 .$$

$$\partial_\mu F_{\mu\nu} = \Sigma_{\nu\alpha} \frac{\partial^2 \partial_\alpha h}{h} - \Sigma_{\nu\alpha} \frac{\partial_{\alpha\mu} h\, \partial_\mu h}{h^2} + \Sigma_{\mu\alpha} \frac{\partial_{\alpha\nu} h\, \partial_\mu h}{h^2}$$

$$+ 2\Sigma_{\mu\nu} \frac{\partial_{\mu\alpha} h\, \partial_\alpha h}{h^2} - \frac{2}{h^3} \Sigma_{\mu\nu}\, \partial_\alpha h\, \partial_\alpha h\, \partial_\mu h$$

$$= \Sigma_{\nu\alpha} \frac{\partial^2 \partial_\alpha h}{h} - 3\,\Sigma_{\nu\alpha} \frac{\partial_{\alpha\beta} h\, \partial_\beta h}{h^2} + \Sigma_{\alpha\beta} \frac{\partial_{\beta\nu} h\, \partial_\alpha h}{h^2}$$

$$+ \frac{2}{h^3} \Sigma_{\nu\alpha}\, \partial_\alpha h\, \partial_\beta h\, \partial_\beta h$$

$$-i\left[\Sigma_{\mu\gamma}, \Sigma_{\nu\alpha} \frac{\partial_{\alpha\mu} h}{h} - \Sigma_{\mu\alpha} \frac{\partial_{\alpha\nu} h}{h} + \Sigma_{\mu\nu} \frac{\partial_\alpha h\, \partial_\alpha h}{h^2}\right] \frac{\partial_\gamma h}{h} =$$

$$= \Sigma_{\mu\nu} \frac{\partial_{\alpha\mu} h}{h} \frac{\partial_\alpha h}{h} - \cancel{\Sigma_{\mu\alpha} \frac{\partial_{\alpha\mu} h}{h}} \frac{\partial_\nu h}{h} - \Sigma_{\gamma\nu} \frac{\partial^2 h}{h} \frac{\partial_\gamma h}{h} +$$

$$+ \Sigma_{\gamma\alpha} \frac{\partial_{\alpha\nu} h}{h} \frac{\partial_\gamma h}{h} + \Sigma_{\mu\alpha} \frac{\partial_{\alpha\nu} h}{h} \frac{\partial_\mu h}{h} + \cancel{\Sigma_{\gamma\mu} \frac{\partial^2 h}{h} \frac{\partial_\gamma h}{h}}$$

$$+ \Sigma_{\gamma\mu} \frac{\partial_{\mu\nu} h}{h} \frac{\partial_\gamma h}{h} - d\, \Sigma_{\gamma\alpha} \frac{\partial_{\alpha\nu} h}{h} \frac{\partial_\gamma h}{h} +$$

(4)

$$+\sum_{\mu\nu} \frac{\partial_\alpha h\, \partial_\alpha h}{h^3}\partial_\mu h - \sum'_{\gamma\mu} \frac{\partial_\alpha h\, \partial_\alpha h}{h^3}\partial_\gamma h +$$

$$+ d \sum_{\gamma\nu} \frac{\partial_\alpha h\, \partial_\alpha h}{h^3}\partial_\gamma h$$

$$\sum_{\nu\alpha} \frac{\partial^2 \partial_\alpha h}{h} - 3\sum_{\nu\alpha} \frac{\partial_{\alpha\beta} h\, \partial_\beta h}{h^2} + \boxed{\cancel{\sum_{\beta\alpha} \frac{\partial_{\alpha\nu} h\, \partial_\beta h}{h^2}}} +$$

$$+ \frac{3}{h^3}\cancel{\sum_{\nu\alpha} \partial_\alpha h\, \partial_\beta h}\, \partial_\beta h - \sum_{\nu\alpha} \frac{\partial_{\beta\alpha} h\, \partial_\beta h}{h^2} + \sum_{\nu\alpha} \frac{\partial^2 h}{h}\frac{\partial_\alpha h}{h}$$

$$+ \boxed{\cancel{\sum_{\beta\alpha} \frac{\partial_{\alpha\nu} h\, \partial_\beta h}{h^2}}} + \boxed{\cancel{\sum_{\beta\alpha} \frac{\partial_{\alpha\nu} h}{h^2}}\partial_\beta h} + \boxed{\cancel{\sum_{\beta\alpha} \frac{\partial_{\alpha\nu} h\, \partial_\beta h}{h^2}}}$$

$$- \boxed{d \cancel{\sum_{\beta\alpha} \frac{\partial_{\alpha\nu} h\, \partial_\beta h}{h^2}}} + \cancel{\sum_{\nu\alpha} \frac{\partial_\beta h\, \partial_\beta h\, \partial_\alpha h}{h^3}} + \cancel{\sum_{\nu\alpha} \frac{\partial_\beta h\, \partial_\beta h\, \partial_\alpha h}{h^3}}$$

$$- d \cancel{\sum_{\nu\alpha} \frac{\partial_\beta h\, \partial_\beta h\, \partial_\alpha h}{h^3}} = 0.$$

$$\sum_{\nu\alpha} \frac{\partial^2 \partial_\alpha h}{h} - 4\sum_{\nu\alpha} \frac{\partial_{\beta\alpha} h\, \partial_\beta h}{h^2} + \sum_{\nu\alpha} \frac{\partial^2 h}{h}\frac{\partial_\alpha h}{h} = 0$$

5

$$D_\mu F^{\mu\nu} = \Sigma_{\nu\alpha}\underbrace{\left(\frac{\partial^2 \partial_\alpha h}{h} - 4\frac{\partial_{\beta\alpha} h\, \partial_\beta h}{h^2} + \frac{\partial^2 h}{h}\frac{\partial_\alpha h}{h} \right)}_{=0.} = 0.$$

$\times h^2$

$$h\partial^2\partial_\alpha h - 4\partial_{\beta\alpha} h\,\partial_\beta h + \partial^2 h\,\partial_\alpha h = 0,$$

$$\partial_\alpha (h\partial^2 h) - 2\partial_\alpha (\partial_\beta h\,\partial_\beta h) = 0.$$

$$\Rightarrow \boxed{h\partial^2 h - 2\partial_\beta h\,\partial_\beta h = C}$$

constant.

one eqn for 1 function ✓.

$h = 1/f$ $\qquad \partial h = -\frac{1}{f^2}\partial f$ $\qquad \partial^2 h = -\frac{\partial^2 f}{f^2} + \frac{2}{f^3}\partial f\cdot\partial f$

$$\frac{1}{f}\left(-\frac{\partial^2 f}{f^2} + \cancel{\frac{2}{f^3}\partial f\partial f} \right) - \cancel{2\frac{\partial f\partial f}{f^4}} = C$$

$$\boxed{\partial^2 f = -Cf^3}$$

if $C = 0$ $\qquad \boxed{\partial^2 f = 0}$ $\qquad$ simplest case.

($f \to \infty$ at some points is allowed. ($f\to\infty \Rightarrow h = 0$)) ✓

$$f = \sum_i \frac{a_i}{(\vec{x}-\vec{x}_i)^2}$$

4d.

$$\operatorname{Tr} F_{\mu\nu} F^{\mu\nu} = \operatorname{Tr}_{UL} \Big[\Big(\Sigma_{\nu\alpha} \frac{\partial_{\mu\alpha} h}{h} - \Sigma_{\mu\alpha} \frac{\partial_{\alpha\nu} h}{h} + \Sigma_{\mu\nu} \frac{\partial_\alpha h \, \partial_\alpha h}{h^2} \Big) \cdot$$

upper left

(X)

$$\cdot \Big(\Sigma_{\nu\beta} \frac{\partial_{\beta\mu} h}{h} - \Sigma_{\mu\beta} \frac{\partial_{\beta\nu} h}{h} + \Sigma_{\mu\nu} \frac{\partial_\beta h \, \partial_\beta h}{h^2} \Big) \Big]$$

$$\operatorname{Tr}_{UL} \Sigma_{\mu\nu} \Sigma_{\alpha\beta} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} + \frac{1}{2} \left(\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha} \right)$$

by Lorentz inv. & adjusting the constants.
η_8 antisymm.

$$\operatorname{Tr} F_{\mu\nu} F_{\mu\nu} = \frac{2}{2} \left(d \frac{\partial_{\mu\alpha} h \, \partial_{\mu\alpha} h}{h^2} - \frac{\partial_{\mu\alpha} h \, \partial_{\mu\alpha} h}{h^2} \right) - \frac{2}{2} \left(\frac{\partial_{\alpha\mu} h \, \partial_{\alpha\mu} h}{h^2} - \frac{\partial^2 h \, \partial^2 h}{h^2} \right)$$

$$+ \frac{4}{2} \left(\frac{\partial^2 h}{h^3} \partial h \cdot \partial h - d \frac{\partial^2 h}{h^3} \partial h \partial h \right) + 6 \frac{(\partial h \partial h)^2}{h^4}$$

$$= 2 \frac{\partial_{\mu\alpha} h \, \partial_{\mu\alpha} h}{h^2} + \frac{\partial^2 h \, \partial^2 h}{h^2} - 6 \frac{\partial^2 h}{h^3} \partial h \partial h + 6 \frac{(\partial h \partial h)^2}{h^4}$$

using c.o.m. w/ $c=0$ i.e. $\partial^2 h = \frac{2 \partial h \cdot \partial h}{h}$

$$\operatorname{Tr} F_{\mu\nu} F^{\mu\nu} = 2 \frac{\partial_{\mu\alpha} h \, \partial_{\mu\alpha} h}{h^2} + 4 \frac{(\partial h \partial h)^2}{h^4} - 12 \frac{(\partial h \partial h)^2}{h^4} + 6 \frac{(\partial h \partial h)^2}{h^4}$$

$$= 2 \left(\frac{\partial_{\mu\alpha} h \, \partial_{\mu\alpha} h}{h^2} - \frac{(\partial h \partial h)^2}{h^4} \right)$$

$$\cancel{\partial_\mu \left(\frac{\partial_\alpha h}{h} \right) \partial_\mu \left(\frac{\partial_\alpha h}{h} \right)}$$

$$\frac{\partial_{\mu\alpha} h}{h} = \partial_\mu\left(\frac{\partial_\alpha h}{h}\right) + \frac{1}{h^2}\partial_\mu h\,\partial_\alpha h$$

$$\frac{\partial_{\mu\alpha} h}{h}\frac{\partial_{\mu\alpha} h}{h} = \partial_\mu\left(\frac{\partial_\alpha h}{h}\right)\partial_\mu\left(\frac{\partial_\alpha h}{h}\right) + \frac{2}{h^2}\partial_\mu h\,\partial_\alpha h\,\partial_\mu\left(\frac{\partial_\alpha h}{h}\right) +$$

$$+ \frac{1}{h^4}(\partial h\,\partial h)(\partial h\,\partial h)$$

$$\mathrm{Tr}\,F_{\mu}F^{\mu} = 2\,\partial_\mu\left(\frac{\partial_\alpha h}{h}\right)\partial_\mu\left(\frac{\partial_\alpha h}{h}\right) + \frac{4}{h^2}\partial_\alpha h\,\partial_\alpha h\,\partial_\mu\left(\frac{\partial_\alpha h}{h}\right)$$

$$h = e^H$$

$$\mathrm{Tr}\,F_{\mu}F^{\mu} = 2\,\partial_{\mu\alpha}H\,\partial_{\mu\alpha}H + 4\,\partial_\mu H\,\partial_\alpha H\,\partial_{\mu\alpha}H$$

$$= 2\,\partial_\alpha(\partial_\mu H\,\partial_{\mu\alpha}H) - 2\,\partial_\mu H\,\partial_\mu\partial^2 H + 4\,\partial_\mu H\,\partial_\alpha H\,\partial_{\mu\alpha}H$$

$$\partial_\alpha h = \partial_\alpha H e^H \qquad \partial^2 h = (\partial^2 H + \partial H\partial H)\,e^H = 2\,\frac{\partial H\cdot\partial H\,\cancel{e^{2H}}}{\cancel{e^H}}\,e^H$$

$$\partial^2 H = \partial H\cdot\partial H$$

$$\mathrm{Tr}\,F_{\mu}F^{\mu} = 2\,\partial_\alpha(\partial_\mu H\,\partial_{\mu\alpha}H) - 4\,\cancel{\partial_\mu H\,\partial_{\mu\alpha}H\,\partial_\alpha H} + 4\,\cancel{\partial_\mu H\,\partial_\alpha H\,\partial_{\mu\alpha}H}$$

$$= \partial_\alpha\partial_\alpha(\partial_\mu H\,\partial_\mu H)$$

$$\boxed{S = \frac{2}{4g^2}\int d^4x\,\partial_\alpha\partial_\alpha(\partial H\cdot\partial H)}$$

$\frac{1}{4g^2}\int F^a_{\mu}F^{a\,\mu} \to \ldots$ in euclidean

because $\mathrm{Tr}\left(\frac{\sigma^a}{2}\frac{\sigma^b}{2}\right) = \frac{1}{2}\delta^{ab}$.

$$H = \ln h = -\ln f$$

$$\partial H = -\frac{\partial f}{f}$$

$$\partial H . \partial H = \frac{\partial f . \partial f}{f^2}$$

$$f = \sum_i \frac{a_i}{(x-x_i)^2} \qquad \partial_\mu f = -\sum_i \frac{a_i}{(x-x_i)^4} 2(x-x_i)_\mu$$

near sing. $\quad \partial_\mu f \sim -\frac{a_i}{(x-x_i)^4} 2(x-x_i)_\mu$ $\quad x_i \;\; S_i$

$$\frac{\partial f . \partial f}{f^2} \underset{x \to x_i}{\simeq} \frac{4 a_i^2}{a_i^2} \frac{(x-x_i)^{\cancel{6}}}{(x-x_i)^8} = \frac{4}{(x-x_i)^2}$$

$$S = \frac{1}{2g^2} \int d^4x \, \partial_a \partial_a \left(\frac{4}{(x-x_i)^2} \right) = \frac{4}{2g^2} \sum_i \oint_{S_i} -\partial_r \left(\frac{1}{r^2} \right) d\hat{n}$$

$$+ \frac{2}{r^3} r^3 d\Omega_3$$

$$= \frac{8}{2g^2} V_{S_3}$$

$\uparrow$ volume

$$\int d^n r \, e^{-r^2} = V(\Omega_{n-1}) \int_0^\infty dr \, r^{n-1} e^{-r^2} = V_{n-1} \int_0^\infty \frac{du}{2} u^{\frac{n}{2}-1} e^{-u}$$

$u = r^2 \qquad 2r dr = du$

$$= \frac{1}{2} V_{n-1} \Gamma(n/2) = \pi^{n/2}$$

$$= (\sqrt{\pi})^n$$

$$\boxed{V_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}}$$

$$V_3 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$$

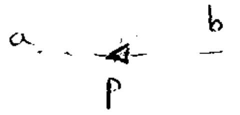
$$\frac{\sigma^a}{2} \cdot \frac{\sigma^b}{2} = \frac{\delta^{ab}}{4} + \frac{i\epsilon^{abc}\sigma^c}{4}$$

$$\mathrm{Tr}\, \frac{\sigma^a}{2}\frac{\sigma^b}{2} = \frac{1}{2}.$$

$$\boxed{S = \frac{8\pi^2}{g^2}}$$ ← for each point

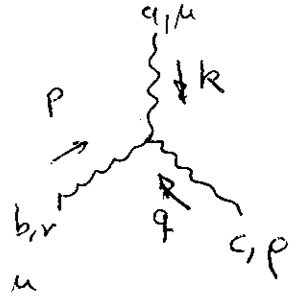
$$S = \frac{8\pi^2}{g^2}(n-1)$$

↑ at ∞ gives (−1)

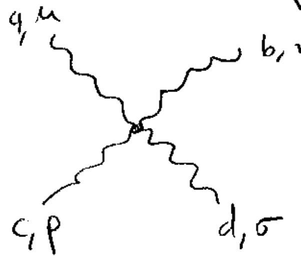


$$\frac{i\delta^{ab}}{p^2+i\varepsilon}$$

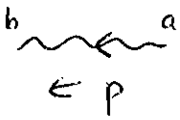
$$= -g f^{abc} p^\mu$$



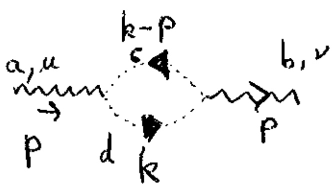
$$g f^{abc}\left[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\right]$$



$$-ig^2\left[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma})\right]$$



$$= -\frac{i\eta_{\mu\nu}}{p^2+i\varepsilon}\delta^{ab}$$



1 loop

$$(-)\ acd \quad bcd$$

$$= \int\frac{d^dk}{(2\pi)^d} g^2 f^{dac} f^{cbd}\frac{k^\mu(k-p)^\nu}{(k^2+i\varepsilon)((k-p)^2+i\varepsilon)}(i)^2$$

(no symmetry factor)

$$= -\frac{1}{2}g^2 C_2(G)\delta^{ab}\int_0^1 d\alpha\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu(k-p)^\nu}{(k^2-2\alpha kp+\alpha p^2+i\varepsilon)^2}$$

$$(k-\alpha p)^2+\alpha(1-\alpha)p^2$$

$$= -\frac{1}{2}g^2 C_2(G)\delta^{ab}\int_0^1 d\alpha\int\frac{(k+\alpha p)^\mu(k-(1-\alpha)p)^\nu}{(k^2+\alpha(1-\alpha)p^2+i\varepsilon)^2}\frac{d^dk}{(2\pi)^d}$$

$$= -\frac{1}{2}g^2 C_2(G)\delta^{ab}\int_0^1 d\alpha\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu-\alpha(1-\alpha)p^\mu p^\nu}{(k^2+\alpha(1-\alpha)p^2+i\varepsilon)^2}$$

$$\Delta = -\alpha(1-\alpha)p^2-i\varepsilon$$

$$= -\frac{1}{2}g^2 C_2(G)\delta^{ab}\int_0^1 d\alpha\left[-\frac{i\eta^{\mu\nu}}{(4\pi)^{d/2}}\frac{1}{2}\frac{\Gamma(1-d/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{1-d/2} - \frac{i\alpha(1-\alpha)p^\mu p^\nu\,\Gamma(2-d/2)}{(4\pi)^{d/2}\,\Gamma(2)\,\Delta^{2-d/2}}\right]$$

$$= + \frac{i}{\cancel{2}} \frac{g^2 C_2(G)\,\delta^{ab}}{(4\pi)^{d/2}} \int_0^1 d\alpha \;\Gamma(1-d/2) \left[\frac{1}{2} \eta^{\mu\nu} \Delta^{d/2-1} + \alpha(1-\alpha)(1-d/2)\, p^\mu p^\nu \cdot \Delta^{d/2-2} \right]$$

$$\int_0^1 d\alpha \;\alpha^{d/2-1} (1-\alpha)^{d/2-1} = B(d/2, d/2) = \frac{\cancel{\Gamma(d)}}{\cancel{\Gamma(d/2)^2}} \; \frac{\Gamma(d/2)^2}{\Gamma(d)}$$

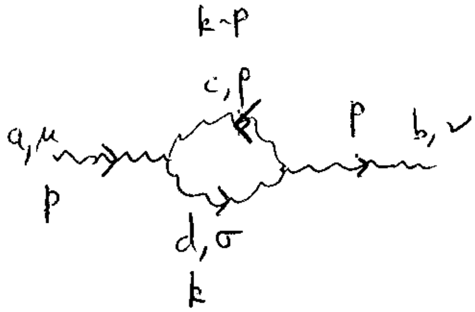
$$= + \frac{i}{\cancel{2}} \frac{g^2 C_2(G)\,\delta^{ab}}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)\,\Gamma(d/2)^2}{\Gamma(d)} \left[\frac{1}{2} \eta^{\mu\nu} (-p^2)^{d/2-1} + (1-d/2)\, p^\mu p^\nu (-p^2)^{d/2-2} \right]$$

$$= + \frac{i}{\cancel{2}}\, g^2 \frac{C_2(G)\,\delta^{ab}}{(4\pi)^{d/2}} B\left(\frac{d}{2}, \frac{d}{2}\right) \Gamma(1-d/2) \frac{(-p^2)^{d/2-1}}{2} \left[\eta^{\mu\nu} - (2-d) \frac{p^\mu p^\nu}{p^2} \right]$$

$$= + \frac{i}{\cancel{2}}\, g^2 \frac{C_2(G)\,\delta^{ab}}{(4\pi)^{d/2}} B\left(\frac{d}{2}, \frac{d}{2}\right) \Gamma\left(1-\frac{d}{2}\right) (-p^2)^{\frac{d}{2}-1} \left[\eta^{\mu\nu} - (2-d) \frac{p^\mu p^\nu}{p^2} \right]$$

 $= 0$ dim. reg.

③



$\int d^dk/(2\pi)^d$

$$= \frac{1}{2} g^2 \frac{f^{adc} f^{bcd} (-i)^2}{(k^2+i\varepsilon)((k-p)^2+i\varepsilon)} \cdot$$

$$\cdot \left[\eta^{\mu\sigma}(p+k)^\rho + \eta^{\sigma\rho}(-k-k+p)^\mu + \eta^{\rho\mu}(k-p-p)^\sigma \right] \cdot$$

$$\cdot \left[\eta^{\sigma\nu}(k+p)^\rho + \eta^{\nu\rho}(-p+k+p)^\sigma + \eta^{\rho\sigma}(-k+p-k)^\nu \right]$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{2} g^2 C_2(G) \delta^{ab} \frac{1}{(k^2+\alpha(1-\alpha)p^2+i\varepsilon)^2} \cdot$$

$$\left[\eta^{\mu\sigma}(k+(1+\alpha)p)^\rho + \eta^{\sigma\rho}(-2k+(1-2\alpha)p)^\mu + \eta^{\rho\mu}(k+(\alpha-2)p)^\sigma \right] \cdot$$

$$\cdot \left[\eta^{\sigma\nu}(k+(1+\alpha)p)^\rho + \eta^{\nu\rho}(k+(\alpha-2)p)^\sigma + \eta^{\rho\sigma}(-2k+(1-2\alpha)p)^\nu \right]$$

$$\eta^{\mu\nu}\left(k^2+(1+\alpha)^2p^2\right) + \left(\cancel{k^\mu k^\nu} + (1+\alpha)(\alpha-2)\,p^\mu p^\nu\right) + \left(-2k^\mu k^\nu + (1+\alpha)(1-2\alpha)p^\mu p^\nu\right)$$

$$\left(-2k^\mu k^\nu + (1+\alpha)(1-2\alpha)p^\mu p^\nu\right) + \left(\cancel{-2k^\mu k^\nu} + (1-2\alpha)(\alpha-2)\,p^\mu p^\nu\right) +$$

$$+ d\left(4k^\mu k^\nu + (1-2\alpha)^2 p^\mu p^\nu\right) + \cancel{k^\mu k^\nu} + (\alpha-2)(1+\alpha)\,p^\mu p^\nu +$$

$$+ \eta^{\mu\nu}\left(k^2+(\alpha-2)^2p^2\right) + \left(-2k^\mu k^\nu\right) + (\alpha-2)(1-2\alpha)\,p^\mu p^\nu$$

$$\eta^{\mu\nu}\left(2k^2 + \left[(1+\alpha)^2 + (\alpha-2)^2\right]p^2\right) + k^\mu k^\nu\left((4d-6)\right) +$$

$$+ p^\mu p^\nu \left(2(1+\alpha)(\alpha-2) + 2(1+\alpha)(1-2\alpha) + 2(1-2\alpha)(\alpha-2) + (1-2\alpha)^2 d\right)$$

$$\eta^{\mu\nu}\left(2k^2+(2\alpha^2-2\alpha+5)p^2\right)+(4d-6)\,k^\mu k^\nu+$$

$$+\,2p^\mu p^\nu\left((\alpha^2+\alpha-2+1+2\alpha^2-\alpha-2\alpha^2+5\alpha-2+2(\alpha-1/2)^2 d\right)$$

$$\left[-3\alpha^2+3\alpha-3+2d(\alpha-1/2)^2\right]$$

$$1\to\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\;;\quad k^2\to-\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}}\;;\quad k^\mu k^\nu\to\frac{1}{d}\eta^{\mu\nu}k^2$$

$$\left(\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}}\right)\left[\;1\to\frac{(1-d/2)}{\Delta}\;,\;k^2\to-\frac{d}{2}\;;\;k^\mu k^\nu=-\frac{1}{2}\eta^{\mu\nu}\right.$$

$$\boxed{\Delta=-\alpha(1-\alpha)p^2}$$

$$= \frac{1}{2}g^2\,C_2(G)\,\delta^{ab}\,\frac{i\,\Gamma(1-d/2)}{(4\pi)^{d/2}\,\Delta^{1-d/2}}\;\times$$

$$\times\;\eta^{\mu\nu}\left[-d+(2\alpha^2-2\alpha+5)(1-d/2)\frac{p^2}{\Delta}+(4d-6)\left(-\frac{1}{2}\right)\right]$$

$$+2p^\mu p^\nu\left[-3\alpha^2+3\alpha-3+2d(\alpha-1/2)^2\right]\frac{(1-d/2)}{\Delta}$$

$$\hookrightarrow\;\eta^{\mu\nu}\left[-d-2d+3-\frac{\overset{2\alpha(\alpha-1)}{(2\alpha^2-2\alpha+5)}(1-d/2)}{\alpha(1-\alpha)}\right]$$

$$-\frac{2p^\mu p^\nu}{p^2}\,\frac{\left[-3\alpha^2+3\alpha-3+2d(\alpha-1/2)^2\right](1-d/2)}{\alpha(1-\alpha)}$$

$$\eta^{\mu\nu}\left[-3d+3+2(1-d/2)-\frac{5}{\alpha(1-\alpha)}(1-d/2)\right]$$

$$-\frac{2p^\mu p^\nu}{p^2}\left[+3-\frac{3}{\alpha(1-\alpha)}+\frac{2d(\alpha^2-\alpha+1/4)}{\alpha(1-\alpha)}\right](1-d/2)$$

$$\eta^{\mu\nu}\left[5-4d-\frac{5(1-d/2)}{\alpha(1-\alpha)}\right]-\frac{2p^\mu p^\nu}{p^2}\left[+3-2d+\frac{d/2-3}{\alpha(1-\alpha)}\right](1-d/2)$$

$$\frac{1}{2}g^2C_2(G)\,\delta^{ab}\,\frac{i\Gamma(1-d/2)}{(4\pi)^{d/2}}\,(-p^2)^{d/2-1}$$

$$\left\{\cdot\left[B(d/2,d/2)\left(\eta^{\mu\nu}(5-4d)+2(-3+2d)(1-d/2)\frac{p^\mu p^\nu}{p^2}\right)\right]\right.$$

$$\left.+\;B(d/2-1,d/2-1)\left(-5(1-d/2)\eta^{\mu\nu}-(d-6)(1-d/2)\frac{p^\mu p^\nu}{p^2}\right)\right\}$$

$$B(d/2-1,d/2-1)=\frac{\cancel{\Gamma(d-2)}\;\Gamma(d/2-1)^2}{\Gamma(d-2)}=$$

$$B(d/2,d/2)=\frac{\Gamma(d/2)^2}{\Gamma(d)}=\frac{d^2}{4}\cancel{\frac{\Gamma(d/2-1)^2}{d(d-1)\Gamma(d-2)}}=\frac{d}{\cancel{4(d-1)}}B(d/2-1,d/2-1)$$

$$B(d/2-1,d/2-1)=\frac{4(d-1)}{d}B(d/2,d/2)\qquad=(d/2-1)^2\frac{\Gamma(d/2-1)^2}{(d-1)(d-2)\Gamma(d-2)}=$$

$$=\frac{1}{4}\frac{(d-2)^2}{(d-1)\cancel{(d-2)}}B(d/2-1,d/2-1)$$

$$B(d/2-1,d/2-1)=\frac{1}{4}\frac{4(d-1)}{d-2}B(d/2,d/2)$$

$$B(d/2-1,\, d/2-1) = 4\,\frac{d-1}{d-2}\; B(d/2,\, d/2)$$

 $$= \frac{i}{2}\, g^2\, C_2(G)\, \delta^{ab}\, \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\; (-p^2)^{d/2-1}\; B(d/2,\, d/2)\,.$$

$$\cdot \Big\{ \eta^{\mu\nu}\Big[(5-4d) + \frac{5}{\cancel{2}}\,\cancel{(d-2)}\,\overset{2}{\cancel{4}}\,\frac{(d-1)}{\cancel{d-2}}\Big] + \frac{p^\mu p^\nu}{p^2}\Big[(3-2d)(d-2) + \frac{(d-6)}{\cancel{2}}\,\cancel{(d-2)}\,\frac{\overset{2}{\cancel{4}}(d-1)}{\cancel{d-2}}\Big]$$

$$5-4d + 10d - 10 \qquad\qquad \cancel{-2d^2} + 4d + 3d - 6 + \cancel{2d^2} - 14d + 12$$

$$6d - 5 \qquad\qquad -7d + 6$$

 $$: \ \frac{i}{2}\, g^2\, \frac{C_2(G)\,\delta^{ab}}{(4\pi)^{d/2}}\; B(d/2,\, d/2)\; \Gamma(1-d/2)\; (-p^2)^{d/2-1} \times \Big[\eta^{\mu\nu} - (2-d)\,\frac{p^\mu p^\nu}{p^2}\Big]$$

$d-2$

 $$: \ \underbrace{\qquad\qquad\qquad\qquad\qquad} \times \Big[(6d-5)\,\eta^{\mu\nu} + (6-7d)\,\frac{p^\mu p^\nu}{p^2}\Big]$$

Sum

$$(6d-4)\,\eta^{\mu\nu} + (-6d+4)\,\frac{p^\mu p^\nu}{p^2}$$

1-loop

 $$: \ \frac{i}{2}\, g^2\, \frac{C_2(G)}{(4\pi)^{d/2}}\,\delta^{ab}\; B\Big(\frac{d}{2},\frac{d}{2}\Big)\,\Gamma\Big(1-\frac{d}{2}\Big)\,(-p^2)^{\frac{d}{2}-1}\,(6d-4)\Big(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big)$$

 $$\boxed{= \ i g^2\, \frac{C_2(G)}{(4\pi)^{d/2}}\,\delta^{ab}\; B\Big(\frac{d}{2},\frac{d}{2}\Big)\,\Gamma\Big(1-\frac{d}{2}\Big)\,(3d-2)\,(-p^2)^{\frac{d}{2}-1}\Big(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big)}$$

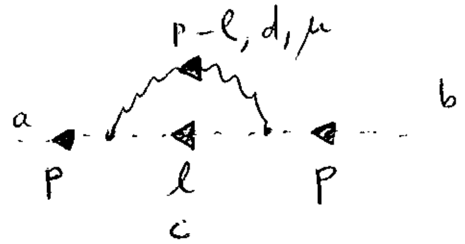
transverse

$$\boxed{\varepsilon = 2 - d/2} \qquad \Gamma(\varepsilon) = \Gamma(2-d/2) \qquad B(2,2) = \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} \qquad (1-d/2)\,\Gamma(1-d/2) = \Gamma(2-d/2)$$

div.

 $$= i g^2\, \frac{C_2(G)}{16\pi^2}\,\delta^{ab}\,\frac{1}{6}\,\frac{\Gamma(\varepsilon)}{-1}\; 10\,(-p^2)\Big(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big) = i g^2\,\frac{C_2(G)}{16\pi^2}\,\delta^{ab}\,\Gamma(\varepsilon)\,\frac{5p^2}{3}\,\mathbb{P}_{\mu\nu}(p)$$

a



$$(-g)^2 \, f^{cdb}_{adc} \, f^{adc}_{bcd} \int \ell^\mu p_\mu \int \frac{i(-i)}{\ell^2 (\ell-p)^2} =$$

$$= g^2 \, (-C_2(G)) \, \delta^{ab} \int \frac{d^d\ell}{(2\pi)^d} \int_0^1 d\alpha \, \frac{(\ell+\alpha p)\cdot p}{(\ell^2-\Delta)^2}$$

$$\Delta = -\alpha(1-\alpha)p^2$$

$$\ell \to \ell + \alpha p$$

$$= -g^2 \, C_2(G) \, \delta^{ab} \, p^2 \int_0^1 d\alpha \, \alpha \, \frac{i}{(4\pi)^{d/2}} \, \frac{\Gamma(2-d/2)}{\Gamma(2)} \, \frac{1}{\Delta^{2-d/2}}$$

$$= +ig^2 \, C_2(G) \, \frac{\delta^{ab}}{(4\pi)^{d/2}} \, \Gamma(2-d/2) \, (-p^2)^{-1+d/2} \int_0^1 d\alpha \, \alpha^{-1+d/2} (1-\alpha)^{-2+d/2}$$

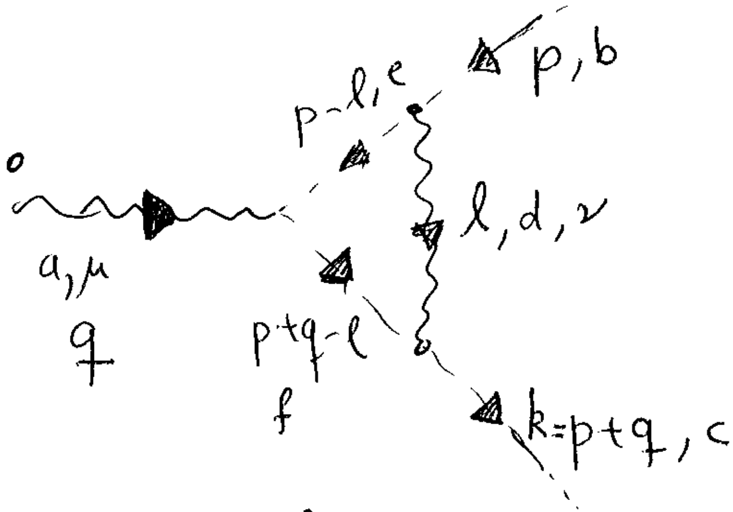
$$B(d/2, d/2-1)$$

 $$= ig^2 \, C_2(G) \, \frac{\delta^{ab}}{(4\pi)^{d/2}} \, \Gamma(2-d/2) \, B(d/2, d/2-1) \, (-p^2)^{d/2-1}$$

$$B(2,1) = \frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}$$

$$\overset{\text{div.}}{=} ig^2 \, C_2(G) \, \frac{\delta^{ab}}{16\pi^2} \, \Gamma(\varepsilon) \, \frac{1}{2} \, (-p^2)$$

(b)



$$(-g)^3 \underbrace{\underbrace{f^{fae}\, f^{cdf}\, f^{edb}}_{fad \quad cef \quad deb}}_{-\frac{1}{2} f^{abc} C_2(G)} \int \frac{d^d l}{(2\pi)^d} \frac{(p+q-l)^\mu (p-l)^\nu (p+q)_\nu \left(\cancel{i^3}(-i)\right)}{(p-l)^2\, (l^2)\, (p+q-l)^2}$$

$$\frac{i}{2} g^3 f^{abc} C_2(G) \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^d l}{(2\pi)^d} \frac{2 k_\nu \left((1-\alpha)p - \beta k - l\right)^\nu \left((1-\beta)k - \alpha p - l\right)^\mu}{(l^2 - \Delta)^3}$$

$$l^2 - 2pl\alpha - 2(p+q)l\beta + \alpha p^2 + \beta (p+q)^2$$

$$(l - \alpha p - \beta(p+q))^2 + \alpha(1-\alpha)p^2 + \beta(1-\beta)k^2 \qquad \Delta = -\alpha(1-\alpha)p^2 - \beta(1-\beta)k^2$$

$$l \to l + \alpha p + \beta k$$

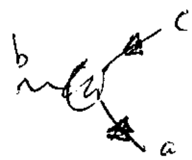
$$\frac{i}{2} g^3 f^{abc} C_2(G) \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^d l}{(2\pi)^d} \frac{2 k_\nu}{(l^2-\Delta)^3} \left[\frac{1}{d} l^2 g^{\mu\nu} + ((1-\alpha)p - \beta k)^\nu ((1-\beta)k - \alpha p)^\mu\right]$$

Divergent piece: (C)

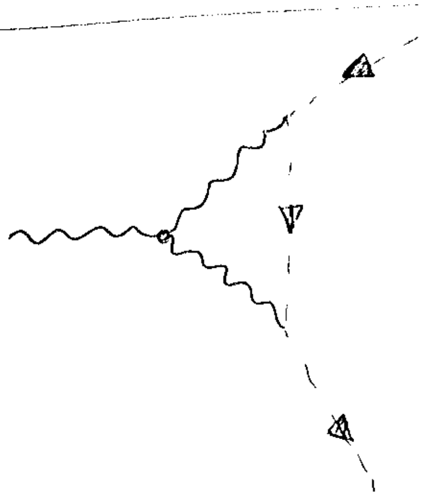
$$\frac{i}{2} g^3 f^{abc} \frac{C_2(G)}{d} k_\mu \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{2i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)} \frac{1}{\Delta^{2-d/2}}$$

$$\xrightarrow{\text{div.}} 2\frac{i}{2} g^3 f^{abc} \frac{C_2(G)}{4} k_\mu \frac{i\,\cancel{2}}{16\pi^2} \frac{\Gamma(\epsilon)}{\cancel{2}} \underbrace{\int_0^1 d\alpha \int_0^{1-\alpha} d\beta}_{1/2}$$

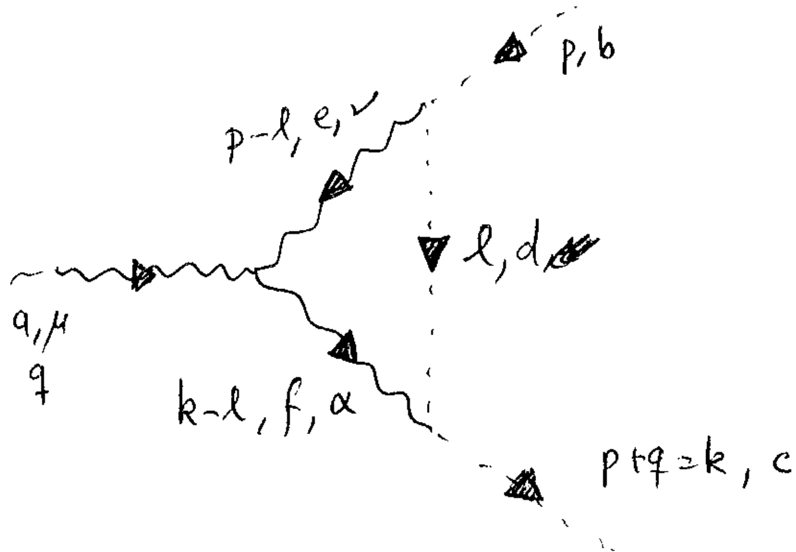
$$-\frac{1}{16\pi^2} \frac{1}{8} C_2(G) f^{abc} g^3 k_\mu \Gamma(\epsilon)$$



f^{bca}



(d)



$$(-g)^2 g\, \underset{+}{f^{deb}_{edb}}\, \underset{+}{f^{cfd}_{cfe}}\, \underset{-}{f^{afe}_{afd}} \left[\eta^{\mu\alpha}(q+k-l)^\nu + \eta^{\alpha\nu}(-k+l-p+l)^\mu + \right.$$

$$\left. + \eta^{\nu\mu}(p-l-q)^\alpha \right] l^\nu k^\alpha \frac{i\, i\, (-i)}{l^2 (k-l)^2 (p-l)^2}$$

$\int \frac{d^dl}{(2\pi)^d}$

$$-ig^3 \frac{1}{2} f^{abc} C_2(G) \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{2}{(l^2-\Delta)^3} \left[\eta^{\mu\alpha}(2k-p-l+\alpha p+\beta k)^\nu + \eta^{\alpha\nu}(2l-(2+\alpha)p-(2+\beta)k) \right.$$

$$\left. + \eta^{\mu\nu}(2p-k-l+\alpha p+\beta k)^\alpha \right].$$

$\Delta = -[\alpha(\alpha-1)p^2 + \beta(1-\beta)k^2 - 2\alpha\beta pk]$

$l \to l - \alpha p - \beta k$

$\cdot\, (l-\alpha p-\beta k)^\nu k^\alpha$

$l^\alpha l^\beta \to \frac{1}{d} \eta^{\alpha\beta}$. divergent piece comes from $l \cdot l$.

<u>div</u>

$$-\frac{i}{2} g^3 f^{abc} C_2(G) \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^dl}{(2\pi)^d} \frac{k^\alpha}{d} \frac{2l^2}{(l^2-\Delta)^3} \left[\eta^{\mu\alpha}(-d) + \eta^{\alpha\nu} 2\eta^{\mu\nu} - \eta^{\mu\nu}\eta^{\alpha\nu} \right]$$

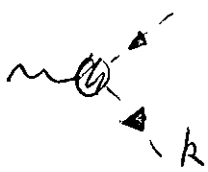
$\eta^{\mu\alpha}(1-d)$

$$= \frac{-i}{2} g^3 f^{abc} k^\mu C_2(G) \frac{(1-d)}{d} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)} \frac{2}{\Delta^{2-d/2}}$$

e

div.

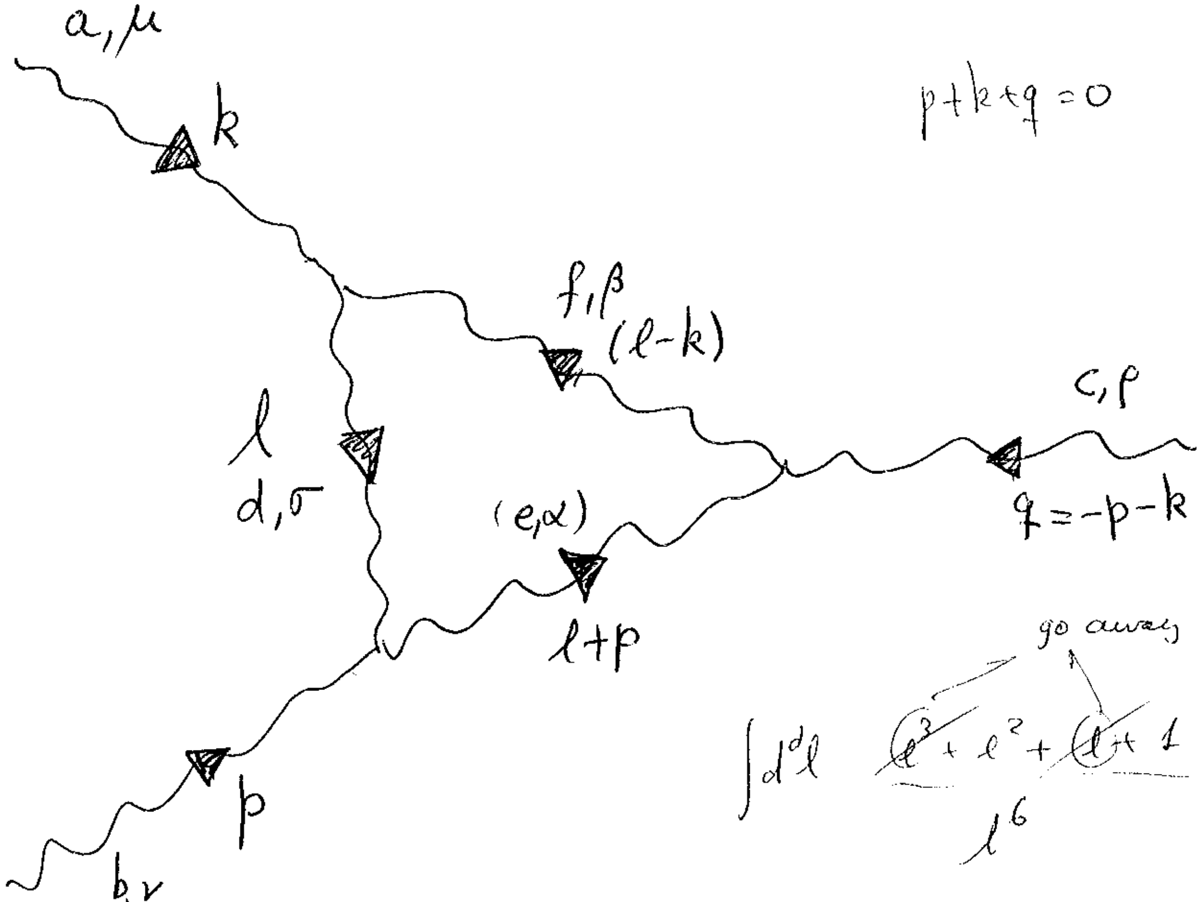
$$= \frac{1}{8} g^3 f^{abc} \frac{k^\mu C_2(G)}{16\pi^2} (-3) \frac{1}{2} \Gamma(\epsilon)$$



$$\frac{1}{16\pi^2} C_2(G) f^{abc} g^3 k_\mu \Gamma(\epsilon) \left[-\frac{1}{8} - \frac{3}{8} \right]$$

$$-\frac{1}{2} \frac{1}{16\pi^2} C_2(G) f^{abc} g^3 k_\mu \Gamma(\epsilon)$$

$p+k+q=0$



go away by Lorentz inv.

$$\int d^d l \; \frac{\cancel{l^3} + l^2 + \cancel{l} + 1}{l^6} \sim \text{log. div.}$$

$$\int d^d k \; \frac{1}{(l-k)^2\, l^2\, (l+p)^2} \rightarrow \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \frac{2}{\left((l+\alpha p-\beta k)^2 + \alpha(1-\alpha)p^2 + \beta(1-\beta)k^2 + 2\alpha\beta\, pk\right)^3}$$

$$\Delta = -\left[\alpha(1-\alpha)\, p^2 + \beta(1-\beta)\, k^2 + 2\alpha\beta\, p\cdot k\right]$$

$$l \rightarrow l - \alpha p + \beta k$$

$$\underbrace{g^3 f^{adf} f^{dbe} f^{ecf}}_{\text{vertices}} \; \underbrace{(-i)^3}_{\text{prop.}} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int d^4 l \; \frac{2}{(l^2-\Delta)^3} \times$$

$$\times [\quad][\quad][\quad]$$

Numerators from vertices

(8)

$$\left.\begin{aligned}&[\eta^{\mu\sigma}(k+\ell)^\beta+\eta^{\sigma\beta}(-2\ell+k)^\mu+\eta^{\beta\mu}(\ell-2k)^\sigma]\\&[\eta^{\sigma\nu}(\ell-p)^\alpha+\eta^{\nu\alpha}(2p+\ell)^\sigma+\eta^{\alpha\sigma}(-2\ell-p)^\nu]\\&[\eta^{\alpha\rho}(\ell+2p+k)^\beta+\eta^{\rho\beta}(\ell-2k-p)^\alpha+\eta^{\alpha\beta}(-2\ell-k-p)^\rho]\end{aligned}\right\}\quad \ell\to\ell-\alpha p+\beta k$$

$$[\eta^{\mu\sigma}\ell^\beta-2\eta^{\sigma\beta}\ell^\mu+\eta^{\beta\mu}\ell^\sigma]+[\eta^{\mu\sigma}((1+\beta)k-\alpha p)^\beta+\eta^{\sigma\beta}(2\alpha p+(1-2\beta)k)^\mu+\eta^{\beta\mu}(-\alpha p+(\beta-2)k)^\sigma]$$

$$[\eta^{\sigma\nu}\ell^\alpha+\eta^{\nu\alpha}\ell^\sigma-2\eta^{\alpha\sigma}\ell^\nu]+[\eta^{\sigma\nu}(-(1+\alpha)p+\beta k)^\alpha+\eta^{\nu\alpha}((2-\alpha)p+\beta k)^\sigma+\eta^{\alpha\sigma}((2\alpha-1)p-2\beta k)^\nu]$$

$$[\eta^{\alpha\rho}\ell^\beta+\eta^{\beta\rho}\ell^\alpha-2\eta^{\alpha\beta}\ell^\rho]+[\eta^{\alpha\rho}((2-\alpha)p+(1+\beta)k)^\beta+\eta^{\beta\rho}((\beta-2)k-(1+\alpha)p)^\alpha+\eta^{\alpha\beta}((2\alpha-1)p-(1+2\beta)k)^\rho]$$

$$\ell^\alpha\ell^\beta\to\frac{1}{d}\eta^{\alpha\beta}\ell^2\quad;\ \ell^3\to0\quad;\ \ell\to0\ .$$

$$\frac{\ell^2}{d}\Big\{\Big(\underset{5}{\eta^{\mu\nu}\eta^{\alpha\beta}}+\underset{(d-2)}{\eta^{\nu\alpha}\eta^{\mu\rho}}-\underset{-4}{2\eta^{\alpha\mu}\eta^{\beta\nu}}-2\eta^{\mu\alpha}\eta^{\nu\beta}-2\eta^{\nu\alpha}\eta^{\mu\beta}+4\eta^{\alpha\beta}\eta^{\mu\nu}+\eta^{\beta\mu}\eta^{\alpha\nu}+\eta^{\beta\mu}\eta^{\nu\alpha}\cdot d-2\eta^{\alpha\nu}\eta^{\beta\mu}\Big)$$

$$\left(\eta^{\alpha\rho}((2-\alpha)p+(1+\beta)k)^\beta+\eta^{\beta\rho}((\beta-2)k-(1+\alpha)p)^\alpha+\eta^{\alpha\beta}((2\alpha-1)p-(1+2\beta)k)^\rho\right)+$$

$$+\Big(\underset{(d-2)}{d\eta^{\alpha\rho}\eta^{\mu\sigma}}+\eta^{\mu\sigma}\eta^{\rho\alpha}-2\eta^{\alpha\rho}\eta^{\mu\sigma}-2\eta^{\sigma\mu}\eta^{\alpha\rho}-\underset{-4}{2\eta^{\rho\sigma}\eta^{\alpha\mu}}+\underset{5}{4\eta^{\alpha\sigma}\eta^{\mu\rho}}+\eta^{\mu\sigma}\eta^{\alpha\rho}+\eta^{\mu\rho}\eta^{\alpha\sigma}-2\eta^{\alpha\mu}\eta^{\rho\sigma}\Big)$$

$$\left(\eta^{\sigma\nu}(-(1+\alpha)p+\beta k)^\alpha+\eta^{\nu\alpha}((2-\alpha)p+\beta k)^\sigma+\eta^{\alpha\sigma}((2\alpha-1)p-2\beta k)^\nu\right)$$

$$+\Big(\underset{(d-2)}{\eta^{\sigma\nu}\eta^{\rho\beta}}+\eta^{\sigma\nu}\eta^{\beta\rho}\cdot d-2\eta^{\sigma\nu}\eta^{\beta\rho}+\underset{5}{\eta^{\nu\rho}\eta^{\beta\sigma}}+\eta^{\beta\rho}\eta^{\nu\sigma}-\underset{-4}{2\eta^{\nu\beta}\eta^{\rho\sigma}}-2\eta^{\rho\sigma}\eta^{\nu\beta}-2\eta^{\rho\beta}\eta^{\sigma\nu}+4\eta^{\beta\sigma}\eta^{\rho\nu}\Big)$$

$$\left(\eta^{\mu\sigma}((1+\beta)k-\alpha p)^\beta+\eta^{\sigma\beta}(2\alpha p+(1-2\beta)k)^\mu+\eta^{\beta\mu}(-\alpha p+(\beta-2)k)^\sigma\right)\Big\}$$

(9)

$$\frac{\ell^2}{d}\Big\{5\eta^{\mu\nu}\big((2-\alpha)p+(1+\beta)k+(\beta-2)k-(1+\alpha)p+d(2\alpha-1)p-d(1+2\beta)k\big)^\rho$$

$$+(d-2)\eta^{\nu\rho}\big((2-\alpha)p+(1+\beta)k\big)^\mu+(d-2)\eta^{\mu\rho}\big((\beta-2)k-(1+\alpha)p\big)^\nu+$$

$$+(d-2)\eta^{\mu\nu}\big((2\alpha-1)p-(1+2\beta)k\big)^\rho$$

$$-4\eta^{\mu\rho}\big((2-\alpha)p+(1+\beta)k\big)^\nu-4\eta^{\rho\nu}\big((\beta-2)k-(1+\alpha)p\big)^\mu-$$

$$-4\eta^{\mu\nu}\big((2\alpha-1)p-(1+2\beta)k\big)^\rho$$

$$+5\eta^{\mu\rho}\big(-(1+\alpha)p+\beta k+(2-\alpha)p+\beta k+d(2\alpha-1)p-2d\beta k\big)^\nu$$

$$+(d-2)\eta^{\mu\nu}\big(-(1+\alpha)p+\beta k\big)^\rho+(d-2)\eta^{\nu\rho}\big((2-\alpha)p+\beta k\big)^\mu+$$

$$+(d-2)\eta^{\mu\rho}\big((2\alpha-1)p-2\beta k\big)^\nu$$

$$-4\eta^{\rho\nu}\big(-(1+\alpha)p+\beta k\big)^\mu-4\eta^{\mu\nu}\big((2-\alpha)p+\beta k\big)^\rho-4\eta^{\mu\rho}\big((2\alpha-1)p-2\beta k\big)^\nu$$

$$+5\eta^{\nu\rho}\big((1+\beta)k-\alpha p+2\alpha dp+(1-2\beta)dk-\alpha p+(\beta-2)k\big)^\mu$$

$$+(d-2)\eta^{\mu\nu}\big((1+\beta)k-\alpha p\big)^\rho+(d-2)\eta^{\rho\nu}\big(2\alpha p+(1-2\beta)k\big)^\mu+(d-2)\eta^{\mu\rho}\big(-\alpha p+(\beta-2)k\big)^\nu$$

$$-4\eta^{\mu\rho}\big((1+\beta)k-\alpha p\big)^\nu-4\eta^{\rho\nu}\big(2\alpha p+(1-2\beta)k\big)^\mu-4\eta^{\mu\nu}\big(-\alpha p+(\beta-2)k\big)^\rho\Big\}$$

$$\frac{l^2}{d}\Big\{ \eta^{\mu\nu} p^\rho \big(10 - 5\alpha - 5 - 5\alpha + 10 d\alpha \boxed{-5d} \cancel{+2d\alpha} \boxed{-d} \cancel{-4\alpha} \cancel{+2} \cancel{-8\alpha} \cancel{+4}$$
$$\boxed{-d} \cancel{-d\alpha} \cancel{+2} \cancel{+2\alpha} \cancel{-8} \cancel{+4\alpha} \cancel{+d\alpha} \cancel{+2\alpha} \cancel{+4\alpha} \big)$$

$$\eta^{\mu\nu} k^\rho \big(5 + 5\beta + 5\beta - 10 - 5d - 10d\beta \cancel{-d} \cancel{+2d\beta} \cancel{+2} \cancel{+4\beta} + 4 \cancel{+8\beta}$$
$$+ \cancel{\beta d} \cancel{-2\beta} \cancel{+4\beta} \cancel{+d} \cancel{+\beta d} \cancel{+2} \cancel{-2\beta} \cancel{+4\beta} + 8 \big)$$

$$\eta^{\nu\rho} p^\mu \big(2d \cancel{-d\alpha} \cancel{+4} \cancel{+2\alpha} \cancel{+4} \cancel{+4\alpha} + 2d \cancel{-\alpha d} \cancel{+4} \cancel{+2\alpha} \cancel{+4} \cancel{+4\alpha} -$$
$$-5\alpha + 10\alpha d - 5\alpha + \cancel{2\alpha d} \cancel{-4\alpha} \cancel{-8\alpha} \big)$$

$$\eta^{\nu\rho} k^\mu \big(d \cancel{+\beta d} \cancel{+2} \cancel{+2\beta} \cancel{+4\beta} \cancel{+8} \cancel{+\beta d} \cancel{-2\beta} \cancel{+4\beta} + 5 + 5\beta$$
$$+ 5d - 10\beta d + 5\beta - 10 \cancel{+4} \cancel{+8\beta} \cancel{+d} \cancel{-2d\beta} \cancel{+2} \cancel{+4\beta} \big)$$

$$\eta^{\mu\rho} p^\nu \big(-d \cancel{-d\alpha} \cancel{+2} \cancel{+2\alpha} \cancel{+8} \cancel{+4\alpha} - 5 - 5\alpha + 10 - 5\alpha + 10d\alpha - 5d$$
$$+ \cancel{2d\alpha} - d \cancel{+4\alpha} \cancel{+2} \cancel{-8\alpha} \cancel{+4} \cancel{-d\alpha} \cancel{+2\alpha} \cancel{+4\alpha} \big)$$

$$\eta^{\mu\rho} k^\nu \big(\cancel{d\beta} - 2d \cancel{-2\beta} \cancel{+4} \cancel{+4} \cancel{-4\beta} + 5\beta + 5\beta - 10d\beta \cancel{+2d\beta} \cancel{+4\beta}$$
$$+ \cancel{8\beta} \cancel{+\beta d} - 2d \cancel{-2\beta} \cancel{+4} \cancel{+4} \cancel{+4\beta} \big) \Big\}$$

$$\frac{l^2}{d}\Big\{ \eta^{\mu\nu} p^\rho (5 - 7d - 10\alpha + 10 d\alpha) + \eta^{\mu\nu} k^\rho (7 + 10\beta - 5d - 10d\beta)$$
$$\eta^{\rho\nu} p^\mu (4d - 10\alpha + 10\alpha d) + \eta^{\nu\rho} k^\mu (-5 + 10\beta + 7d - 10d\beta)$$
$$\eta^{\mu\rho} p^\nu (5 - 10\alpha - 7d + 10d\alpha) + \eta^{\mu\rho} k^\nu (10\beta - 4d - 10d\beta) \Big\}$$

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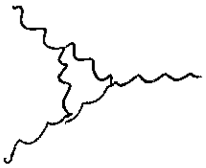

$$g^3 \frac{1}{2} f^{abc} C_2(G)\, i \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; 2 \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)} \frac{1}{\Delta^{2-d/2}}$$

$$\times \Big[\eta^{\mu\nu} p^\rho (5-7d-10\alpha+10d\alpha) + \eta^{\mu\nu} k^\rho (7+10\beta-5d-10d\beta)$$

$$\eta^{\rho\nu} p^\mu (4d-10\alpha+10\alpha d) + \eta^{\nu\rho} k^\mu (-5+10\beta+7d-10d\beta)$$

$$\eta^{\mu\rho} p^\nu (5-10\alpha-7d+10d\alpha) + \eta^{\mu\rho} k^\nu (10\beta-4d-10d\beta) \Big]$$

Divergent piece $d = 4-2\varepsilon$ $\Gamma(\varepsilon)$; $d \to 4$

$$-g^3 f^{abc} C_2(G) \frac{\cancel{4}}{\cancel{4}} \frac{\Gamma(\varepsilon)}{(4\pi)^2} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \Big[\eta^{\mu\nu} p^\rho (-23+30\alpha) + \eta^{\mu\nu} k^\rho (-13-30\beta)$$

$$\eta^{\rho\nu} p^\mu (16+30\alpha) + \eta^{\nu\rho} k^\mu (23-30\beta)$$

$$\eta^{\mu\rho} p^\nu (-23+30\alpha) + \eta^{\mu\rho} k^\nu (-16-30\beta) \Big]$$

$$\int_0^1 d\alpha \int_0^{1-\alpha} d\beta = \int_0^1 (1-\alpha)\, d\alpha = 1 - \frac{1}{2} = 1/2$$

$$\alpha = \int \alpha (1-\alpha) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\beta \quad \frac{(1-\alpha)^2}{2} = \frac{1}{2}\left(\cancel{1} - \cancel{1} + \frac{1}{3}\right) = \frac{1}{6}$$

$$\boxed{-\frac{g^3}{16\pi^2} f^{abc} C_2(G) \Gamma(\varepsilon) \Big[\eta^{\mu\nu} p^\rho (-13) + \eta^{\mu\nu} k^\rho (-23) + \eta^{\rho\nu} p^\mu\, 26 + \eta^{\nu\rho} k^\mu (13) + \eta^{\mu\rho} p^\nu (-13) + \eta^{\mu\rho} k^\nu (-26) \Big]}$$