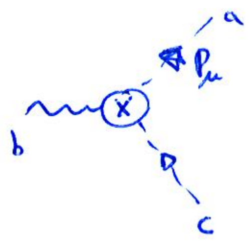
"Counter vertices."

 $= -ig^2 \frac{C_2(G)}{16\pi^2}\delta^{ab}\frac{1}{\varepsilon}\frac{5}{3}p^2\mathbb{P}_{\mu\nu}(p) = -ip^2\mathbb{P}_{\mu\nu}\,\delta_3\,\delta^{ab}$

 $= +ig^2\frac{C_2(G)}{32\pi^2}\delta^{ab}\frac{1}{\varepsilon}p^2 \qquad = +ip^2\delta_{2c}$

 $= \frac{g^3}{32\pi^2}C_2(G)\,p_\mu\,\frac{1}{\varepsilon} \qquad = \cancel{-}ig\,p_\mu\,\delta_{1c}$

Counter terms:

$$S = -\int\frac{1}{4g^2}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} = -\int\frac{1}{2}(\partial_\mu A_\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

∂ by parts.

∂_μ

$$= -\int\frac{1}{2}\partial_\mu A_\nu\,\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A^\mu\,\partial_\nu A^\nu$$

$V \to \quad \mathcal{L} = T - V \qquad e^{-iHt} \quad \to \quad \underline{\underline{iV}}$

$AA \to \quad 2iV$

$p^2\mathbb{P}_{\mu\nu} = p^2\delta_{\mu\nu} - p_\mu p_\nu \quad \to$

$$V = \delta_3\left(\frac{1}{2}(\partial_\mu A_\nu\,\partial^\mu A^\nu - \partial_\mu A_\mu\,\partial^\nu A_\nu)\right)$$

$\downarrow$

$$i(-p^2\delta_{\mu\nu} + p_\mu p_\nu)\delta_3 = -ip^2\mathbb{P}_{\mu\nu}\,\delta_3$$

$$\text{---}\ \delta_{2c}\, \partial^\mu \bar{c}^a\, \partial_\mu c_a \quad \rightarrow \quad +ip^2\,\delta_{2c}$$

$$-ip \qquad ip \qquad\qquad c\,\bar{c}\,A.$$

$$g\,\delta_{k}\, f^{abc}\, \partial_\mu \bar{C}^a A^{\mu\, b} C^c \;\rightarrow\; g\, i\, \delta_{1c}\, i p_\mu\, f^{abc} \;=\; -g\,\delta_{1c}\, P_\mu\, f^{abc}$$

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + \partial^\mu \bar{c}^a \partial_\mu c^a + g\,\partial^\mu \bar{c}^a f^{abc} A^b_\mu C^c$$

$$-\frac{1}{4}\delta_3\, \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + \delta_{2c}\, \partial^\mu \bar{c}^a \partial_\mu c_a + g\,\delta_{1c}\, \partial^\mu \bar{c}^a f^{abc} A^b_\mu C^c$$

$$\mathcal{L}_{(0)} = -\frac{1}{4}(1+\delta_3)\, \mathrm{Tr}\, F F^{\nu} + (1+\delta_{2c})\, \partial^\mu \bar{c}^a \partial_\mu c^a_a + g(1+\delta_{1c})\, \partial^\mu \bar{c}^a f^{abc} A^b_\mu c^c$$

$$\overset{\text{bare}}{A^{(0)}_\mu} = \sqrt{1+\delta_3}\ \overset{\text{renormalized}}{A^R_\mu}$$

$$C^a_{(0)} = \sqrt{1+\delta_{2c}}\ C^{aR}$$

$$\bar{C}^a_{(0)} = \sqrt{1+\delta_{2c}}\ \bar{C}^{aR}$$

$$g_R\,(1+\delta_{1c})\,(1+\delta_{2c})^{-1}\left(\sqrt{1+\delta_3}\right)^{-1} \partial^\mu \bar{C}^a_{(0)}\, A^b_{\mu(0)}\, C^c_{(0)}\, f^{abc}$$

$$g_0 = \left(1+\delta_{1c} - \delta_{2c} - \frac{1}{2}\delta_3\right) g_R$$

$$\delta_3 = g^2 \frac{C_2(G)}{32\pi^2} \frac{2}{\varepsilon} \frac{5}{3}$$

$$\delta_{2c} = g^2 \frac{C_2(G)}{32\pi^2} \frac{1}{\varepsilon}$$

$$\delta_{1c} = - g^2 \frac{C_2(G)}{32\pi^2} \frac{1}{\varepsilon}$$

$$g_0 = \left(1 + \frac{C_2(G)}{32\pi^2} \frac{g_R^2}{\varepsilon} \left(-1-1-\frac{5}{3}\right)\right) g_R$$

$$\boxed{g_0 = \left(1 - \frac{11}{3} \frac{C_2(G)}{32\pi^2} \frac{g_R^2}{\varepsilon}\right) g_R}$$

We introduce a scale. g_R is adimensional but g_0 is <u>not</u>

$$S^{(d)} = \int \overset{-d}{d^d x} \left(\overset{+1}{\partial} \underset{\frac{d-2}{2}}{A} \overset{+1}{\partial} A + g \underbrace{\partial A A A}_{\substack{\frac{3}{2}(d-2)+1 \\ \frac{3}{2}d-2}} + g^2 AAAA - \cdots \right)$$

$$[g_0] = \mu^{-\frac{3}{2}d+2+d = -\frac{d}{2}+2 = \frac{4-d}{2} = \varepsilon}$$

mass

$$[g_0) = \mu^{\varepsilon}$$

So we should have used:

$$\int d^d x \,(- - \; (\mu^{\varepsilon} g_R) + -$$

as coupling const.

$$g_0 = \mu^\varepsilon g_R \left(1 - \frac{11}{3}\frac{C_2(G)}{32\pi^2}\frac{g_R^2}{\varepsilon}\right)$$

we keep this.

we can put $\mu^{2\varepsilon} g_R^2$

but $\mu^{2\varepsilon} = 1 + \underbrace{2\varepsilon \log\mu}$
goes to finite piece
and we choose ~~to~~ not to ~~remove it~~ subtract it.

this is a definition of g_R. We could use another $\tilde{g}_R = g_R + .\varepsilon + .\varepsilon^2 + - -$

We cannot do that if we require only poles.

$$g_0 = \mu^\varepsilon \left(g_R + \sum_{r=1}^{\infty} \frac{a_r(g_R)}{\varepsilon^r}\right)$$

~~g_0~~ How does the definition of g_R changes if we change μ? Notice g_0 is the same.

$\mu \to (1+\delta)\mu \quad \delta \ll 1$

$\mu'^\varepsilon = (1+\delta)^\varepsilon \mu^\varepsilon \simeq (1+\varepsilon\delta)\mu$ (approximation for small δ) (not ε)

$$\boxed{\mu' = (1+\delta)\mu}$$

$$g_0 = (\hat{\mu})^\varepsilon \left(\tilde{g}_R + \sum_{r=1}^{\infty} \frac{a_r(\tilde{g}_R)}{\varepsilon^r}\right)$$

but

$$g_0 = (1+\varepsilon\delta)\mu^\varepsilon \left(\tilde{g}_R + \sum_{r=1}^{\infty} \frac{a_r(\tilde{g}_R)}{\varepsilon^r}\right)$$

$$= \mu^\varepsilon \left(\tilde{g}_R + \frac{a_1(\tilde{g}_R)}{\varepsilon} + \frac{a_2(\tilde{g}_R)}{\varepsilon^2} + \dots + \cancel{\varepsilon\delta}\right.$$

$$\left. + \varepsilon\delta\tilde{g}_R + \delta . a_1(\tilde{g}_R) + \frac{\delta . a_2(\tilde{g}_R)}{\varepsilon} + \dots \right)$$

gives the same.

$\Downarrow$

Also

$$(1+\rho\varepsilon) f(\lambda_R + \rho\,\delta\lambda_R) =$$

$\rho \leftrightarrow \delta$

$$= f(\lambda_R) + \rho\,\delta\lambda_R f'(\lambda_R) + \rho\varepsilon f(\lambda_R)$$

$$\delta\lambda_R f'(\lambda_R) + \varepsilon f(\lambda_R) = 0$$

$$\delta\lambda_R = -\varepsilon \frac{f(\lambda_R)}{f'(\lambda_R)}$$

$$f(\lambda_R) = \lambda_R + \sum a_r/\varepsilon^r$$

$$f'(\lambda_R) = 1 + \sum a'_r/\varepsilon^r$$

$$\delta\lambda_R = -\varepsilon\lambda_R - \varepsilon b_1 + \sum \frac{b_r}{\varepsilon^r}$$

$(b_{r\geq 2} = 0)$

~~is~~ not acceptable we redefine $\tilde{g}_R \to g_R$

$$\tilde{g}_R = g_R + \delta . g_R^{(1)}$$

$$= g_R + \delta(d_0 + \varepsilon d_1 + \varepsilon^2 d_2 + \dots)$$

$$g_0 = \mu^\varepsilon \left(g_R + \delta d_0 + \varepsilon\delta d_1 + \varepsilon^2 \delta d_2 + \dots \right.$$

$$\left. + \dots \right)$$

We ~~show~~ it's enough with d_0, d_1. If so adding $d_2, \dots$ etc. would ~~only~~ give rise to another $g(g_R)$ that ~~is finite~~. gives finite results. $\Rightarrow$ should be unique.

$$g_0 = \mu^{\varepsilon}\Big(\underline{\underline{g_R}} + \delta d_0 + \varepsilon \delta d_1 + \boxed{\frac{a_1(g_R)}{\varepsilon}} + \frac{\delta a'_1}{\varepsilon}(d_0 + \underline{\underline{\varepsilon d_1}}) + \frac{\delta a'_2}{\varepsilon^2}(d_0 + \varepsilon d_1) + \varepsilon \delta g_R + \delta a_1 + \delta \frac{a_2}{\varepsilon} + \ldots \Big)$$

$$d_1 = -g_R$$

$$\delta d_0 + \delta d_1 a'_1 + \delta a_1 = 0$$

$$a_1 + \delta a'_1 d_0 + \delta a'_2 d_1 + \delta a_2 = 0$$

$$\frac{1}{\varepsilon^r} : \quad \frac{a_r}{\varepsilon^r} + \frac{\delta a'_r}{\varepsilon^r} d_0 + \frac{\delta a'_{r+1}}{\varepsilon^{r+1}} \varepsilon d_1 + \delta \frac{a_{r+1}}{\varepsilon^r} = 0$$

$$\cancel{d_1 = (-a_1 - d_0)\frac{1}{a'_1}} \qquad d_0 = -a_1 - a'_1 d_1 = -a_1 + a'_1 g_R$$

$$a'_r d_0 + a'_{r+1} d_1 + a_{r+1} = 0$$

$$\cancel{a'_{r+1} = \frac{1}{d_1}(-a_r - a'_r d_0)}$$

$$(a_{r+1} + d_1 a'_{r+1}) = -a'_r d_0$$

$$\tilde{g}_R = g_R + \delta\,(d_0 + \varepsilon d_1)$$

$$\frac{1}{\delta}\Big(g_R(\tilde{\mu}) - g_R(\mu)\Big) = d_0 + \varepsilon d_1$$

$$\tilde{\mu} = (1+\delta)\mu = \mu + \delta\mu$$

$$\tilde{\mu} - \mu = \mu \cdot \delta$$

$$\mu \, \frac{g_R(\tilde{\mu}) - g_R(\mu)}{\tilde{\mu} - \mu} = d_0 + \varepsilon d_1$$

$$\boxed{\mu \partial_\mu g_R = d_0} \qquad \varepsilon \to 0$$

$$\boxed{\mu \partial_\mu g_R = -a_1 + g_R \frac{\partial}{\partial g_R} a_1} \; \boxed{= \beta(g_R)}$$

$$a_1 = -\frac{11}{3} \frac{C_2(G)}{32\pi^2} g_R^3 = b_1 g_R^3$$

$$\cancel{-b_1} \quad \beta = -b_1 g_R^3 + 3 g_R^3 b_1 = 2 b_1 g_R^3$$

$$\boxed{\beta = -\frac{11}{3} \frac{C_2(G)}{16\pi^2} g_R^3}$$

$$\mu \partial_\mu g_R = -\beta_0 g^3$$

$$\int \frac{\partial g}{g^3} = -\int \frac{\beta_0}{\mu}$$

$$\cancel{+}\frac{1}{2g^2}\Big|_{g_0}^{g} = \cancel{+}\beta_0 \ln \mu \Big|_{\mu_0}^{\mu}$$

$$\frac{1}{g^2} - \frac{1}{g_0^2} = 2\beta_0 \ln \frac{\mu}{\mu_0}$$

$$\frac{1}{g^2} = \frac{1}{g_0^2} + 2\beta_0 \ln \mu/\mu_0$$

$$g^2 = \frac{1}{\frac{1}{g_0^2} + 2\beta_0 \ln \frac{\mu}{\mu_0}} = \frac{g_0^2}{1 + 2\beta_0 g_0^2 \ln \frac{\mu}{\mu_0}}$$

$\mu \to \infty \quad g^2(\mu) \to 0$

$$1 + 2\beta_0 g_0^2 \ln \frac{\Lambda}{\mu_0} = 0 \qquad \ln \frac{\Lambda}{\mu_0} = -\frac{1}{2\beta_0 g_0^2}$$

$$\boxed{\Lambda = \mu_0 \, e^{-\frac{1}{2\beta_0 g_0^2}}}$$

$$g^2 = \frac{g_0^2}{1 + 2\beta_0 g_0^2 \ln\left(\frac{\Lambda}{\mu_0}\frac{\mu}{\Lambda}\right)} = \frac{\cancel{g_0^2}}{2\beta_0 \cancel{g_0^2} \ln\left(\frac{\mu}{\Lambda}\right)} = \frac{1}{2\beta_0 \ln\left(\frac{\mu}{\Lambda}\right)}$$

$$\boxed{g^2(\mu) = \frac{1}{2\beta_0 \ln(\mu/\Lambda)}}$$

Λ is the only parameter!!
but it just sets the scale.

no parameter at all!

Also notice:

$$a_{r+1} + d_1 a'_{r+1} = -a'_r d_0$$

$$a_{r+1} - g_R a'_{r+1} = -\beta a'_r$$

$$a_{r+1} = \sum a_{r+1\,p}\, g_R^p$$

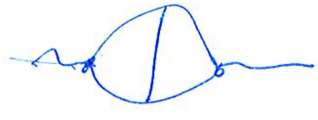
$$\sum_p a_{r+1\,p}\,(1-p)\, g_R^p$$

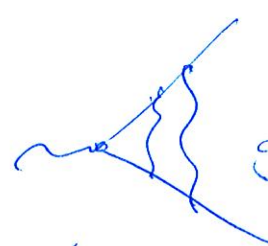
$$\sum_{p\geq 1} (1-p)\, a_{r+1\,p}\, g_R^p = -\underset{\sim g_R^3}{\beta}\, \underset{g_R^{p-1}}{a'_r}$$

<u>e.g.</u>

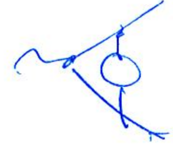
$$a_2 - g_R a'_2 = -\underset{g^3}{\beta}\, \underset{g^2}{a'_1} \sim g^5$$

$a_2 \sim g^5$





g^5



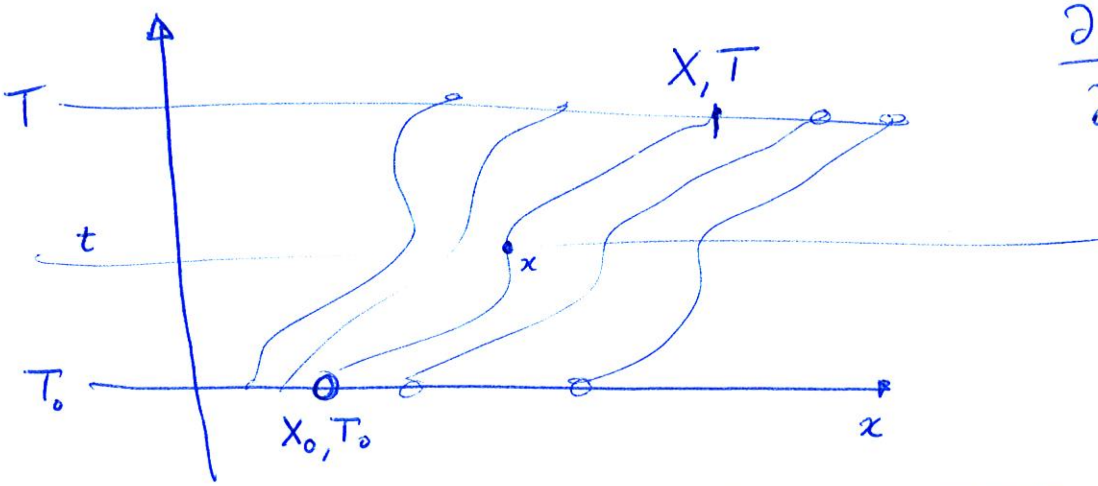
g^5

<u>2 loops</u>

2loops $\frac{1}{\varepsilon^2}$ given by 1-loop $\frac{1}{\varepsilon}$

$$a_3 \sim \beta a'_2 \sim g^3 g^4 \sim g^7$$ ~ 3 loops etc.

"Bacterial model"



$$\frac{\partial \rho(X,T)}{\partial T} + v(X)\,\partial_X \rho = L(X)$$

$x(t; X, T)$
final point

$$\boxed{\begin{aligned} &x(T; X, T) = X \\ &\frac{dx}{dt}(t; X, T) = v(x(t; X, T)) \end{aligned}}$$

equivalent

$$\boxed{\int_x^X \frac{dy}{v(y)} = T - t}$$

•) change $t \to t + \delta t$; $\int_{x+\delta x}^{X} \frac{dy}{v(y)} = T - t - \delta t \Rightarrow -\frac{\delta x}{v(x)} = -\delta t$

$$\Rightarrow \boxed{\partial_t x = v(x)}$$

•) change $T \to T + \delta T \to -\frac{\delta X}{v(X)} = \delta T \to \boxed{\partial_T x = -v(x)}$

•) change $X \to X + \delta X$. $\int_{x+\delta x}^{X+\delta X} \frac{dy}{v(y)} = T - t$ $\quad \frac{\delta X}{v(X)} - \frac{\delta x}{v(x)} = 0$

$$\boxed{\partial_X x = \frac{v(x)}{v(X)}}$$

depend on X, T

$$\rho(X,T) = \rho(X_0, T_0)\, e^{\int_{T_0}^{T} L(x(t;X,T))\,dt}$$

fixed.

$$\frac{\partial \rho}{\partial T} = \partial_{X_0}\rho\, \partial_T X_0\, e^{\int} + \rho(X_0,T_0)\, L(x(T,X,T))\, e^{\int}$$

$$+ \rho_0 \int_{T_0}^{T} \partial_T L(x(t;X,T))\,dt\; e^{\int}$$

$$= -\partial_{X_0}\rho_0\, \partial_1 v(X_0)\, e^{\int} + \rho(X,T) L(X) +$$

$$+ \rho(X,T)\int_{T_0}^{T} \underbrace{\partial_T L(x(t;\cancel{X},T))}_{-\partial_x L\, v(x)}\,dt$$

$$\frac{\partial \rho}{\partial X} = \partial_{X_0}\rho\, \partial_X X_0\, e^{\int} + \rho(X_0,T_0)\, e^{\int} \cdot \int_{T_0}^{T} \partial_X L(x(t;X,T))\,dt$$

$$= \partial_{X_0}\rho\, \frac{v(x)}{v(X_0)}\, e^{\int} + \rho \cdot \int_{T_0}^{T} \partial_x L\, \frac{v(x)}{v(X)}\,dt$$

$$v(X)\frac{\partial \rho}{\partial X} = \partial_{X_0}\rho_0\, v(x)\, e^{\int} + \rho \int_{T_0}^{T} \partial_x L\, v(x)\,dt$$

$$\boxed{\frac{\partial \rho}{\partial T} + v(X)\frac{\partial \rho}{\partial X} = \rho(X,T)\, L(X)}$$

$$\cancel{\sigma\partial_\sigma - \beta\frac{\partial}{\partial\lambda} + n\gamma + (n-4)}$$

$$\left(\sigma\partial_\sigma - \beta\,\partial_\lambda - (\gamma_m - 1)m\frac{\partial}{\partial m} + n\gamma + (n-4)\right)\Gamma_R(\sigma p_i, m, \lambda, \mu) = 0.$$

$t = \ln\sigma$

$v = -\beta$

$L = -n\gamma - (n-4)$

$$-\int_{\tilde\lambda}^{\lambda}\frac{d\lambda}{\beta(\lambda)} = \ln\left(\frac{\sigma}{\tilde\sigma}\right) \quad \rightarrow \quad \lambda(\sigma;\tilde\lambda,\tilde\sigma)$$

$$\beta(\lambda) = \frac{3}{16\pi^2}\lambda^2 = b_1\lambda^2$$

$$-\frac{1}{b_1}\int_{\tilde\lambda}^{\lambda}\frac{d\lambda}{\lambda^2} = \ln(\sigma/\tilde\sigma)$$

$$\boxed{\tilde\lambda = \frac{\lambda}{1 - b_1\lambda\ln(\sigma/\tilde\sigma)}}$$

$$\frac{1}{b_1}\left.\frac{1}{\lambda}\right|_{\tilde\lambda}^{\lambda} = \frac{1}{b_1}\left(\frac{1}{\lambda} - \frac{1}{\tilde\lambda}\right) = \ln(\sigma/\tilde\sigma)$$

$$\frac{1}{\lambda} = \frac{1}{\tilde\lambda} + b_1\ln(\sigma/\tilde\sigma) \Rightarrow \boxed{\lambda = \frac{\tilde\lambda}{1 + b_1\tilde\lambda\ln(\sigma/\tilde\sigma)}}$$

$$\Gamma(\sigma p_i) = \Gamma(p_i, \tilde\lambda, -)\, e^{\int_{\tilde\sigma}^{\sigma}(-n\gamma - (n-4))\frac{d\sigma}{\sigma}}$$

$$= \Gamma(p_i, \tilde\lambda(\lambda,\sigma), --)\, e^{-n\int_{\tilde\sigma}^{\sigma}\frac{d\sigma}{\sigma}\gamma(\tilde\lambda(\sigma)) - (n-4)\ln\sigma/\tilde\sigma}$$

$$\Gamma(\sigma p_i ; \lambda) = \Gamma(\tilde{\sigma} p_i , \bar{\lambda}(\lambda,\sigma), \bar{m}(\lambda,\sigma)) \left(\frac{\sigma}{\tilde{\sigma}}\right)^{-(n-4)} e^{-n\int_{\tilde{\sigma}}^{\sigma} \frac{d\sigma}{\sigma} \gamma(\bar{\lambda}(\sigma))}$$ (4)

$\tilde{\sigma} = 1$

$$\Gamma(\cancel{\sigma} p_i ; \lambda, m) = \sigma^{4-n} \Gamma(p_i , \bar{\lambda}(\lambda,\sigma), \bar{m}(\lambda,\sigma)).$$

$$\cdot e^{-n\int_{\lambda}^{\bar{\lambda}} \frac{d\sigma}{d\bar{\lambda}} \underbrace{\frac{d\bar{\lambda}}{\cancel{d\sigma}} \frac{1}{\sigma}}_{\cancel{\beta}\ -1/\beta} \gamma(\bar{\lambda})}$$

$$\sigma \frac{\partial \bar{\lambda}}{\partial \sigma} = -\beta$$

$$\Gamma(\sigma p_i ; \lambda, m) = \sigma^{4-n} \Gamma(p_i , \bar{\lambda}(\lambda,\sigma), \bar{m}(\lambda,\sigma)) \, e^{-n\int_{\lambda}^{\bar{\lambda}} \frac{\gamma(\bar{\lambda})}{\beta(\bar{\lambda})} d\bar{\lambda}}$$

<u>Example of QCD</u>

$$g^2 = \frac{g_0^2}{1+2b_1 g_0^2 \ln \mu/\mu_0} = \frac{g_0^2}{2b_1 g_0^2 \ln\mu + \ln_{QCD}} = \frac{1}{2b_1 \ln \mu/\mu_{QCD}}$$

$m \to 0$

Assume $\gamma(g) = c g^2$

$\bar{g}^2 =$

$$\beta = -b_1 g^3$$

$$\frac{1}{b_1}\int_{\bar{g}}^{g} \frac{dg}{g^3} = \ln(\sigma/\tilde{\sigma}) ; \quad -\frac{1}{2g^2} + \frac{1}{2\bar{g}^2} = b_1 \ln(\sigma/\tilde{\sigma})$$

$$\frac{1}{\bar{g}^2} = \frac{1}{g^2} + 2b_1 \ln\frac{\sigma}{\tilde{\sigma}} ; \quad \bar{g}^2 = \frac{g^2}{1 + 2b_1 g^2 \ln \sigma/\tilde{\sigma}}$$

$\tilde{\sigma} = 1$

$$\bar{g}^2 = \frac{g^2}{1+2b_1 g^2 \ln\sigma}$$

$$\bar{g}^2 = \frac{1}{\quad}$$

$$\Gamma(\sigma p_i ; \lambda, \mu) = \sigma^{4-n}\, \Gamma(p_i, \bar{g}^2, \mu)\, e^{+n\int_g^{\bar{g}} \frac{c\bar{g}^2}{b_1 \bar{g}^3} d\bar{g}}$$

$$e^{n\frac{c}{b_1}\ln\frac{\bar{g}}{g}}$$

$$\sigma \to \infty \qquad \bar{g}^2 \simeq \frac{1}{2b_1 \ln\sigma}$$

$$\Gamma(\sigma p_i ; \lambda, \mu) = \sigma^{4-n}\, \Gamma\left(p_i ; \frac{1}{2b_1\ln\sigma} ; \mu\right) \left(\frac{\bar{g}^2}{g^2}\right)^{\frac{nc}{2b_1}}$$

$$= \sigma^{4-n} \frac{1}{(2b_1 g^2)^{\frac{nc_1}{b_1}}} (\ln\sigma)^{-\frac{nc}{2b_1}}\, \Gamma\left(p_i ; \frac{1}{2b_1\ln\sigma} ; \mu\right)$$

$$\bar{g}^2 = \frac{g^2}{1 + 2b_1 g^2 \ln\sigma} = \frac{1}{2b_1 \ln \mu/\bar{\Lambda}_{QCD}}$$

↑ running.

$$2b_1 \ln\frac{\mu}{\bar{\Lambda}_{QCD}} = \frac{1}{g^2} + 2b_1 \ln\sigma$$

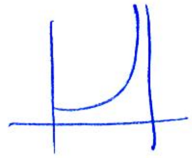
$$\frac{1}{g^2} = 2b_1 \ln\frac{\mu}{\sigma\bar{\Lambda}_{QCD}} = 2b_1 \ln\frac{\mu}{\Lambda_{QCD}}$$

$$\boxed{\bar{\Lambda}_{QCD} = \frac{1}{\sigma}\Lambda_{QCD}}$$

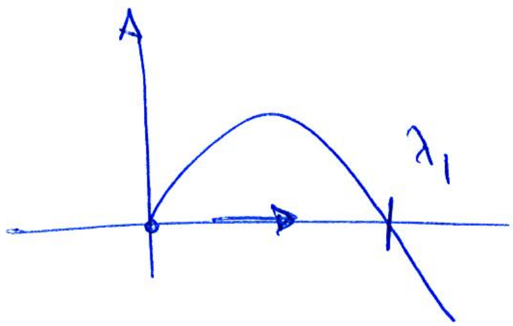
$\bar{\Lambda}_{QCD} \to 0$

$\sigma \to \infty$

ϕ^4 ; $\beta > 0$



Suppose instead



as $\sigma \to \infty$ $\underline{\lambda \to \lambda_1}$

$$\bar{\lambda}(\lambda, \sigma \to \infty) = \lambda_1$$

$$\beta(\bar{\lambda}) \simeq -\beta_0 (\bar{\lambda} - \lambda_1)$$

$$\Gamma(\sigma p_i ; \lambda, m) = \sigma^{4-n} \; \Gamma(p_i) \, \lambda, \ldots) \, . \, e^{+n \int_\lambda^{\bar{\lambda}} \cancel{\frac{\gamma(\lambda)}{\beta_0(\bar{\lambda}-\lambda)}} d\bar{\lambda}}$$

$$e^{n \gamma_1 \frac{\ln(\bar{\lambda}-\lambda)}{\beta_0} \Big|^{\bar{\lambda}}}$$

$$\cdot \, e$$

$$(\bar{\lambda}-\lambda_1)^{n\gamma_1/\beta_0}$$

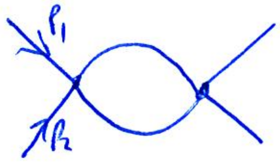
$$e^{-n \int_{\tilde{\sigma}}^{\sigma} \frac{d\sigma}{\sigma} \gamma_1} = e^{-\gamma_1 \ln \sigma} = \sigma^{-\gamma_1 n}$$

$$\Gamma(\sigma p_i ; \lambda, m) = \sigma^{4-n-n\gamma_1} \; \Gamma(p_i ; \lambda_1 \ldots)$$

$\underline{\text{scaling behaviour}}$ — anomalous dim. indep. of λ!!

1 - loop $\boxed{\text{massless } \phi^4}$ 2 - loops.

1



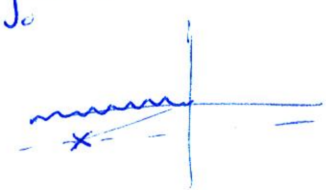
$p = p_1 + p_2$

$$\frac{(-i\lambda)^2}{2} \int \frac{d^d l}{(2\pi)^d} \frac{i}{l^2 + i\varepsilon} \frac{i}{(p-l)^2 + i\varepsilon}$$

$$= \frac{\lambda^2}{2} \left(\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 + i\varepsilon} \frac{1}{(p-l)^2 + i\varepsilon} \right) = \frac{\lambda^2}{2} \int_0^1 d\alpha \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + \alpha(1-\alpha)p^2 + i\varepsilon)^2}$$

$$\Delta = -\alpha(1-\alpha)p^2 - i\varepsilon$$

$$= \frac{\lambda^2}{2} \int_0^1 d\alpha \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \Delta^{d/2-2} = \frac{\lambda^2}{2} \frac{i\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 d\alpha \left(-\alpha(1-\alpha)p^2 - i\varepsilon\right)^{d/2-2}$$



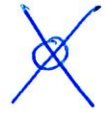
$$= \frac{\lambda^2}{2} \frac{i\Gamma(2-d/2)}{(4\pi)^{d/2}} B(d/2-1, d/2-1) (-s-i\varepsilon)^{d/2-2}$$

$$= \frac{i\lambda^2}{2} \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} B(1-\varepsilon, 1-\varepsilon) (-s-i0)^{-\epsilon}$$

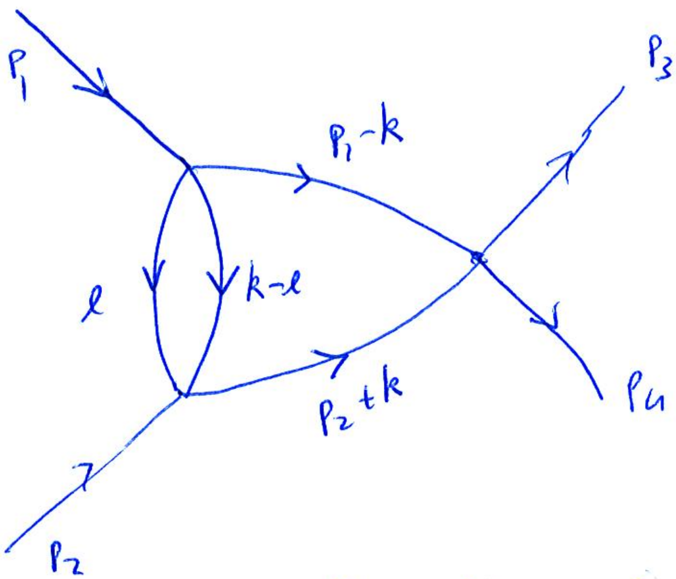
$$= \frac{i\lambda^2}{32\pi^2} \left(\frac{1}{\epsilon} + (2 - \gamma - \ln(-s-i0) + \mathcal{O}(\varepsilon) \right)$$

All contributions 

$$\boxed{= \frac{3i\lambda^2}{32\pi^2} \frac{1}{\varepsilon} + \frac{3i\lambda^2}{32\pi^2} (2-\gamma+\ln 4\pi) - \frac{i\lambda^2}{32\pi^2} (\ln(-s-i0) + \ln(-t-i0) + \ln(-u-i0)).}$$

 $- \dfrac{3i\lambda^2}{32\pi^2} \dfrac{1}{\varepsilon}$

2-loops



$$\frac{(-i\lambda)^3}{2}\int\frac{d^d l}{(2\pi)^d}\frac{d^d k}{(2\pi)^d}\frac{i}{l^2+i\epsilon}\frac{i}{(k-l)^2+i\epsilon}\frac{i}{(p_1-k)^2+i\epsilon}\frac{i}{(p_2-k)^2+i\epsilon}$$

$$\frac{(-i\lambda)^3}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i\Gamma(2-d/2)}{(4\pi)^{d/2}}B(d/2-1,d/2-1)\frac{(-k^2-i\epsilon)^{d/2-2}}{((p_1-k)^2+i\epsilon)((p_2+k)^2+i\epsilon)}$$

$$\frac{(-i\lambda)^3}{2}\frac{i\Gamma(2-d/2)}{(4\pi)^{d/2}}B(d/2-1,d/2-1)\cdot\int\frac{d^dk}{(2\pi)^d}\cdot$$

$$\int_0^\infty \frac{d\alpha_1\,\alpha_1^{2-d/2-1}\,e^{\alpha_1 i(k^2+i\epsilon)}}{\Gamma(2-d/2)}\, i^{\cancel{2}-d/2}\int_0^\infty d\alpha_2\,\frac{e^{\alpha_2 i((p_1-k)^2+i\epsilon)}}{\cancel{i}}\int_0^\infty d\alpha_3\,\frac{e^{\alpha_3 i((p_2+k)^2+i\epsilon)}}{\cancel{i}}$$

$$-\frac{\lambda^3}{2}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}B(d/2-1,d/2-1)\frac{i^{-d/2}}{\Gamma(2-d/2)}\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\;\alpha_1^{1-d/2}\cdot$$

$$\int\frac{d^dk}{(2\pi)^d}\, e^{i(\alpha_1+\alpha_2+\alpha_3)k^2-2i(\alpha_2p_1-\alpha_3p_2)k+i\alpha_2p_1^2+i\alpha_3p_2^2-(\alpha_1+\alpha_2+\alpha_3)\epsilon}$$

$$\int_0^\infty d\mu \quad \delta(\alpha_1+\alpha_2+\alpha_3-\mu) = 1 .$$

$$\int_0^\infty d\mu \, d\alpha_1 d\alpha_2 d\alpha_3 \quad \delta(\alpha_1+\alpha_2+\alpha_3-\mu)$$

$$\rightarrow \quad \alpha_i = \mu\beta_i \quad \Rightarrow \quad \int_0^\infty d\mu \, d\beta_1 d\beta_2 d\beta_3 \, \mu^3 \, \frac{1}{\mu} \, \delta(\Sigma\beta_i - 1)$$

$$-\frac{\lambda^3}{2} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} B(d/2-1, d/2-1) \frac{i^{-d/2}}{\Gamma(2-d/2)} \int_0^\infty d\mu \, \mu^{3-d/2} d\beta_1 d\beta_2 d\beta_3 \, \delta(\Sigma\beta_i - 1) \beta_1^{1-d/2}$$

$$\int \frac{d^dk}{(2\pi)^d} \quad e^{i\mu k^2 - 2i\mu(\beta_2 P_1 - \beta_3 P_2)k + i\mu\beta_2 P_1^2 + i\mu\beta_3 P_2^2 - \mu\varepsilon}$$

$$i\mu(k - \beta_2 P_1 + \beta_3 P_2)^2 - i\mu(\beta_2 P_1 - \beta_3 P_2)^2 + i\mu\beta_2 P_1^2 + i\mu\beta_3 P_2^2 - \mu\varepsilon$$

$$i\mu k^2 + i\mu(-\beta_2^2 P_1^2 - \beta_3^2 P_2^2 + 2\beta_2\beta_3 P_1 \cdot P_2 + \beta_2 P_1^2 + \beta_3 P_2^2) - \mu\varepsilon$$

$$\int \frac{d^dk}{(2\pi)^d} e^{i\mu k^2} = \int \frac{dk_0}{(2\pi)} e^{i\mu k_0^2} \int \frac{dk_j}{(2\pi)} e^{-i\mu k_j^2} = \sqrt{\frac{\pi}{-i\mu}} \left(\sqrt{\frac{\pi}{i\mu}}\right)^{d-1} = \frac{\pi^{d/2}}{\mu^{d/2}} \frac{(-i)^{\frac{d-2}{2}}}{(2\pi)^d}$$

$$= \left(\frac{\pi}{\mu}\right)^{d/2} \frac{e^{-\frac{i\pi}{4}(d-2)}}{(2\pi)^d}$$

$$-\frac{\lambda^3}{2} \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} B(d/2-1, d/2-1) \frac{i^{-d/2}}{\Gamma(2-d/2)} \int_0^\infty d\mu \frac{\mu^{3-d/2}}{(2\pi)^d} d\beta_j \, \delta(\Sigma\beta_j - 1) \beta_1^{1-d/2} \cdot \pi^{d/2} \mu^{-d/2} (-i)^{\frac{d-2}{2}}$$

$$e^{i\mu(\beta_2(1-\beta_2)P_1^2 + \beta_3(1-\beta_3)P_2^2 + 2\beta_2\beta_3 P_1 P_2) - \mu\varepsilon}$$

(4)

$$-\frac{\lambda^3}{2}\frac{\cancel{\Gamma(2-d/2)}}{(4\pi)^{d/2}} B(d/2-1,d/2-1) \frac{i^{-d/2}(-i)^{d/2-1}\pi^{d/2}}{\cancel{\Gamma(2-d/2)}(2\pi)^d}$$

$$\int_0^\infty d\beta_i\, \Gamma(4-d)\, (\varepsilon - i(\beta_2(1-\beta_2)p_1^2+\beta_3(1-\beta_3)p_2^2+2\beta_2\beta_3 p_1 p_2)^{d-4} \beta_1^{1-d/2}\delta(\Sigma\beta_i-1)$$

$$-\frac{\lambda^3}{2}\frac{B(d/2-1,d/2-1)}{2^{2d}\pi^d} \underbrace{i^{-d/2}(-i)^{d/2-1}(i)^{d-4}}_{e^{\frac{i\pi}{2}(-\frac{d}{2}+d-4-\frac{d}{2}+1)}=i}\Gamma(4-d)\cdot$$

$$\cdot\int_0^\infty d\beta_i\,\delta(\Sigma\beta_i-1)\,\beta_1^{1-d/2}\,(-(2\beta_2\beta_3 p_1p_2+\beta_2(1-\beta_2)p_1^2+\beta_3(1-\beta_3)p_2^2-i\varepsilon)^{d-4}$$

$\Big\downarrow$ on-shell $\equiv (p_1^2=p_2^2=0)$

$$-\frac{i\lambda^3}{2}\frac{B(d/2-1,d/2-1)}{2^{2d}\pi^d}\Gamma(4-d) \underbrace{\int_0^\infty d\beta_i\,\delta(\Sigma\beta_i-1)\,\beta_1^{1-d/2}}\,(-s-i0)^{d-4}\beta_2^{d-4}\beta_3^{d-4}$$

$$\int_0^1 d\beta_3\int_0^{1-\beta_3} d\beta_2\,(1-\beta_2-\beta_3)^{1-d/2}\,\beta_2^{d-4}\,\beta_3^{d-4}$$

$$\beta_2=(1-\beta_3)\tilde\beta_2 \qquad \int_0^1 d\beta_3\int_0^1 d\tilde\beta_2\,(1-\beta_3)^{2-d/2+d-4=-2+d/2}\cdot$$

$$\cdot(1-\tilde\beta_2)^{1-d/2}\,\tilde\beta_2^{d-4}\,\beta_3^{d-4}$$

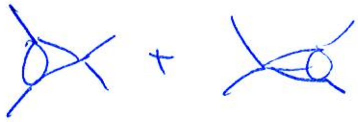
$$B(d/2-1,\,d-3)\,B(2-d/2,d-3)$$

$d=4-2\varepsilon \qquad d/2=2-\varepsilon$

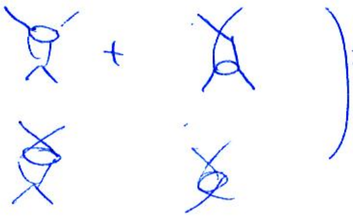
$$-\frac{i\lambda^3}{2}\frac{B(1-\varepsilon,1-\varepsilon)}{(4\pi)^{4-2\varepsilon}}\Gamma(2\varepsilon)\,B(1-\varepsilon,\underset{1-2\varepsilon}{\cancel{d-2\varepsilon}})\,B(\varepsilon,1-2\varepsilon)\,(-s-i0)^{-2\varepsilon}$$

$$=-\frac{i\lambda^3}{2^9\pi^4}\left(\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(-\ln(-s-i0)+\frac{5}{2}-\gamma+\ln 4\pi\right)+\left(\ln(-s-i0)\right)^2-2\left(\frac{5}{2}-\gamma+\ln(4\pi)\right)\ln(-s-i0)\right.$$

$$\left.+C_0+O(\varepsilon)\right)$$



: factor of 2.



$s \leftrightarrow t \leftrightarrow u$

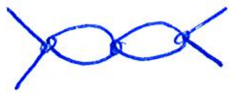
Total contribution.

factor 6

$$-\frac{3i\lambda^3}{2^9\pi^4}\frac{1}{\varepsilon^2} - \frac{3i\lambda^3}{2^8\pi^4\varepsilon}\left(\frac{5}{2}-\gamma+\ln 4\pi\right)$$

$$+\frac{i\lambda^3}{2^8\pi^4\varepsilon}\left(\ln(-s-i\epsilon)+\ln(-t-i\epsilon)+\ln(-u-i\epsilon)\right)-$$

$$-\frac{i\lambda^3}{2^8\pi^4}\left(\ln^2(-s-i\epsilon)+\ln^2(-t-i\epsilon)+\ln^2(-u-i\epsilon)\right)+\cdots$$



$$\frac{(-i\lambda)^3}{4}\int\frac{d^d l}{(2\pi)^d}\frac{i}{l^2}\frac{i}{(p-l)^2}\int\frac{d^d k}{(2\pi)^d}\frac{i}{k^2}\frac{i}{(p-k)^2}$$

$$\frac{i\lambda^3}{4}\left(\frac{i\Gamma(2-d/2)}{(4\pi)^{d/2}}B(d/2-1,d/2-1)(-s-i0)^{d/2-2}\right)^2$$

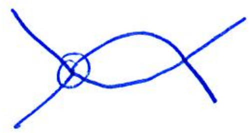
$$-\frac{i\lambda^3}{4}\frac{(\Gamma(\epsilon))^2}{(4\pi)^{4-2\epsilon}}\left(B(1-\epsilon,1-\epsilon)\right)^2(-s-i0)^{d-4}$$

$$-\frac{i\lambda^3}{2^{10}\pi^4}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(4-2\gamma+2\ln(4\pi)-2\ln(-s)\right)+2(\ln(-s))^2-2(4+2\ln(4\pi)-2\gamma)\ln s+c_0\right.$$

∘ All contributions

$$-\frac{3i\lambda^3}{2^{10}\varepsilon^2\pi^4}-\frac{3i\lambda^3}{2^{10}\varepsilon\pi^4}\left(4-2\gamma+2\ln(4\pi)\right)+\frac{i\lambda^3}{2^9\pi^4\epsilon}\left(\ln(-s-i0)+\ln(-t+i0)+\ln(-u-i0)\right)$$

$$-\frac{i\lambda^3}{2^9\pi^4}\left[(\ln(-s-i0))^2+(\ln(-t-i0))^2+(\ln(-u-i0))^2\right]+\ldots$$



$$-\frac{3i\lambda^2}{32\pi^2}\frac{1}{\varepsilon}\frac{(+i\lambda)}{2}\int\frac{d^d l}{(2\pi)^d}\frac{i}{l^2}\frac{i}{(p-l)^2}$$

$$\frac{3\lambda^3}{2^6\pi^2\varepsilon}\frac{i\Gamma(2-d/2)}{(4\pi)^{d/2}}B(d/2-1,d/2-1)(-s-i\varepsilon)^{d/2-2}$$

$$\frac{3i\lambda^3}{2^6\pi^2\varepsilon}\frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}B(1-\varepsilon,1-\varepsilon)(-s-i0)^{-\epsilon}$$

$$\frac{3i\lambda^3}{2^{10}\pi^4}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}(2-\gamma+\ln(4\pi)-\ln(-s-i0))+\frac{1}{2}\ln^2(-s)-\ln(-s)(-2-\ln(4\pi)\right)$$

All contributions:

$$\frac{9i\lambda^3}{2^9\pi^4\epsilon^2}+\frac{9i\lambda^3}{2^9\pi^4}\frac{1}{\epsilon}(2-\gamma+\ln(4\pi))-\frac{3i\lambda^3}{2^9\pi^4\epsilon}(\ln(-s-i0)+\ln(-t-i0)+\ln(-u-i0))$$

$$+\text{finite}+\frac{3i\lambda^3}{2^{10}\pi^4}\left(\ln^2(-s)+\ln^2(-t)+\ln^2(-u)\right)$$

 $-\frac{3i\lambda^3}{2^9\pi^4}\frac{1}{\varepsilon^2} - \frac{3i\lambda^3}{2^8\pi^4\varepsilon}\left(\frac{5}{2}-\gamma+\ln 4\pi\right)$

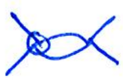
$+\frac{i\lambda^3}{2^8\pi^4\varepsilon}\left(\ln(-s-i0)+ ---\right)$

$-\frac{i\lambda^3}{2^8\pi^4}\left(\ln^2(-s-i0)+ ---\right)$

 $-\frac{3i\lambda^3}{2^{10}\varepsilon^2\pi^4} - \frac{3i\lambda^3}{2^{10}\varepsilon\pi^4}\left(4-2\gamma+2\ln(4\pi)\right) +$

$+\frac{i\lambda^3}{2^9\pi^4\varepsilon}\left(\ln(-s)+ ---\right)$

$-\frac{i\lambda^3}{2^9\pi^4}\left[\ln^2(-s)+ ---\right)$

 $\frac{9i\lambda^3}{2^9\pi^4\varepsilon^2} + \frac{9i\lambda^3}{2^9\pi^4}\frac{1}{\varepsilon}\left(2-\gamma+\ln(4\pi)\right)$

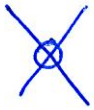
$-\frac{3i\lambda^3}{2^9\pi^4\varepsilon}\left(\ln(-s)+ ---\right) + \frac{3i\lambda^3}{2^{10}\pi^4}\left(\ln^2(-s)+ ---\right)$

$-\frac{i\lambda^3}{2^{10}\pi^4}\frac{1}{\varepsilon^2}(6+3-18) = \frac{9i\lambda^3}{2^{10}\pi^4}\frac{1}{\varepsilon^2}$

$-\frac{3i\lambda^3}{2^9\pi^4\varepsilon}\Big(5-2\gamma+2\ln 4\pi+2-\gamma+\ln 4\pi$
$-6+3\gamma-3\ln 4\pi\Big)$
$-\frac{3i\lambda^3}{2^9\pi^4\varepsilon}$

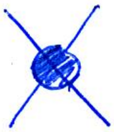
$\frac{i\lambda^3}{2^9\pi^4\varepsilon}\left(\ln(-s)+ ---\right)(2+1-3)=0$ ✓

$-\frac{i\lambda^3}{2^{10}\pi^4}\left(\ln^2(-s)+ ---\right)(4+2-3) = -\frac{3i\lambda^3}{2^{10}\pi^4}\left(\ln^2(-s)+\ln^2(-t)+\ln^2(-u)\right)$



$$= -\frac{3i\lambda^2}{32\pi^2}\frac{1}{\varepsilon} + \frac{3i\lambda^3}{2^9\pi^4\varepsilon} - \frac{9i\lambda^3}{2^{10}\pi^4}\frac{1}{\varepsilon^2}$$

leading logs



$$= -i\lambda - \frac{i\lambda^2}{32\pi^2}\left(\ln(-s-i\delta) + \ln(-t-i\delta) + \ln(-u-i\delta)\right) -$$

$$- \frac{3i\lambda^3}{2^{10}\pi^4}\left[(\ln(-s-i\delta))^2 + (\ln(-t-i\delta))^2 + (\ln(-u-i\delta))^2\right] + \cdots$$

In the deep euclidean region

$p^2 < 0 \quad \sigma p_i \; ; \; \sigma \to \infty \qquad -s \to \sigma^2 \quad -t \to \sigma^2 \quad -u \to \sigma^2$

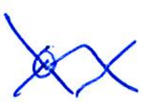
10

total x6.

$$-\frac{3i\lambda^3}{2^8\pi^4}\left(\frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}(-2\ln\sigma) + \dots + 4(\ln\sigma)^2 + \dots\right) \checkmark$$

 same as before

$$-\frac{3i\lambda^3}{2^{10}\varepsilon^2\pi^4} + \frac{3i\lambda^3}{2^9\varepsilon\pi^4}2\ln\sigma - \frac{3i\lambda^3}{2^9\pi^4}4(\ln\sigma)^2$$



$$\frac{9i\lambda^3}{2^9\pi^4\varepsilon^2} - \frac{9i\lambda^3}{2^9\pi^4\varepsilon}2\ln\sigma + \frac{9i\lambda^3}{2^{10}\pi^4}4(\ln\sigma)^2$$

$$-\frac{3i\lambda^3}{2^{10}\varepsilon^2\pi^4}(2+1-6) = \frac{9i\lambda^3}{2^{10}\varepsilon^2\pi^4}$$

$$+\frac{3i\lambda^3}{2^8\pi^4}\frac{1}{\varepsilon}(+2\ln\sigma)\left(1+\frac{1}{2}-\frac{3}{2}\right) = 0 \checkmark$$

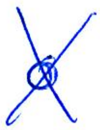
$$-\frac{3i\lambda^3}{2^8\pi^4}4(\ln\sigma)^2\left(1+\frac{1}{2}-\frac{3}{4}\right) = -\frac{9i\lambda^3}{2^{10}\pi^4}4(\ln\sigma)^2$$

$$\frac{3}{2}-\frac{3}{4} = \frac{6-3}{4} = +\frac{3}{4}$$

$$\sigma\to\infty \quad (-i\lambda) - \frac{3i\lambda^2}{32\pi^2}2\ln\sigma + \frac{9i\lambda^3}{2^{10}\pi^4}4(\ln\sigma)^2 + \dots$$

 $$-i\left(\lambda + \frac{3\lambda^2}{2^4\pi^2}\ln\sigma + \frac{9\lambda^3}{2^8\pi^4}(\ln\sigma)^2 + \dots\right)$$

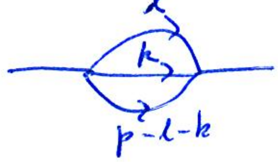
$$-i\lambda\left(1+\frac{3\lambda}{16\pi^2}\ln\sigma + \left(\frac{3\lambda}{16\pi^2}\right)^2(\ln\sigma)^2+\dots\right)$$

 $$-i\,\frac{\lambda}{1-\frac{3\lambda}{16\pi^2}\ln\sigma}$$

$\sigma \to \infty$ euclidean region.

$$\boxed{-i\bar{\lambda}}$$ ✓

β-function controls the leading logs.



wave-function renorm.

$$\frac{(-i\lambda)^2}{6}\int\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\;\frac{i}{l^2+i\epsilon}\;\frac{i}{(k^2+i\epsilon)}\;\frac{i}{(p-l-k)^2+i\epsilon}$$

$$\frac{i\lambda^2}{6}\int\frac{d^dk}{(2\pi)^d}\;\frac{1}{k^2+i\epsilon}\;\frac{i\Gamma(2-d/2)}{(4\pi)^{d/2}}\;B(d/2-1,d/2-1)\;\left(-(p-k)^2+i\epsilon\right)^{d/2-2}$$

$$-\frac{\lambda^2}{6}\;\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\;B(d/2-1,d/2-1)\int\frac{d^dk}{(2\pi)^d}\int_0^\infty d\alpha_1\;\frac{e^{i\alpha_1(k^2+i\epsilon)}}{i}\int_0^\infty d\alpha_2\;\frac{\alpha_2^{1-d/2}\,i^{2-d/2}}{\Gamma(2-d/2)}\;e^{i\alpha_2(i\epsilon+(p-k)^2)}$$

$$e^{i\alpha_1k^2+i\alpha_2k^2-2i\alpha_2pk+i\alpha_2p^2}$$

$$\int_0^\infty d\mu\int d\beta_1d\beta_2\;\mu\;\delta(\Sigma\beta_i-1)\;e^{i\mu k^2-2i\mu\beta_2pk+i\mu\beta_2p^2}$$

$$e^{i\mu(k-\beta_2p)^2-i\mu\beta_2^2p^2}$$

$$e^{i\mu k^2}$$

$$-\frac{\lambda^2}{6}\;\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\;B(d/2-1,d/2-1)\;\frac{1}{i}\;\frac{i^{2-d/2}}{\Gamma(d/2-2)}\int_0^\infty d\mu\,\mu\int d\beta_1d\beta_2\;\delta(\Sigma\beta_i-1)\;\mu^{1-d/2}\beta_2^{1-d/2}\left(\frac{\pi}{\mu}\right)^{d/2}\frac{e^{-\frac{i\pi}{4}(d-2)}}{(2\pi)^d}\;e^{i\mu\beta_2(1-\beta_2)p^2-\mu\epsilon}$$

$$-\frac{\lambda^2}{6}\;\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\;B(d/2-1,d/2-1)\;\frac{i^{1-d/2}}{(4\pi)^{d/2}}\;\frac{e^{-\frac{i\pi}{4}(d-2)}}{\Gamma(d/2-2)}\int_0^\infty d\mu\;\mu^{2-d}\int_0^1 d\beta\;\beta^{1-d/2}\;e^{i\mu\beta(1-\beta)p^2-\mu\epsilon}$$

$$-\frac{\lambda^2}{6}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} B(d/2-1, d/2-1)\, \frac{i\, e^{-\frac{i\pi}{4}d-\frac{i\pi d}{4}+\frac{i\pi}{2}}}{(4\pi)^{d/2}\Gamma(d/2-2)}\, \Gamma(3-d)\int_0^1 d\beta\, \beta^{1-d/2}(\varepsilon - i\beta(1-\beta)p^2)^{d-3}$$

(13)

$i(-(\beta-\beta^2-i\varepsilon$

$$-\frac{\lambda^2}{6}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} B(d/2-1, d/2-1) \quad \frac{(-)\, e^{-\frac{i\pi d}{2}}}{(4\pi)^{d/2}\Gamma(d/2-2)}\Gamma(3-d)\; i^{d-3}\int_0^1 d\beta\, \beta^{1-d/2}(-p^2-i\varepsilon)^{d-3}$$

$$\beta^{d-3}(1-\beta)^{d-3}$$

$$\beta^{d/2-2}\;(1-\beta)^{d-3}$$

$e^{\frac{i\pi d}{2}}$ i

$(i^3)\; -i$

$$\frac{(-i)\,\Gamma(3-d)}{(4\pi)^{d/2}\Gamma(d/2-2)}(-p^2-i\varepsilon)^{d-3}\, B(d/2-1, d-2)$$

$$\frac{i\lambda^2}{6}\frac{B(d/2-1, d/2-1)}{(4\pi)^{d}}\,\Gamma(3-d)\,(-p^2-i\varepsilon)^{d-3}\, B(d/2-1, d-2)$$

$d=4-2\varepsilon \qquad d/2=2-\varepsilon$

$$\frac{i\lambda^2}{6}\frac{B(1-\varepsilon,1-\varepsilon)}{(4\pi)^{4-2\varepsilon}}\,\Gamma(-1+2\varepsilon)\,(-p^2-i\varepsilon)^{1-2\varepsilon}\, B(1-\varepsilon, 2-2\varepsilon)$$

$$\frac{\Gamma(1)\Gamma(2)}{\Gamma(3)}$$

$$\Gamma(-1+2\varepsilon)=\frac{\Gamma(2\varepsilon)}{-1+2\varepsilon}\simeq -\frac{1}{2\varepsilon}$$

$$-\frac{i\lambda^2}{6}\,\frac{1}{2\varepsilon}\,\frac{1}{2^8\pi^4}\,(-p^2-i\varepsilon)\,(1-2\varepsilon\ln(-p^2-i\varepsilon))\,\frac{1}{2}$$

$$=-\frac{i\lambda^2}{3\times 2^{11}\pi^4\varepsilon}(-p^2-i\varepsilon)\,(1-2\varepsilon\ln(-p^2-i\varepsilon))$$

definition of Z_ϕ, Z_m, Z_λ

Counterterms:

$$\mathcal{L} = \frac{1}{2} Z_\phi (\partial\phi_R)^2 - \frac{1}{2} Z_\phi m_R^2 Z_m \phi_R^2 - \frac{\lambda}{4!} Z_\lambda \phi_R^4$$

$$= \frac{1}{2} (\partial\phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4$$

$$m_0^2 = Z_m \cancel{Z_\phi} m_R^2 \qquad \lambda_0 = Z_\lambda Z_\phi^{-2} \lambda$$

$$\phi_0 = Z_\phi^{1/2} \phi_R$$

$$G_{(2)}^{-1} = -i Z_\phi (p^2 - m^2 Z_m) - \Gamma^{(2)} \qquad = -i(-)^{-1} - 0$$

If $\Gamma^{(2)} = C_0 + C_2 p^2 + \text{finite}(p^2)$; $\Gamma^{(4)} = C_4 + \text{finite}(p^2)$

∞ we cancel.

then $-i Z_\phi p^2 + i m_R^2 Z_\phi Z_m - C_0 - C_2 p^2 = -i p^2 + i m_R^2$

$$-i Z_\phi - C_2 = -i \qquad Z_\phi Z_m i m_R^2 - C_0 = i m_R^2$$

$$Z_\phi - i C_2 = 1 \qquad Z_\phi Z_m m_R^2 + i C_0 = m_R^2$$

$$\boxed{Z_\phi = 1 + i C_2} \qquad Z_\phi Z_m = 1 - i C_0 / m_R^2$$

$$\boxed{Z_m = \frac{1 - i C_0/m_R^2}{1 + i C_2}}$$

$$\boxed{Z_\lambda = 1 - \frac{i C_4}{\lambda}}$$

$$\times \; -i\lambda Z_\lambda = -i\lambda - C_4$$

$$\lambda Z_\lambda = \lambda - i C_4$$

$$\boxed{Z_\lambda = 1 - \frac{i C_4}{\lambda}}$$

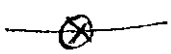
Counterterms.

$$\mathcal{L}_c = \frac{1}{2}(Z_\phi - 1)(\partial\phi_R)^2 - \frac{1}{2}(Z_\phi Z_m - 1)\, m_R^2 \phi_R^2 - \frac{\lambda}{4!}(Z_\lambda - 1)\phi_R^4$$

$$Z_\phi - 1 = iC_2 \qquad Z_\phi Z_m - 1 = -i\frac{C_0}{m_R^2} \qquad Z_\lambda - 1 = -\frac{iC_4}{\lambda}$$

 $-C_4$ ✓

$$\mathcal{L}_c = \frac{iC_2}{2}(\partial\phi_R)^2 + \frac{iC_0}{2}\phi_R^2 + \frac{iC_4}{4!}\phi_R^4$$

 $-C_0 + p^2 C_2$ ✓

$(T-V)$ $\quad -\mathcal{L}_c \quad e^{iVt}$

$i\mathcal{L}_c$

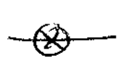
$$-\frac{C_2}{2}(\partial\phi)^2 - \frac{C_0}{2}\phi^2 - \frac{C_4}{4!}\phi_R^4$$

$(ip)(-ip)$

$\phi_p \phi_{-p}$

Summary.

If. $\quad \Gamma^{(2)} = C_0 + C_2 p^2 + \text{finite}(p^2) \quad ; \quad \Gamma^{(4)} = C_4 + \text{finite}(p^2)$

$\Rightarrow$  $-C_4$ $\qquad$  $-C_0 - C_2 p^2$

$$Z_\phi = 1 + iC_2 \qquad Z_m = \frac{1 - iC_0/m_R^2}{Z_\phi} \qquad Z_\lambda = 1 - iC_4/\lambda$$

$$\lambda_B = \frac{(1 - iC_4/\lambda)}{(1 + iC_2)^2}\,\lambda_R$$

We have:

$$C_4 = \frac{3i\lambda^2}{32\pi^2}\frac{1}{\varepsilon} - \frac{3i\lambda^3}{2^9\pi^4\varepsilon} + \frac{9i\lambda^3}{2^{10}\pi^4}\frac{1}{\varepsilon^2}$$

$$C_0 = \frac{i\lambda}{32\pi^2}\frac{m^2}{\varepsilon} \qquad \text{(1-loop massive)}$$

$$C_2 = \frac{i\lambda^2}{3\times 2^{11}\pi^4\varepsilon}\cancel{p^2}$$

$$Z_\phi = \left(1 - \frac{\lambda^2}{3\times 2^{11}\pi^4\varepsilon}\right)$$

$$Z_{m^2} = \left(1 + \frac{\lambda}{32\pi^2}\cancel{m}\frac{1}{\varepsilon}\right) \qquad \text{1-loop.} \qquad ; \; Z_m = \sqrt{Z_{m^2}}$$

$$\lambda_B = \left(1 + \frac{3\lambda}{32\pi^2}\frac{1}{\varepsilon} - \frac{3\lambda^2}{2^9\pi^4\varepsilon} + \frac{9\lambda^2}{2^{10}\pi^4\varepsilon^2}\right)\left(1 - \frac{\lambda^2}{3\times 2^{11}\pi^4\varepsilon}\right)^{-2}\lambda_R$$

$$= \left(1 + \frac{3\lambda}{32\pi^2}\frac{1}{\varepsilon} - \frac{3\lambda^2}{2^9\pi^4\varepsilon} + \frac{9\lambda^2}{2^{10}\pi^4\varepsilon^2} + \frac{\lambda^2}{3\times 2^{10}\pi^4\varepsilon}\right)\lambda$$

$$-\frac{3}{3\times 2^{10}}(18-1) = -\frac{17}{3\times 2^{10}}$$

$$\lambda_B = \left(1 + \frac{3\lambda}{32\pi^2}\frac{1}{\varepsilon} - \frac{17\lambda^2}{3\times 2^{10}\pi^4\varepsilon} + \frac{9\lambda^2}{2^{10}\pi^4\varepsilon^2}\right)\lambda$$

β-function:

$$a_1 = \frac{3\lambda^2}{32\pi^2} - \frac{17}{3\times 2^{10}}\frac{\lambda^3}{\pi^4}$$

$$\beta = 2\left(-a_1 + \lambda \frac{\partial}{\partial \lambda} a_1\right)$$

$$a_1 = \sum a_{1n}\lambda^n \qquad \beta = 2\sum_n (n-1)\lambda^n a_n$$

$$\beta = 2\left(\frac{3}{32\pi^2}\lambda^2 - \frac{17}{3\times 2^9}\frac{\lambda^3}{\pi^4}\right)$$

$$\boxed{\beta = \frac{3}{16\pi^2}\lambda^2 - \frac{17}{3\times 2^8}\frac{\lambda^3}{\pi^4}}$$

$$\frac{\beta}{16\pi^2} = 3g^2 - \frac{17}{3}g^3 \qquad g = \frac{\lambda}{16\pi^2}$$

$$\boxed{\beta_g = 3g^2 - \frac{17}{3}g^3} \checkmark$$

Renormalization of ϕ^4 massive $\qquad d=4-2\varepsilon \qquad$ ①

$$S=\int\left(\frac{1}{2}\partial\phi\partial\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4\right)d^dx$$

$$[\phi]=\mu^{\frac{d-2}{2}}=\mu^{1-\varepsilon} \quad ; \quad [\lambda]=\mu^{2\varepsilon} \leftarrow d-4(1-\varepsilon)=4-2\varepsilon-4+4\varepsilon=2\varepsilon$$

$$\lambda_0=\mu^{2\varepsilon}\left(\lambda_R+\frac{a_1}{\varepsilon}+\frac{a_2}{\varepsilon^2}+\cdots\right)$$

$$m_0=m\left(1+\frac{b_1}{\varepsilon}+\frac{b_2}{\varepsilon^2}+\cdots\right)$$

$$\phi_0=\phi\left(1+\frac{c_1}{\varepsilon}+\frac{c_2}{\varepsilon^2}+\cdots\right)=Z_\phi^{1/2}\phi$$

$\mu\to(1+\delta)\mu \qquad \lambda\to\lambda+\delta(d_0+\varepsilon d_1)$

$$\lambda_0=(1+2\varepsilon\delta)\mu^{2\varepsilon}\left(\lambda_R+\delta(d_0+\varepsilon d_1)+\frac{a_1}{\varepsilon}+\frac{a_1'\delta(d_0+\varepsilon d_1)}{\varepsilon}+\frac{a_2}{\varepsilon^2}+\cdots\right)$$

$$=\mu^{2\varepsilon}\left(\lambda_R+\delta d_0+\delta\varepsilon d_1+\frac{a_1}{\varepsilon}+a_1'\frac{\delta d_0}{\varepsilon}+\underline{a_1'\delta d_1}+\cdots\right.$$
$$\left.+2\varepsilon\delta\lambda_R+2\delta a_1+\cdots\right)$$

$$d_1+2\lambda_R=0 \qquad \boxed{d_1=-2\lambda_R} \qquad \delta d_0+a_1'\delta d_1+2\delta a_1=0$$

$$d_0=-2a_1-d_1a_1'=-2a_1+2\lambda_R a_1'=2(-a_1+\lambda_R a_1')$$

$$\beta=\mu\frac{\partial}{\partial\mu}\lambda=-2\lambda_R\varepsilon+2(-a_1+\lambda_R a_1')$$

$$m_0=m\left(1+\frac{b_1}{\varepsilon}+\frac{b_2}{\varepsilon^2}+\cdots\right)\to m_0=\tilde{m}\left(1+\frac{b_1}{\varepsilon}+\frac{b_1'\delta(d_0+\varepsilon d_1)}{\varepsilon}+\cdots\right)$$

$$=\tilde{m}\left(1+\delta\frac{b_1'd_1}{\ }+\cdots\right)$$

$\mu\to(1+\delta)\mu \qquad m=\tilde{m}(1+\delta b_1'd_1+\cdots)$

$$\tilde{m}=m-m\,b_1'd_1\delta \qquad \mu\frac{\partial\ln m}{\partial\mu}=-d_1b_1'=2\lambda b_1'=\gamma_m$$

②

$$\phi_0 = \overset{\infty}{\phi}\underbrace{(1+\delta c_1' d_1 + \dots)^{1/2}}_{\tilde{Z}_\phi^{1/2}} = \overset{\infty}{\phi}(1+\delta c_1' d_1)\ \overset{\infty}{Z}_\phi^{1/2}$$

$$\frac{\delta Z_\phi^{1/2}}{Z_\phi^{1/2}} = \frac{1}{2}\delta c_1' d_1 = -\frac{2}{2}\lambda c_1'$$

$$\frac{1}{2}\mu\partial_\mu \ln Z_\phi = -\frac{2}{2}\lambda c_1' = -\lambda c_1'$$

$$\gamma = \frac{1}{2}\mu\frac{\partial \ln Z_\phi}{\partial\mu} = -\lambda c_1'$$

$$\beta = \mu\frac{\partial\lambda}{\partial\mu} = -2\lambda\varepsilon + 2(-a_1 + \lambda a_1')$$

$$\gamma_m = 2\lambda b_1'$$

$$\gamma = -\lambda_0 c_1'$$

$$c_1 = -\frac{\lambda^2}{3\times 2^{11}\pi^4} = -\frac{g^2}{24} \Rightarrow \boxed{\gamma = \frac{g}{12}}$$

$$\gamma_m \quad b_1 = \frac{\lambda}{2\times 32\pi^2} \qquad \boxed{\gamma_m = \frac{\lambda}{2\times 16\pi^2} = \frac{g}{2}}$$

$/Z_{m^2} \leftrightarrow Z_m$

$$\overbrace{\phi \quad \phi}^{4}\,\overbrace{\phi\phi}^{1}\,\overbrace{\phi \quad \phi}^{3} \to 1/2$$

mass renormalization.

$d = 4-2\varepsilon \qquad d/2 = 2-\varepsilon$

$$\frac{(-i\lambda)}{2}\int\frac{d^4k}{(2\pi)^d}\,\frac{i}{(k^2-m^2+i\varepsilon)} = -\frac{i\lambda}{2}\,\frac{i(-1)i}{(4\pi)^{d/2}}\,\frac{\Gamma(1-d/2)}{\Gamma(1)}\,\Delta^{d/2-1}$$

$\Delta = m^2 - i\varepsilon$

$$= -\frac{i\lambda}{2}\,\frac{\Gamma(-1+\varepsilon)}{(4\pi)^{2-\varepsilon}}\,(m^2)^{1-\varepsilon} = \frac{i\lambda}{32\pi^2}\,(4\pi)^{\varepsilon}\,\frac{\Gamma(\varepsilon)}{1-\varepsilon}\,m^{2-2\varepsilon}$$

$$\simeq \frac{i\lambda}{32\pi^2}\,\frac{m^2}{\varepsilon} \quad\longrightarrow\quad -\!\otimes\!- \;-\; \frac{i\lambda}{32\pi^2}\,\frac{m^2}{\varepsilon}$$

$$(\ )^{-1} - \bigcirc \;\Rightarrow\; -ip^2 + im^2 - \!-\!\otimes\!-$$

$$im^2 + \frac{i\lambda}{32\pi^2}\,\frac{m^2}{\varepsilon} = i\,m^2\left(1+\frac{\lambda}{32\pi^2}\,\frac{1}{\varepsilon}\right)$$

Since $Z_\phi = 1$ at 1-loop $\Rightarrow$ $m_0^2 = m^2\left(1+\frac{\lambda}{32\pi^2}\frac{1}{\varepsilon}\right)$

$m_0 = m\left(1+\frac{\lambda}{64\pi^2}\frac{1}{\varepsilon}\right)$

$$b_1 = \frac{\lambda}{64\pi^2} \;\Rightarrow\; \gamma_m = \frac{\lambda}{32\pi^2}$$

$$\gamma_m = \frac{1}{2}g \;\checkmark \quad \left(g = \frac{\lambda}{16\pi^2}\right)$$

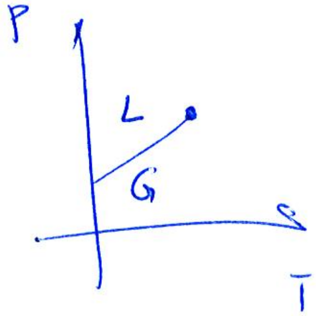
Critical behavior

near a critical point, second order phase transition we have that the correlation length diverges:

$$\langle m(\vec{r})\, m(0) \rangle \sim \frac{1}{r^p}\, e^{-r/\xi} \qquad \xi \to \infty$$

this gives scale invariance at the critical point:

$$\langle m(\vec{r})\, m(0) \rangle \sim \frac{1}{r^p}$$



magnets, etc

Consider magnets.

Critical exponents

$$Z = e^{-\beta A} = \mathrm{Tr}\, e^{-\beta \mathcal{H}}$$

(A: free energy, $\mathcal{H}$: hamiltonian.)

H: magnetic field.

$$M = -\frac{\partial A}{\partial H} \qquad \chi = \frac{1}{V}\frac{\partial M}{\partial H} \qquad C = T^2 \frac{\partial^2 A}{\partial T^2}$$

$$t = \frac{T - T_c}{T_c} \qquad t \to 0 \text{ : critical point.}$$

T_c: critical temperature

critical exponents

$$\langle m(r) m(0) \rangle \sim r^{-p} e^{-r/\xi}$$

$$p = d-2+\eta \qquad \xi \sim t^{-\nu}$$

$$M \sim |t|^{\beta} \qquad \chi \sim |t|^{-\gamma} \qquad M \sim H^{1/\delta} \ (t=0)$$

$$C \sim |t|^{-\alpha}$$

Scaling hypothesis. ξ: only length scale. (near conformal point).

$$m(r) = \frac{M}{V}$$

$$m \sim L^{-p/2} = L^{-\frac{1}{2}(d-2+\eta)} \sim \frac{M}{V}$$

βA : dimensionless $(e^{-\beta A})$.

$$-\tfrac{1}{2}d+1-\tfrac{\eta}{2}+\tfrac{d}{2} \quad 1-\eta/2$$

$$M = -\frac{\partial A}{\partial H} \qquad m = -\frac{1}{V}\frac{\partial A}{\partial H} = -\frac{1}{V}\frac{\beta \partial A}{\beta \partial H}$$

$$\beta H \sim L^{-d} L^{\frac{1}{2}(d-2+\eta)} = L^{\frac{1}{2}(-d-2+\eta)}$$

$$\chi = \frac{1}{V}\frac{\partial M}{\partial H} \sim \frac{L^{-\frac{1}{2}(d-2+\eta)}}{L^{\frac{1}{2}(-d-2+\eta)}}$$

$$\chi \sim L^{2-\eta}$$

Assume $L \sim \xi \sim t^{-\nu}$

$$\Rightarrow \quad m \sim t^{-\frac{\nu}{2}(d-2+\eta)} \qquad H \sim t^{-\frac{\nu}{2}(-d-2+\eta)}$$

$$\beta A \sim t^{+\nu d} \qquad C \sim t^{+\nu d-2} \qquad \chi \sim \xi^{-\nu(2-\eta)}$$

$$\frac{\beta A}{V}$$

$$\xi^{-d}$$

$\alpha = 2 - \nu d$ ✓

$\beta = -\frac{\nu}{2}(2-d-\eta)$ ✓

$\gamma = \nu(2-\eta)$ ✓

$$t^{-\frac{\nu}{2}(d-2+\eta)} \sim t^{\frac{\nu}{2}(-d-2+\eta)\frac{1}{\delta}}$$

$$\delta = \frac{-d-2+\eta}{-(d-2+\eta)} = \frac{d+2-\eta}{d-2+\eta} \checkmark$$

! we only need η, ν

3d model based on ϕ^4 theory.

ϕ^4 with $m^2=0$ flows to an IR fixed point. for $d \neq 4$.

$$\beta = -2\lambda\varepsilon + \frac{3\lambda^2}{16\pi^2} = 0 \qquad \frac{3\lambda}{16\pi^2} = 2\varepsilon$$

$g = \frac{\lambda}{16\pi^2}$ $\qquad \boxed{g_* = \frac{2\varepsilon}{3}}$ $\qquad 2\varepsilon = 1 \qquad g_* = 1/3$ in $\underline{d=3}$.

$m^2 \sim t$ assumption $\Rightarrow$ $t=0$ is critical point.

$$\Gamma(x/\sigma, m^2, \lambda) \simeq \sigma^{d-n-n\gamma}\,\Gamma(x; \bar{m}^2, \bar{\lambda})$$

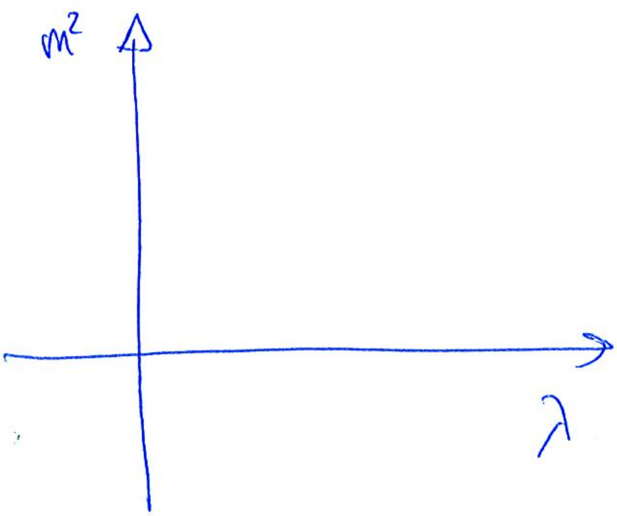
$x \sim 1/p$ $\qquad$ take $\bar{\lambda} \simeq \lambda_*$

At the critical point

$$\langle \phi(r)\, \phi(0) \rangle \sim \frac{1}{r^{2\Delta}}$$

Δ : conf. dim. $= \frac{d-2}{2} + \gamma$ $\qquad p = 2\Delta = \boxed{d-2+2\gamma}$

$$\boxed{\eta = 2\gamma}$$



$$\sigma \frac{\partial}{\partial \sigma} \ln \bar{m} = \gamma_m - 1 = -1 + g/2 \simeq -1 + \frac{g_*}{2} = -1 + \epsilon/3$$

$$\bar{m}^2 = \sigma^{-2+\frac{2\epsilon}{3}} m_0^2$$

$$\Gamma(\sigma p_i, m^2) \sim \sigma^{4-n-n\gamma}\, \Gamma\left(p_i, \bar{m}^2 = \sigma^{-2+\frac{2\epsilon}{3}} m^2\right)$$

$\sigma \to 0 \qquad \sigma \sim 1/x \qquad m^2 \sim t$ by hypothesis $\qquad$ $2\epsilon = 1 \quad \boxed{\nu = 3/5} \simeq 0.6$

log distance $\qquad$ $2 - 1/3 = \frac{5}{3}$

$$\Gamma\left(p_i,\ t\, x^{2-\frac{2\epsilon}{3}} = \left(x\, t^{\frac{1}{2-2\epsilon/3}}\right)^{2-2\epsilon/3}\right.$$

fixed large momentum

$$\xi = t^{-\frac{1}{2-2\epsilon/3}} \Rightarrow \nu = \frac{1}{2-2\epsilon/3}$$

Normal coordinates

$$\phi^a = \bar{\phi}^a + \xi^a - \frac{1}{2}\Gamma^a_{bc}\xi^b\xi^c - \frac{1}{3!}\Gamma^a_{bcd}\xi^b\xi^c\xi^d + \dots$$

$$\Gamma^a_{bcd} = \nabla_d \Gamma^a_{bc}$$

$$g_{ab}(\phi+\delta\phi) = g_{ab} - \frac{1}{3}R_{acbd}\xi^c\xi^d - \frac{1}{6}D_c R_{adbe}\xi^c\xi^d\xi^e \dots$$

$$\partial_\mu\phi^a = \partial_\mu\bar{\phi}^a + D_\mu\xi^a + \left[\frac{1}{3}R^a{}_{cdb}\xi^c\xi^d + \dots\right]\partial_\mu\bar{\phi}^b$$

$$I = I[\bar{\phi}] + \int d^2x\, g_{ab}\partial_\mu\phi^a D_\mu\xi^b +$$

$$+ \frac{1}{2}\int dx\, \{ g_{ab}D_\mu\xi^a D^\mu\xi^b + R_{acdb}\xi^c\xi^d\partial_\mu\phi^a\partial^\mu\phi^b \}$$

$$+ \dots$$

$$I^{(2)} = \frac{1}{2}\int d^2x\, \left\{ D_\mu\xi^a D_\mu\xi^a + \left(R_{iabj}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{\partial I}{\partial\phi^i}\Gamma^i_{ab}\right)\xi^a\xi^b \right\}$$

$n = 2+\varepsilon$

$$I^{\text{counter}}(\phi) = \frac{1}{4\pi\varepsilon}\int d^2x\, \left[R_{ab}\partial_\mu\phi^a\partial^\mu\phi^b + g^{ab}\Gamma^c_{ab}\frac{\delta I}{\delta\phi^c} \right]$$

②

$$\bar{g}_{ab} = \delta_{ab} \qquad \bar{\Gamma}^a_{bc} = 0$$

$$\int \mathcal{D}\phi \; e^{i\int \frac{1}{2}\partial_\mu \xi^a \partial^\mu \xi^a + \frac{1}{2} R_{iabj}\, \partial_\mu \phi^i \partial_\mu \phi^j \, \xi^a \xi^b}$$

$$e^{i\int \frac{1}{2}\partial_\mu \xi^a \partial^\mu \xi^a + \frac{1}{2} M_{ab}\, \xi^a \xi^b}$$

$$K_{ab} = -\frac{i}{2}\partial^2 \delta^{ab} + \frac{i}{2} M^{ab}$$

$$M^{ab} v^{(l)}_b = \lambda^{(l)} v^a$$

$$\xi^{(l)}_a = v^{(l)}_a e^{ikx} \quad \to \text{eigenvalue} \quad \frac{i}{2}p^2 + \frac{i}{2}\lambda^{l}$$

$$\mathrm{Tr}\ln = \sum_{(l)} \int \frac{d^2p}{(2\pi)^2} \ln\left(\frac{i}{2}p^2 + \frac{i}{2}\lambda^{(l)}\right)$$

$$= L^2 \sum_l \int \frac{d^2p}{(2\pi)^2} \int_0^\infty \frac{d\mu\, \mu^{s-1}}{\Gamma(s)} e^{-\mu(\varepsilon - \frac{i}{2}p^2 + \frac{i}{2}\lambda(l))}$$

$$\int \frac{d^2p}{(2\pi)^2} e^{\frac{i}{2}\mu p^2} = \frac{1}{4\pi^2}\sqrt{\frac{\pi}{-\frac{i\mu}{2}}}\sqrt{\frac{\pi}{\frac{i\mu}{2}}} = \frac{2\pi}{4\pi^2 \mu} = \frac{1}{2\pi\mu}$$

$$= \frac{L^2}{2\pi} \int_0^\infty \frac{d\mu\, \mu^{s-2}}{\Gamma(s)} e^{-\mu(\varepsilon + \frac{i}{2}\lambda(l))} = \frac{L^2}{2\pi} \frac{(\varepsilon + \frac{i}{2}\lambda(l))^{1-s}}{\Gamma(s)} \Gamma(s-1)$$

$$= \frac{L^2}{2\pi} \frac{(\frac{i}{2}\lambda(l))^{1-s}}{s-1} \Rightarrow \zeta_K(s) = \frac{L^2}{2\pi} \sum_l \frac{1}{s-1}\left(\frac{i}{2}\lambda(l)\right)^{s-1}$$

③

$$\zeta_K(0) = +\frac{iL^2}{2\pi}\sum_{(e)}\frac{\delta_{(e)}}{2}e + \frac{iL^2}{4\pi}\mathrm{Tr}\,M_{ab}$$

$$= +\frac{i}{4\pi}\int d^2x\; R_{iaaj}\,\partial_\mu\phi^i\partial^\mu\phi^j$$

$$\boxed{= -\frac{i}{4\pi}\int d^2x\; R_{ij}\,\partial_\mu\phi^i\partial^\mu\phi^j}$$

$$\det K = e^{-\zeta_K'(0)} \qquad \det^{-\frac{1}{2}} K = e^{\frac{1}{2}\zeta_K'(0)}$$

~~$L \to L/a$ $\quad K \to K/a^2$~~

$$x \to x/a \qquad K \to a^2 K$$

$$\det^{-1/2}(a^2K) = e^{-\ln a\,\zeta(0)}\det^{-1/2}K$$

$$= e^{+\ln a\frac{i}{4\pi}\int d^2x\, R_{ij}\partial_\mu\phi^i\partial^\mu\phi^j}$$

$$g_{ij} = g_{ij} + \frac{1}{4\pi}\ln a\, R_{ij}$$

~~wrong~~ sign?

$$\mu\partial_\mu g_{ij} = \frac{1}{4\pi}R_{ij}$$

$$g_{ij} = \frac{1}{g^2}\,\mathring{g}^{(n)}_{ij} \quad \leftarrow \text{sphere}$$

$$R_{ij} = (N-2)\,\mathring{g}_{ij}$$

$$-\frac{2}{g^3}\beta_g = \frac{N-2}{4\pi} \qquad \boxed{\beta_g = -\frac{N-2}{8\pi}g^3}$$

4π?