$$\ln \det A = \sum_n \ln a_n$$

eigenvalues

$$\zeta_A(s) = \sum_n \frac{1}{a_n^s} = \sum_n a_n^{-s}$$

$$\partial_s \zeta_A(s) = \sum_n (-\ln a_n)\, a_n^{-s}$$

$$\zeta_A'(0) = -\sum_n \ln a_n$$

$$\boxed{\det A = e^{-\zeta_A'(0)}}$$

$$(\det A)^{-1/2} = e^{\frac{1}{2}\zeta_A'(0)}$$

$$A \to \mu A \qquad a_n \to \mu a_n$$

$$\zeta_{\mu A}(s) = \sum_n \mu^{-s} a_n^{-s} = \mu^{-s} \zeta_A(s)$$

$$\zeta_{\mu A}'(s) = -\ln\mu\, \mu^{-s} \zeta_A(s) + \mu^{-s} \zeta_A'(s)$$

$$\zeta_{\mu A}'(0) = -\ln\mu\, \zeta_A(0) + \zeta_A'(0)$$

$$\det(\mu A) = \det A\, e^{\ln\mu\, \zeta_A(0)}$$

$$\boxed{\det(\mu A) = \mu^{\zeta_A(0)} \det A}$$

$$\det{}^{1/2}(\mu A) = e^{\frac{1}{2}\zeta_A(0)} \det A$$

$$S^{(2)} = -i \int \left(\frac{1}{2} \phi\, \partial^2 \phi + \frac{\lambda}{4} \bar{\phi}_c^2 \phi^2 \right) - \varepsilon \int \phi^2$$

$$= - \int d^4x\, d^4y\ \phi(x)\, K(x,y)\, \phi(y)$$

$$K(x,y) = \frac{i}{2} \partial^2 \delta(x-y) + \frac{i}{4} \lambda \bar{\phi}_c^2 \delta(x-y) + \varepsilon\, \delta(x-y)$$

$$\det{}^{-1/2} K(x,y) = e^{-\frac{1}{2} \ln \det K} = e^{i \frac{i}{2} \ln \det K}$$

$$S_{eff}^{(2)} = \frac{i}{2} \ln \det K$$

space-time
Volume $V = L^4$ $\qquad \psi = \frac{1}{\sqrt{V}} e^{ipx} \qquad p_j = \frac{2\pi}{L} n_j$

$$\boxed{\varepsilon_p = -\frac{i}{2} p^2 + \frac{i}{4} \lambda \bar{\phi}_c^2 + \varepsilon}$$

$$\boxed{S_{eff}^{(2)} = \frac{i}{2} L^4 \int \frac{d^4p}{(2\pi)^4} \ln \left(\varepsilon - \frac{i}{2} p^2 + \frac{i}{4} \lambda \phi_c^2 \right)}$$

↳ divergent

•) $\dfrac{\partial S_{eff}^{(2)}}{\partial \phi_c^2} \sim \displaystyle\int \frac{d^4p}{(p^2+\phi^2)} \qquad \partial^2 \to \int \frac{1}{(p^2+\phi^2)^2}$

$\dfrac{\partial^3 S_{eff}^{(2)}}{(\partial(\phi_c^2))^3}$ is finite

Call $\frac{\lambda}{4}\bar{\Phi}_c^2 = A$

$$\partial_A \int \frac{d^4p}{(2\pi)^4} \ln\left(\varepsilon - \frac{i}{2}p^2 + iA\right) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\varepsilon - \frac{i}{2}p^2 + iA}$$

$$\partial_A^2 = +\int \frac{d^4p}{(2\pi)^4} \frac{1}{\left(\varepsilon - \frac{i}{2}p^2 + iA\right)^2}$$

$$\partial_A^3 = -2\int \frac{d^4p}{(2\pi)^4} \frac{i}{\left(\varepsilon - \frac{i}{2}p^2 + iA\right)^3} = -2i \int \frac{d^4p}{(2\pi)^4} \int_0^\infty \frac{d\mu\, \mu^{3-1}}{\Gamma(3)} e^{-\mu\varepsilon + \frac{i}{2}\mu p^2 - \mu iA}$$

$$= +i \int_0^\infty d\mu\, \mu^{3-1} \left(1 - \frac{i}{4\pi^2\mu^2}\right) e^{-\mu\varepsilon - i\mu A}$$

$$= -\frac{1}{4\pi^2} \int_0^\infty d\mu\, \mu^{1-1} e^{-\mu\varepsilon - i\mu A} = -\frac{1}{4\pi^2} \frac{\overset{=1}{\Gamma(1)}}{(\varepsilon + iA)}$$

$$\boxed{\partial_A^3 = \frac{i}{4\pi^2}\frac{1}{A}}$$

$$\int_A = \frac{i}{4\pi^2}\ln A + C_1 \xrightarrow{\int_A} \frac{i}{4\pi^2}(A\ln A - A) + C_1 A + C_2 \to$$

$$\xrightarrow{\int_A} \frac{i}{4\pi^2}\left(\frac{1}{2}A^2\ln A - \frac{A^2}{4} - \frac{A^2}{2}\right) + C_1\frac{A^2}{2} + C_2 A + C_3$$

$$A\ln A + \frac{1}{2}A - \frac{A}{2} - A$$

So $$\int_A\iint = \frac{i}{8\pi^2}A^2\ln A + \underbrace{\bar{c}_1 + \bar{c}_2 A + \bar{c}_3 A^2}_{\text{fixed by renormalization.}}$$

$$S_{eff}^{(2)} = \frac{i}{2} L^4 \frac{i}{8\pi^2} A^2 \ln A + \cdots$$

$$= -\frac{1}{16\pi^2} L^4 \left(\frac{\lambda}{4}\phi_c^2\right)^2 \ln\left(\frac{\lambda}{4}\phi_c^2\right) + \cdots$$

$$= -\frac{\lambda^2 L^4 \phi_c^4}{2^8 \pi^2} \ln(\phi_c^2) + a + b\phi_c^2 + c\phi_c^4$$

undetermined.

Take $a = b = 0$

by ~~se~~

$\phi_c \to a\phi_c$.

$\phi_c^4 \to a^4 \phi_c^4$ can be compensated by $L \to L/a$.

but we have $\ln(a^2\phi_c^2) = 2\ln a + \ln(\phi_c^2)$

$$\delta S_{eff}^{(2)} = -\frac{\lambda^2 L^4}{2^7 \pi^2} \phi_c^4 \ln a.$$

$$S_{cl} = -\frac{\lambda}{4!} \underset{1234}{\phi_c^4} L^4$$

$$\bar{\lambda} = \lambda + \frac{\lambda^2}{2^7\pi^2} 4! \ln a = \lambda + \frac{3\lambda^2}{16\pi^2} \ln a$$

$$\frac{\partial \lambda}{\partial \ln a} = \frac{3\lambda^2}{16\pi^2} \Rightarrow \beta\text{-function.}$$

ζ-function reg.

$$\zeta_K(s) = L^4 \int \frac{d^4p}{(2\pi)^4} \int_0^\infty \frac{d\mu\, \mu^{s-1}}{\Gamma(s)}\, e^{-\mu\left(\varepsilon - \frac{i}{2}p^2 + \frac{i\lambda}{4}\bar{\phi}_c^2\right)}$$

$$= \frac{L^4}{\Gamma(s)} \int_0^\infty d\mu\, \overset{s-2}{\mu^{s-1}} \left(-\frac{i}{4\pi^2\mu^2}\right) e^{-\mu\left(\varepsilon + \frac{i\lambda}{4}\bar{\phi}_c^2\right)}$$

$$= -\frac{i}{4\pi^2} \frac{L^4}{\Gamma(s)} \left(\varepsilon + \frac{i\lambda}{4}\bar{\phi}_c^2\right)^{2-s} \Gamma(s-2)$$

$$\frac{\Gamma(s-2)}{\Gamma(s)} = \frac{\cancel{\Gamma(s-2)}}{(s-2)(s-1)\cancel{\Gamma(s-2)}} = \frac{1}{(s-2)(s-1)}$$

$$\zeta_K(s) = -\frac{i}{4\pi^2} \frac{L^4}{(s-1)(s-2)} \left(\frac{i\lambda}{4}\bar{\phi}_c^2\right)^{2-s}$$

$$\zeta_K(0) = +\frac{i}{4\pi^2} \frac{L^4}{2} \frac{\lambda^2}{16} \bar{\phi}_c^4 = \frac{i\lambda^2 L^4 \bar{\phi}_c^4}{2^7\pi^2}$$

$$\det K = e^{-\zeta_K'(0)} \qquad \det^{-1/2} K = e^{\frac{1}{2}\zeta_K'(0)}$$

$$\zeta_{a^2K} = a^{-2s}\zeta_K \qquad \zeta'_{a^2K}(0) = -2\ln a\, \zeta(0) + \zeta_K'(0)$$

$$\det^{-1/2}(a^2K) = e^{-\ln a\, \zeta(0)} \det^{-1/2} K = e^{-\frac{i\lambda^2}{2^7\pi^2} L^4 \bar{\phi}_c^4 \ln a}$$

same as before

$$e^{-S_{eff}(\bar{\phi}_c)} = \int \mathcal{D}\phi \; e^{-S_{cl}(\bar{\phi}_c) - \frac{1}{2}\int \partial\phi\partial\phi + \frac{\lambda}{4!}(6\bar{\phi}^2\phi^2 + 4\phi^3\bar{\phi} + \phi^4)}$$

(2)

$$\overset{\text{1-loop}}{\underset{\sim}{\downarrow}} \; e^{-S_{cl}(\bar{\phi}_c)} \det{}^{-1/2}\left(\frac{1}{2}\partial^2 - \frac{1}{4}\lambda\bar{\phi}_c^2\right)$$

$$S_{cl}(\bar{\phi}_c) = \frac{1}{2}\int (\partial\bar{\phi})^2 + \int \frac{1}{4!}\lambda\bar{\phi}^4 d^4x$$

$$\tilde{\phi}_c = \mu\bar{\phi}_c(\mu x) \qquad \partial\tilde{\phi}_c = \mu^2\partial\bar{\phi}_c(\mu x)$$

$$S_{cl}(\tilde{\phi}_c(x)) = \cancel{\frac{1}{2}}\int d^4x\,\mu^4\left[\frac{1}{2}(\partial\bar{\phi}_c(\mu x))^2 + \frac{1}{4!}\lambda\bar{\phi}^4(\mu x)\right]$$

$$= \int d^4\tilde{x}\,[\;-\;-\;] = S_{cl}(\bar{\phi}_c)$$

classical action is scale invariant.

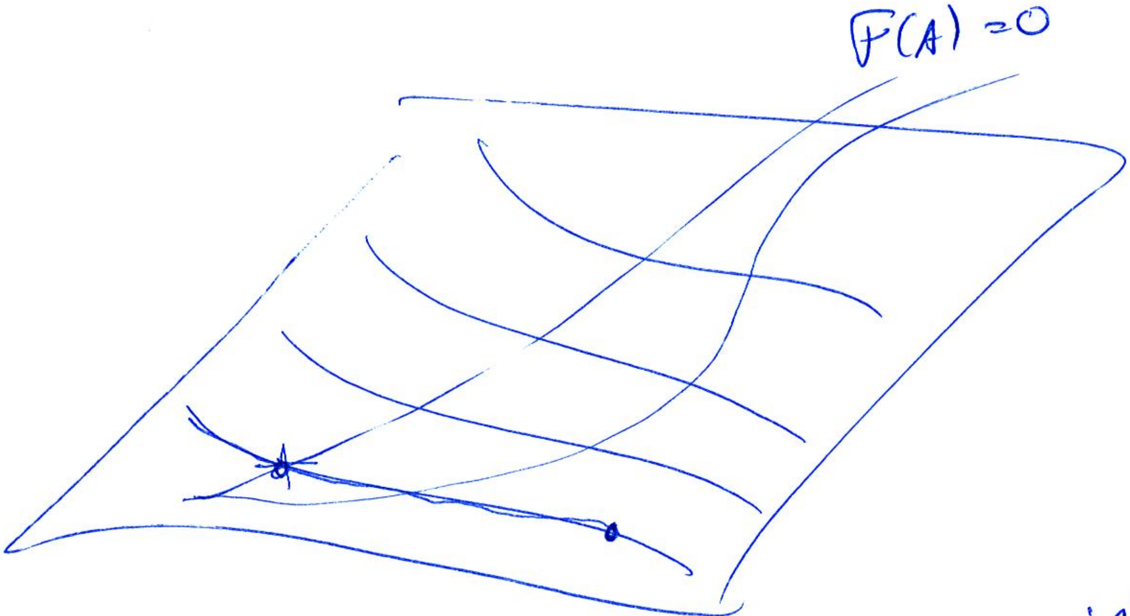
$$A = \frac{1}{2}\partial_x^2 - \frac{1}{4}\lambda\mu^2\bar{\phi}_c^2(\mu x)$$

$\tilde{x} = \mu x$

$$= \mu^2\left(\frac{1}{2}\partial_{\tilde{x}}^2 - \frac{1}{4}\lambda\bar{\phi}_c^2(x)\right)$$

$$\mu\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x}$$

$$\det{}^{-1/2}(\mu^2 A) = \underbrace{\mu^{-\zeta_A(0)}}\det A$$



$$A_\mu \to U A_\mu U^\dagger - i\, \partial_\mu U\, U^\dagger \qquad U = e^{i\lambda^a t^a}$$

$$A^a_\mu \to \partial_\mu \lambda^a + f^{abc} \lambda^b A^c_\mu \quad \text{infinitesimal.}$$

$$\Delta_f^{-1}[A_\mu] = \int \mathcal{D}\lambda^a(x)\; \delta(\mathcal{F}(A^\lambda_\mu))$$

$$\Delta_f^{-1}[A^{\tilde\lambda}_\mu] = \int \mathcal{D}\lambda^a \;\delta(\mathcal{F}(A^{\overbrace{\tilde\lambda\circ\lambda}^{\tilde{\tilde\lambda}}}))$$

$$= \int \mathcal{D}\tilde{\tilde\lambda}^a \;\delta(\mathcal{F}(A^{\tilde{\tilde\lambda}})) = \Delta_f^{-1}[A_\mu]$$

gauge invariant functional.

$$\int \mathcal{D}A_\mu \int \mathcal{D}\lambda\; \delta(\mathcal{F}(A^\lambda_\mu))\; \Delta_f(A_\mu)\, e^{iS[A_\mu]}\, \mathcal{O}(A_\mu)$$

$$\int \mathcal{D}\lambda \int \mathcal{D}A^\lambda_\mu\; \delta(\mathcal{F}(A^\lambda_\mu))\; \Delta_f(A^\lambda_\mu)\, e^{iS[A^\lambda_\mu]}\, \mathcal{O}(A^\lambda_\mu)$$

$$= \underbrace{\int \mathcal{D}\lambda}_{V_G} \int \mathcal{D}A_\mu \; \delta(F(A_\mu)) \, \Delta_f(A_\mu) \, e^{iS[A_\mu]} \, \mathcal{O}(A_\mu)$$

drop.

$$\int \mathcal{D}\lambda \; \delta(F^a(A_\mu^\lambda))$$

$$\int \mathcal{D} \quad \int d\xi_a \; \delta(f^b(x_a)) = \int d\eta_a \; \delta(\partial_a f^b \eta^a) =$$

$$f^b(\bar{x}_a) = 0 \qquad \xi_a^b = \partial_a f^b \eta^a \qquad \frac{\partial_c \xi_\theta^b}{\partial \eta_a} = \partial_a f^b$$

$$= \int d\xi_a \; \frac{\partial \eta_a}{\partial \xi_b} \; \delta(\xi_a) = \det{}^{-1}(\partial_a f^b)$$

$$\int \mathcal{D}\lambda^a(x) \; \delta(F^a(A_\mu^\lambda)) = \det{}^{-1}\left(\frac{\delta F^a(A_\mu^\lambda)(x)}{\delta \lambda^b(y)} \right)$$

$$\Delta_f = \det\left(\frac{\delta F^a(x)}{\delta \lambda^b(y)} \right) \Bigg|_{F(A_\mu)=0}$$

$$= \int \mathcal{D}\bar{c} \, \mathcal{D}c \; e^{\int \bar{c}^a(x) \frac{\delta F^a(x)}{\delta \lambda^b(y)} c^b(y)}$$

Background field gauge

$$A_\mu = \hat{A}_\mu + \mathcal{A}_\mu$$

$\hat{A}_\mu$: classical "fixed" $\mathcal{A}_\mu$: fluctuations.

gauge transf.

$$\delta A_\mu = D_\mu \lambda = \partial_\mu \lambda + i [A_\mu, \lambda]$$

now

$$\delta(\hat{A}_\mu + \mathcal{A}_\mu) = \partial_\mu \lambda + i [\hat{A}_\mu, \lambda] + i [\mathcal{A}_\mu, \lambda]$$

$$= \hat{D}_\mu \lambda + i [\mathcal{A}_\mu, \lambda]$$

gauge transf. changes $\mathcal{A}_\mu$, leaves $\hat{A}_\mu$ fixed. Then we should have

$$\delta \hat{A}_\mu = 0 \qquad \boxed{\delta \mathcal{A}_\mu = \hat{D}_\mu \lambda + i [\mathcal{A}_\mu, \lambda]}$$

For matter fields, e.g. fermion

$$\delta\psi = i\lambda\psi \quad ; \quad \delta(\hat{\psi} + \psi) = i\lambda(\hat{\psi} + \psi)$$

$$\delta\hat{\psi} = 0 \quad \delta\psi = i\lambda(\hat{\psi} + \psi)$$

We need to fix this gauge symmetry.

$$\mathcal{F} = \hat{D}_\mu A^\mu = \partial_\mu A^\mu + i[\hat{A}_\mu, A^\mu]$$

treated as ordinary adjoint field.

$$\delta\mathcal{F} = \hat{D}_\mu(\hat{D}_\mu \lambda + i[A_\mu, \lambda])$$

not gauge invariant, as it should.

ghost term:

$$i\int \bar{c}\,\hat{D}_\mu(\hat{D}_\mu c + i[A_\mu, c])$$

$$S_{\bar{c}c} = i\int \hat{D}_\mu \bar{c}\left(\hat{D}_\mu c + i[A_\mu, c]\right)$$

$$c \to \hat{c} + c \qquad \bar{c} \to \hat{\bar{c}} + \bar{c}$$

$$S = i\int \hat{D}_\mu(\hat{\bar{c}} + \bar{c})(\hat{D}_\mu(\hat{c} + c) + i[A_\mu, \hat{c} + c])$$

Up to know everything is as usual.

However in this gauge (i.e. in this choice of $\mathcal{F}$)

we can do a gauge transf. of the external fields:

$$\delta \hat{A}_\mu = \hat{D}_\mu \lambda$$

$$\delta \hat{\varphi} = i\lambda\hat{\varphi}$$

$$\delta \hat{c} = i[\lambda, \hat{c}]$$

$$\delta \hat{\bar{c}} = i[\lambda, \hat{\bar{c}}]$$

supplemented by.

$$\delta A_\mu = i[A_\mu, \lambda] \quad \leftarrow \text{"adjoint field"}$$

$$\delta \varphi = i[\varphi, \lambda]$$

$$\delta c = i[\lambda, c]$$

$$\delta \bar{c} = i[\lambda, \bar{c}]$$

then $S_0[A]$ is invariant since this is a gauge transf.

Let's see gauge fixing term:

$$\mathrm{Tr}\, \mathcal{F}^2 = \mathrm{Tr}\big(\hat{D}_\mu A^\mu\big)^2$$

covariant derivative, adjoint field $\}$ $\to$ invariant!

ghost:

$$\hat{D}_\mu(\hat{\bar{c}}+\hat{c})\, \hat{D}_\mu(\hat{c}+c) \quad \text{also invariant.}$$

$$i\, \hat{D}_\mu(\hat{\bar{c}}+\hat{\bar{c}})\, [A_\mu, \hat{c}+c] \quad \underline{\text{also}}$$

everything is invariant! $\to$ effective action is gauge invariant!!

$$A_\mu = \hat{A}_\mu + \mathcal{A}_\mu$$

$$\delta A_\mu = D_\mu \lambda = \partial_\mu \lambda + i [A_\mu, \lambda]$$

$$\delta(\hat{A}_\mu + \mathcal{A}_\mu) = \partial_\mu \lambda + i [\hat{A}_\mu, \lambda] + i [\mathcal{A}_\mu, \lambda]$$

$$= \hat{D}_\mu \lambda + i [\mathcal{A}_\mu, \lambda]$$

gauge $\qquad \delta \hat{A}_\mu = 0 \qquad \delta \mathcal{A}_\mu = \hat{D}_\mu \lambda + i [\mathcal{A}_\mu, \lambda]$

$$\cancel{F_{\mu\nu} F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A^\mu}$$

$$F^{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu +$$

$$+ [\hat{A}_\mu, \hat{A}_\nu] + [\hat{A}_\mu, \mathcal{A}_\nu] + [\mathcal{A}_\mu, \hat{A}_\nu] + [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

$$= \hat{F}_{\mu\nu} + \mathcal{F}_{\mu\nu} + [\hat{A}_\mu, \mathcal{A}_\nu] + [\mathcal{A}_\mu, \hat{A}_\nu]$$

$$F^2 = \hat{F}^2 + \mathcal{F}^2 + \hat{F}_{\mu\nu} ([\hat{A}_\mu, \mathcal{A}_\nu] + [\mathcal{A}_\mu, \hat{A}_\nu])$$

$$F^{\mu\nu} = \hat{F}_{\mu\nu} + \hat{D}_\mu \mathcal{A}_\nu - \hat{D}_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

$$F^2 = \hat{F}^2 + \hat{F}_{\mu\nu} \hat{D}_\mu \mathcal{A}_\nu - D_\nu \mathcal{A}_\mu$$

$$F^2 = \hat{F}^2 + 2\hat{F}_{\mu\nu}\left(\overset{\text{drop}}{\hat{D}_\mu A_\nu} + [A_\mu, A_\nu]\right) +$$

$$+ \left(\hat{D}_\mu A_\nu - \hat{D}_\nu A_\mu + [A_\mu, A_\nu]\right)^2$$

$$= \hat{F}^2 + 2\hat{F}_{\mu\nu}[A_\mu, A_\nu] + F_{\mu\nu}F^{\mu\nu}$$

$$\hat{F}_{\mu\nu}\hat{D}_\mu A_\nu$$

$$\mathbb{F} = \hat{D}_\mu A^\mu = \partial_\mu A^\mu + i[\hat{A}_\mu, A^\mu]$$

$$\delta \mathbb{F} = \hat{D}_\mu\left(\hat{D}_\mu \lambda + i[A_\mu, \lambda]\right)$$

$$\int \hat{D}_\mu \bar{c}\left(\hat{D}_\mu c + i[A_\mu, c]\right)$$

$$\mathrm{Tr}\left(\hat{F}^{\mu\nu}\hat{D}_\mu A_\nu\right)$$

linear term

invariant