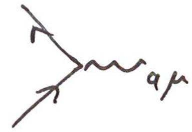
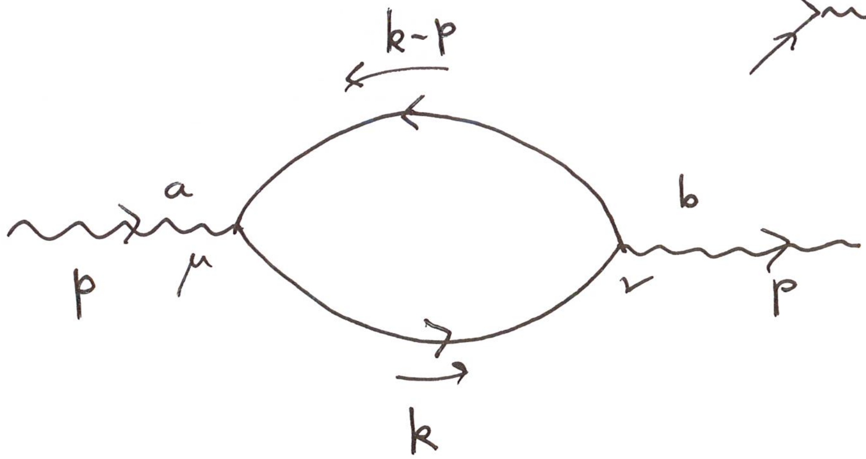
Quark loop correction to β-function



$ig\gamma^\mu t^a$



fermion loop

$$-(ig)^2\,\mathrm{Tr}(t^a t^b)\, i^2 \int \frac{d^dk}{(2\pi)^d}\, \frac{\mathrm{Tr}\left(\gamma^\mu (\not{k}-\not{p}+m)\,\gamma^\nu (\not{k}+m)\right)}{\left((k-p)^2-m^2+i\varepsilon\right)\left(k^2-m^2+i\varepsilon\right)}$$

propagators

$$\int_0^1 d\alpha\, \frac{1}{\left(\cancel{\alpha k^2}-2\alpha pk+\alpha p^2-\cancel{\alpha m^2}+(1-\cancel{\alpha})k^2-(1-\cancel{\alpha})m^2\right)^2}$$

$$(k-\alpha p)^2+\alpha(1-\alpha)p^2-m^2+i\varepsilon$$

$$k\to k+\alpha p$$

$$-g^2\,\mathrm{Tr}(t^a t^b)\int \frac{d^dk}{(2\pi)^d}\, \frac{\mathrm{Tr}\left(\gamma^\mu(\not{k}-(1-\alpha)\not{p}+m)\,\gamma^\nu(\not{k}+\alpha\not{p}+m)\right)}{\left(k^2+\alpha(1-\alpha)p^2-m^2+i\varepsilon\right)^2}$$

Numerator: $\mathrm{Tr}\left(\gamma^\mu(\not{k}-(1-\alpha)\not{p})\,\gamma^\nu(\not{k}+\alpha\not{p})\right)+m^2\,\overbrace{\mathrm{Tr}\,\gamma^\mu\gamma^\nu}^{4\eta^{\mu\nu}}$

$$\mathrm{Tr}(\gamma^\mu\not{k}\gamma^\nu\not{k})-\alpha(1-\alpha)\,\mathrm{Tr}(\gamma^\mu\not{p}\gamma^\nu\not{p})+4m^2\eta^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu\not{k}\gamma^\nu\not{k}) = -\mathrm{Tr}(\gamma^\mu\gamma^\nu\underbrace{\not{k}\not{k}}_{k^2})+2k^\nu\,\mathrm{Tr}(\gamma^\mu\not{k})$$

$$= -k^2\,4\eta^{\mu\nu}+8k^\mu k^\nu$$

$$-g^2 \mathrm{Tr}(t^a t^b) \int \frac{d^dk}{(2\pi)^d} \frac{(-4k^2\eta^{\mu\nu} + 8k^\mu k^\nu + \alpha(1-\alpha)4p^2\eta^{\mu\nu} - 8\alpha(1-\alpha)p^\mu p^\nu + 4m^2\eta^{\mu\nu})}{(k^2-\Delta)^2} \quad (2)$$

$$\Delta = -\alpha(1-\alpha)p^2 + m^2$$

$$k^\mu k^\nu \to \frac{1}{d}\eta^{\mu\nu}k^2$$

$$\frac{8}{d}\eta^{\mu\nu}k^2 - 4k^2\eta^{\mu\nu} = \frac{4}{d}(2-d)\,k^2\eta^{\mu\nu} = -\frac{4}{d}(d-2)k^2\eta^{\mu\nu}$$

$$-g^2 \mathrm{Tr}(t^a t^b)\left[\int \frac{d^dk}{(2\pi)^d}\left(-\frac{4}{d}(d-2)\eta^{\mu\nu}\right)\frac{k^2}{(k^2-\Delta)^2} + \right.$$

$$\left. + \int \frac{d^dk}{(2\pi)^d} \frac{(\alpha(1-\alpha)4p^2\eta^{\mu\nu} - 8\alpha(1-\alpha)p^\mu p^\nu + 4m^2\eta^{\mu\nu})}{(k^2-\Delta)^2}\right]$$

$$(2-d) = 2(1-d/2) \qquad \Gamma(2-d/2)$$

$$= -g^2 \mathrm{Tr}(t^a t^b)\left\{\left(\cancel{-}\frac{4}{\cancel{d}}(d-2)\eta^{\mu\nu}\right)\frac{(-)i}{(4\pi)^{d/2}}\frac{\cancel{d}}{2}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} + \right.$$

$$\cancel{(1-d/2)\Gamma(1-d/2)}$$

$$\left. + (\alpha(1-\alpha)4p^2\eta^{\mu\nu} - 8\alpha(1-\alpha)p^\mu p^\nu + 4m^2\eta^{\mu\nu})\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\right\}$$

$$= -g^2 \mathrm{Tr}(t^a t^b)\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\left\{-4\eta^{\mu\nu}\Delta + 4\alpha(1-\alpha)p^2\eta^{\mu\nu} - 8\alpha(1-\alpha)p^\mu p^\nu \cancel{+4m^2\eta^{\mu\nu}}\right\}$$

$$\cancel{-4\eta^{\mu\nu}m^2} + 4\alpha(1-\alpha)p^2\eta^{\mu\nu}$$

$$2 \qquad 0$$

$$= -g^2 \,\mathrm{Tr}(t^a t^b) \frac{i}{(4\pi)^{d/2}} \int_0^1 \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \; 8\alpha(1-\alpha) \; (p^2 \eta^{\mu\nu} - p^\mu p^\nu)$$

↑ transverse ✓.

$$= -g^2 \,\mathrm{Tr}(t^a t^b) \frac{i}{(4\pi)^{d/2}} \, 8\Gamma(2-d/2) \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{(m^2-\alpha(1-\alpha)p^2)^{2-d/2}} (p^2 \eta^{\mu\nu} - p^\mu p^\nu)$$

divergent piece

$d = 4-\varepsilon$ $\qquad \Gamma(2-2+\varepsilon/2) = \Gamma(\varepsilon/2)$

replace $d \to 4$

$$J_{div} = -g^2 \,\mathrm{Tr}(t^a t^b) \frac{i}{16\pi^2} \, 8\Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 d\alpha \, \alpha(1-\alpha) \; (p^2 \eta^{\mu\nu} - p^\mu p^\nu)$$

$\alpha - \alpha^2$

$\left.\frac{\alpha^2}{2} - \frac{\alpha^3}{3}\right|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$$= -g^2 \,\mathrm{Tr}(t^a t^b) \frac{i}{16\pi^2} \frac{2}{\varepsilon} \frac{8}{6} (p^2 \eta^{\mu\nu} - p^\mu p^\nu)$$

$$= -\frac{i}{\varepsilon} g^2 \,\underbrace{\mathrm{Tr}(t^a t^b)}_{\frac{1}{2}\delta^{ab}} \frac{1}{6\pi^2} (p^2 \eta^{\mu\nu} - p^\mu p^\nu)$$

if many fermions.

$$\Big)_{div} = -\frac{i}{\varepsilon} g^2 \, \frac{1}{2} \delta^{ab} \frac{1}{6\pi^2} \left(p^2 \eta^{\mu\nu} - p^\mu p^\nu\right) \cdot N_f$$

$$\frac{1}{\varepsilon} \, i g^2 \, \frac{5}{24\pi^2} N \delta^{ab} \left(\eta_{\mu\nu} p^2 - p_\mu p_\nu\right)$$

$$\frac{ig^2}{\varepsilon} \delta^{ab} \left(\eta_{\mu\nu} p^2 - p_\mu p_\nu\right) \frac{1}{6\pi^2} \left(\frac{5}{4} N - \frac{1}{2} N_f\right)$$

$$\frac{5}{4} \left(N - \frac{2}{5} N_f\right)$$

$$\delta_3 = \frac{1}{\varepsilon} \frac{5}{24\pi^2} g^2 \left(N - \frac{2}{5} N_f\right)$$

$$g_0 = g\left(1 - \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} N \left(1 + 1 + \cancel{8}\frac{5}{3}\left(\cancel{N} - \frac{2}{5} \frac{N_f}{N}\right)\right)\right)$$

$$\frac{11}{3} - \frac{2}{3} \frac{N_f}{N}$$

$$g_0 = g\left(1 - \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} N_f\right)\right)$$

$$\beta(g_R) = -\frac{g_R^3}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} N_f\right) = -\frac{g_R^3}{48\pi^2} \left(11 N - 2 N_f\right) \checkmark$$