Path-integral quantization of gauge theories

Initial comments (/problems)

We need

•) Unitarity: positive definite norm states → probability interpretation

unitary S-matrix.

$\epsilon^{\mu}_{(\sigma)}$ polarization vector. For (manifest) Lorenz inv. we keep all ($\sigma = 0,1,2,3$) and not just the physical ones.

Then we can have op. $a^{\dagger}_{\vec{k},\sigma}$ and expect.

$$[a_{\vec{k},\sigma_1}, a^{\dagger}_{\vec{k}',\sigma_2}] = -\delta^{(3)}(\vec{k}-\vec{k}')\, \epsilon^{\mu}_{\sigma_1} \epsilon_{\mu\sigma_2}$$

For space like polarizations this gives $[a_{k}, a^{\dagger}_{k'}] = \delta(k-k')$ but for a temporal polarization $\epsilon = (1,0,0,0)$ it gives

$[a_0, a_0^{\dagger}] = -1$ (ignore momentum here).

But $|1\rangle = a_0^{\dagger}|0\rangle$; $\langle 1|1\rangle = \langle 0| a_0 a_0^{\dagger} |0\rangle =$

$$= \langle 0| [a_0, a_0^{\dagger}] + a_0^{\dagger} \underbrace{a_0 |0\rangle}_{0} = -\langle 0|0\rangle$$

$$= -1$$

negative norm state!!

We have to eliminate them from the spectrum and make sure they are not produced

If we scatter physical polarizations no unphysical one is produced (If we include an unphysical one then it does not matter).

·) Global symmetries are preserved (including Poincare invariance).

Gauge invariance is a way to make Lorentz symmetry manifest but it is not physical. It relates different descriptions of the same physical situation. A global symmetry relates different but equivalent physical situation (e.g. translation inv. : the result of an experiment does not depend on position but position labels different physical situations).

◦) Renormalizability (gauge invariance also helps with this).

Problems

.) we mentioned negative norm states.

.) constraints.

$U(1)$ gauge theory

$$S = \frac{1}{4g^2}\int d^4x\, (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \frac{1}{2g^2}\int d^4x\, \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right)$$

there is no $\partial_0 A_0 \Rightarrow \Pi_0 = 0 \quad \left(= \frac{\delta h}{\delta \dot{A}_0}\right)$

the equation for A_0 is a constraint.

$$D_\mu F^{\mu 0} = J^0 \Rightarrow \quad D_i F^{i0} = J^0 \qquad \text{(Gauss law).}$$

no time derivatives
constraints values of the field. (Normally you give initial values and the eqns. constrain the evolution in time)

.) In the path integral approach one might try

$$\int \mathcal{D}A_\mu\, e^{iS[A_\mu]}$$

but this integrates over redundant configurations.

As usual one can use path integrals or canonical quantization.

Redundancy of configurations also shows up as a quadratic operator in the action that is not invertible. If you change A_μ by a gauge transformation the action does not change.

$$S^{(2)} = -\frac{1}{4}\int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

(for $U(1)$ or sum over A_μ^{a}, $a=1\ldots N^2-1$ for $SU(N)$)

$$= -\frac{1}{2}\int d^4x \, (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$$

$$= \frac{1}{2}\int d^4x \, A_\mu \underbrace{(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)}_{\text{quadratic operator}} A_\nu$$

For a gaussian integral we have to invert

$$R_{\mu\nu} = \eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu$$

$$(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\, G_\nu{}^{\lambda}(x-y) = \eta^{\mu\lambda}\, \delta^{(4)}(x-y)$$

In Fourier space $\Delta_\nu{}^\lambda(k) = \int \frac{d^4x}{(2\pi)^4} e^{ikx}\, G_\nu{}^\lambda(x)$

this gives

$$(-k^2\eta^{\mu\nu} + k^\mu k^\nu)\Delta_\nu{}^\lambda = \eta^{\mu\lambda}$$

But $(-k^2\eta^{\mu\nu} + k^\mu k^\nu)\,k_\nu = -k^2 k^\mu + k^2 k^\mu = 0$

zero eigenvalue → not invertible.

we can try $\underbrace{\Delta_\nu{}^\lambda = a\,\delta_\nu{}^\lambda + b\,k_\nu k^\lambda}_{\text{functions of } k^2}$

$$(-k^2\eta^{\mu\nu} + k^\mu k^\nu)(a\,\delta_\nu{}^\lambda + b\,k_\nu k^\lambda) =$$

$$= -a k^2 \eta^{\mu\lambda} \cancel{- b k^2 k^\mu k^\lambda} + a\,k^\mu k^\lambda + \cancel{b k^2 k^\mu k^\lambda} = \eta^{\mu\lambda}$$

$a = -1/k^2$ but then we get $\underbrace{-\frac{1}{k^2}k^\mu k^\lambda}_{\text{we cannot avoid this.}} + \eta^{\mu\lambda}$

⇒ We have to fix the gauge.

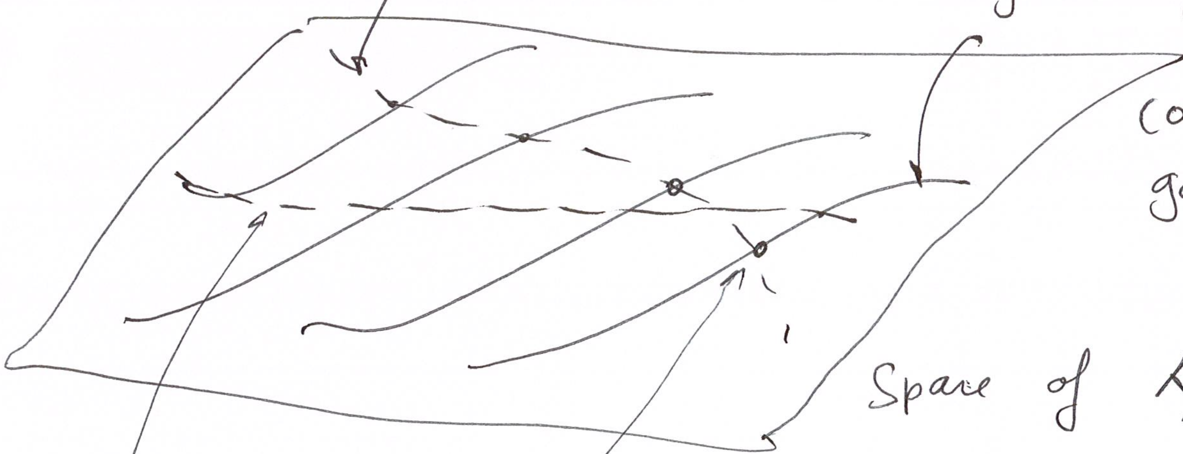
Solves redundancy and then ~~an~~ quadratic operator should be invertible. But we need to choose an appropriate gauge.

Consider

$F[A_\mu] = 0$ gauge condition. $F[A_\mu]$ should <u>not</u> be gauge invariant for $SU(N)$ $F^a[A_\mu^b] = 0$ $a = 1 \ldots N^2-1$ 1 cond. for each gauge field.

However we want to do this such that the answer does not depend on the gauge fixing.

<u>Idea</u>



with the appropriate Jacobian we can separate the integration over the orbit ($\equiv$ Volume of the gauge group) and remove it.

•) Assumption: For every configuration of the gauge field $A^a_\mu(x)$ there is one and only one gauge transformation $U(x)$ such that for

$$\tilde{A}_\mu = U A_\mu U^\dagger - i\,\partial_\mu U U^\dagger \quad \text{one has} \quad F[\tilde{A}_\mu]=0$$

That is the gauge condition picks one and only one representative of each orbit or class.

If we write $U(x) = e^{-i\theta_a(x) T^a}$

then

$$\int \mathcal{D}\theta(x)\ \delta(F^a(\overset{U A_\mu U^{-1} - \frac{i}{g}\partial_\mu U U^\dagger}{A^\theta_\mu})) = \int \mathcal{D}\theta(x) \frac{\delta(\theta - \theta_A(x))}{\left|\frac{\delta F^a}{\delta\theta^b}\right|_{\theta=\theta_A}} = \frac{1}{\left|\frac{\delta F^a}{\delta\theta^b}\right|_{\theta=\theta_A}}$$

θ_A is the gauge transf. that takes $A_\mu \to \hat{A}_\mu \ /\ F^a(\hat{A}_\mu) = 0$

$$\left|\frac{\delta F^a(A^\theta)}{\delta\theta^b}\right|_{\theta=\theta_A} = \left|\frac{\delta F^a(\tilde{A}^\theta)}{\delta\theta^b}\right|_{\theta=0}$$

: gauge inv. since it depends only on $\hat{A}_\mu$

$$\frac{\delta F^a(A^\theta)}{\delta\theta^b} = M_{ab}(x,y)$$

e.g. $F^a(A) = \partial^\mu A^a_\mu$

$$F^a(A^\theta) = \partial^\mu\left(A^a_\mu - \partial_\mu\theta^a - ig f^{abc}\theta^b A^c_\mu\right)$$

$$= \partial^\mu A^a_\mu - \partial^2\theta^a - ig f^{abc}\partial^\mu\theta^b A^c_\mu - ig f^{abc}\theta^b\partial^\mu A^c_\mu$$

$$\frac{\delta F^a}{\delta\theta^b} = -\partial^2\delta(x-y)\,\delta^{ab} - ig f^{abc}A^c_\mu\partial^\mu\delta(x-y) - ig f^{abc}\underbrace{\partial^\mu A^c_\mu}_{0 \text{ for } \theta=0}$$

$$= -\delta^{ab}\partial^2\delta(x-y) - g f^{abc}A^c_\mu\partial^\mu\delta(x-y)$$

$$\det M_{ab}(x,y) = \int \mathcal{D}\bar{c}^a \mathcal{D}c^b \; e^{-\int \bar{c}^a \partial^2 c^a + g f^{abc} \bar{c}^a A_\mu^c \partial^\mu c^b}$$

$$= \int \mathcal{D}\bar{c}^a \mathcal{D}c^b \; e^{\int \partial_\mu \bar{c}^a (\partial^\mu c^a - g f^{abc} A_\mu^c c^b)}$$

$$= \int \mathcal{D}\bar{c}^a \mathcal{D}c^b \; e^{\int \partial_\mu \bar{c}^a D^\mu c^a}$$

$D_\mu c^a = \partial_\mu c^a - g f^{abc} A_\mu^c c^b$

$$\int \mathcal{D}A_\mu \; \Delta_{FP}(\tilde{A}) \int \mathcal{D}\theta \, \delta(\mathcal{F}^a(A_\mu^\theta)) \; e^{iS[A_\mu]}$$

$$\int \mathcal{D}\theta \int \mathcal{D}A_\mu \; \Delta_{FP}(\hat{A}) \; \delta(\mathcal{F}^a(A_\mu^\theta)) \; e^{iS[A_\mu]}$$

change of variables $A_\mu^\theta \to A_\mu$ $\qquad S[A_\mu] = S[A_\mu^\theta]$

$$\underbrace{\int \mathcal{D}\theta}_{\text{volume of gauge group.}} \int \mathcal{D}A_\mu \; \Delta_{FP}(A_\mu) \; \delta(\mathcal{F}^a(A_\mu)) \; e^{iS[A_\mu]}$$

$$Z = \int \mathcal{D}A_\mu \; \Delta_{FP}(A_\mu) \; \delta(\mathcal{F}^a(A_\mu)) \; e^{iS[A_\mu]}$$

Now we put a source:

$$Z[J] = \int \mathcal{D}A_\mu \; \Delta_{FP}(A_\mu) \; \delta(F^a(A_\mu)) \, e^{iS[A_\mu] - i\int J_\mu A^{\mu a}}$$

It is independent of F^a gauge fixing.

$\Rightarrow$ Average over gauge fixings.

$$\int \mathcal{D}B^a(x) \; e^{-\frac{i}{2\xi}\int (B^a(x))^2} \; \delta(F^a(A_\mu) - B^a(x))$$

$$= e^{-\frac{i}{2\xi}\int d^4x \, (F^a(A_\mu))^2}$$

For a covariant gauge $\partial_\mu A^{\mu a} = 0$

we get finally

$$Z[J] = \int \mathcal{D}\bar{c}\,\mathcal{D}c \int \mathcal{D}A_\mu \; e^{i\int \partial_\mu \bar{c}^a D^\mu c^a - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 - \frac{1}{2}\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}} \times$$

$$\times e^{i\int (J_\mu(x) A^\mu(x) + \bar{\eta}^a c^a + \eta^a \bar{c}^a)}$$

Generating functional allows computation of correlators (Green functions) including for ghosts.

Comments:

- ghosts $\bar{c}^a, c^a$ are spin 0 fermions. Wrong spin-statistics. but they are not physical. If one scatters physical fields only physical fields are produced → no ghosts or unphysical polarizations → BRST symmetry (see later).
- $\bar{c} = c^+$ for hermiticity of $\mathcal{L}$. One can rescale ghosts, change sign in front, etc.
- the gauge field now has an invertible quadratic operator since the full action is not gauge invariant anymore.

Let's check.

$$S_0^{(2)} = \int d^4x \left[-\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 \right.$$

$$\left. + \bar{c}^a \partial^2 c^a \right]$$

gauge field. → $-\frac{1}{2} \partial_\mu A_\nu^a \partial^\mu A^{a\nu} + \frac{1}{2} \partial_\mu A_\nu^a \partial^\nu A^{a\mu}$

$$S^{(2)} = \int d^4x \; \frac{1}{2} A_\nu^a (\eta^{\nu\mu} \partial^2 - \partial^\mu \partial^\nu) A_\mu + \frac{1}{2\xi} A_\mu^a \partial^\mu \partial^\nu A_\nu^a$$

$$= \int d^4x \; \frac{1}{2} A_\nu^a \left(\eta^{\mu\nu} \partial^2 - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu\right) A_\mu^a$$

Simplest choice $\xi = 1$ (Feynman gauge)

then $S^{(2)} = \frac{1}{2}\int d^4x\, A^a_\nu\, \eta^{\nu\mu}\, \partial^2 A^a_\nu$

It is like 4 scalar fields but A^a_0 has the wrong sign. It still leads to negative norm states.

the idea is that unphysical polarizations and ghosts can be produce inside a diagram by quantum fluctuations but they cancel each other. (Fermion loops have a $-$ sign).

Propagator for general ξ: (We are going to use $\xi = 1$ in actual calculations).

$$\left(\eta^{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)\Delta_{\nu\alpha}(x-y) = \delta^\mu_\alpha\, \delta^{(4)}(x-y)$$

$$\left(-\eta^{\mu\nu}k^2 + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right)\Delta_{\nu\alpha}(k) = \delta^\mu_\alpha$$

$$\left(-\eta^{\mu\nu}k^2 + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right)\left(a\,\eta_{\nu\alpha} + b\,k_\nu k_\alpha\right) = \delta^\mu_\alpha$$

$$-ak^2\,\delta^\mu_\alpha \cancel{-bk^2 k^\mu k_\alpha} + a\left(1-\frac{1}{\xi}\right)k^\mu k_\alpha + b\left(\cancel{1}-\frac{1}{\xi}\right)k^2 k^\mu k_\alpha = \delta^\mu_\alpha$$

$a = -1/k^2$ $\qquad a\left(1-\frac{1}{\xi}\right) - \frac{b}{\xi}k^2 = 0 \qquad b = \frac{a}{k^2}(\xi-1) = -\frac{1}{k^4}(\xi-1)$

$$\Delta_{\nu\alpha}(k) = -\frac{1}{k^2}\eta_{\nu\alpha} + \frac{(1-\xi)}{k^4}(k_\nu k_\alpha)$$

$$= -\frac{1}{k^2}\left(\eta_{\nu\alpha} - (1-\xi)\frac{k_\nu k_\alpha}{k^2}\right)$$

$$i\Delta^{ab}_{\nu\alpha} = -\frac{i}{k^2+i\varepsilon}\left(\eta_{\nu\alpha} - (1-\xi)\frac{k_\nu k_\alpha}{k^2}\right)\delta^{ab}$$ including gauge indices.

$k \to \infty \qquad \Delta \sim 1/k^2$

$\boxed{\xi = 1}$ $$i\Delta^{ab}_{\nu\alpha}(k) = -\frac{i}{k^2+i\varepsilon}\eta_{\nu\alpha}\,\delta^{ab}$$

same sign for α = spatial = 1, 2, 3
wrong sign for $\underline{\alpha = 0}$.

Recall a scalar $$i\Delta = \frac{i}{k^2-m^2+i\varepsilon}$$

ghost. propagator

$$\partial^2\Delta_c = \delta^{(4)}(x-y)$$

$$-k^2\Delta = 1 \qquad \Delta = -\frac{1}{k^2}$$

$$\boxed{i\Delta^{ab}_{\text{ghost}} = -\frac{i}{k^2+i\varepsilon}\delta^{ab}}$$

Spontaneous symmetry breaking.

U(1) abelian Higgs model.

$$\mathcal{L} = (D_\mu\phi)^\dagger (D^\mu\phi) + \mu^2 \phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu\phi = (\partial_\mu - igA_\mu)\phi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \qquad \mu^2 < 0$$

$$v = \left(-\frac{\mu^2}{\lambda}\right)^{1/2} \qquad |\phi| = v/\sqrt{2}.$$

Instead of the previous (unitary) parameterization we write

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \langle 0|\phi_1|0\rangle = v \qquad \langle 0|\phi_2|0\rangle = 0$$

$$\phi_1 = v + \tilde{\phi}_1$$

$$D_\mu\phi = \frac{1}{\sqrt{2}}(\partial_\mu - igA_\mu)(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}}\partial_\mu\phi_1 + \frac{i}{\sqrt{2}}\partial_\mu\phi_2 - \frac{ig}{\sqrt{2}}A_\mu\phi_1 + \frac{g}{\sqrt{2}}A_\mu\phi_2$$

$$= \left(\frac{1}{\sqrt{2}}\partial_\mu\tilde{\phi}_1 + \frac{g}{\sqrt{2}}A_\mu\phi_2\right) + i\left(\frac{1}{\sqrt{2}}\partial_\mu\phi_2 - \frac{g}{\sqrt{2}}A_\mu v - \frac{g}{\sqrt{2}}A_\mu\tilde{\phi}_1\right)$$

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu\tilde{\phi}_1)^2 + \frac{g^2}{2}A_\mu A^\mu\phi_2^2 + gA_\mu\phi_2\partial_\mu\tilde{\phi}_1$$

$$+\frac{1}{2}\left[(\partial_\mu\phi_2)^2 + g^2v^2A_\mu A^\mu + g^2\tilde{\phi}_1^2A_\mu A^\mu\right]$$

$$+\frac{2}{2}\left[-gA_\mu\partial^\mu\phi_2\, v - gA_\mu\tilde{\phi}_1\partial_\mu\phi_2 + g^2A_\mu A^\mu v\tilde{\phi}_1\right]$$

Potential

$$\mu^2 = -\lambda v^2$$

$$V(\phi) = -\frac{\lambda v^2}{2}\left((v+\tilde{\phi}_1)^2 + \phi_2^2\right) + \frac{\lambda}{4}\left((v+\tilde{\phi}_1)^2 + \phi_2^2\right)^2$$

$$= -\frac{\lambda v^2}{2}\left(v^2 + 2v\tilde{\phi}_1 + \tilde{\phi}_1^2 + \phi_2^2\right) + \frac{\lambda}{4}\left(v^2 + 2v\tilde{\phi}_1 + \tilde{\phi}_1^2 + \phi_2^2\right)^2$$

$$= -\frac{\lambda v^4}{2} - \lambda v^3\tilde{\phi}_1 - \frac{\lambda v^2}{2}\tilde{\phi}_1^2 - \frac{\lambda v^2}{2}\phi_2^2 +$$

$$+ \frac{\lambda}{4}\left(v^4 + 4v^2\tilde{\phi}_1^2 + \tilde{\phi}_1^4 + \phi_2^4 + 4v^3\tilde{\phi}_1 + 2v^2\tilde{\phi}_1^2 + 2v^2\phi_2^2 +\right.$$

$$\left. + 4v\tilde{\phi}_1^3 + 4v\tilde{\phi}_1\phi_2^2 + 2\tilde{\phi}_1^2\phi_2^2\right)$$

$$= -\frac{\lambda v^4}{2} \cancel{- \lambda v^3\tilde{\phi}_1} \cancel{- \frac{\lambda v^2}{2}\tilde{\phi}_1^2} \cancel{- \frac{\lambda v^2}{2}\phi_2^2} + \frac{\lambda v^4}{4} + \lambda v^2\tilde{\phi}_1^2$$

$$+ \frac{\lambda}{4}\tilde{\phi}_1^4 + \frac{\lambda}{4}\phi_2^4 \cancel{+ \lambda v^3\tilde{\phi}_1} \cancel{+ \frac{\lambda}{2}v^2\tilde{\phi}_1^2} \cancel{+ \frac{\lambda}{2}v^2\phi_2^2} +$$

$$+ \lambda v\tilde{\phi}_1^3 + \lambda v\tilde{\phi}_1\phi_2^2 + \frac{\lambda}{2}\tilde{\phi}_1^2\phi_2^2$$

$$= -\frac{\lambda v^4}{4} + \lambda v^2\tilde{\phi}_1^2 + \lambda v\tilde{\phi}_1^3 + \lambda v\tilde{\phi}_1\phi_2^2 +$$

$$+ \frac{\lambda}{4}\left(\tilde{\phi}_1^2 + \phi_2^2\right)^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi}_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{1}{2} g^2 v^2 A_\mu A^\mu + g A_\mu \phi_2 \partial^\mu \tilde{\phi}_1$$
$$+ \frac{1}{2} A_\mu A^\mu \phi_2^2 + \frac{1}{2} g^2 \tilde{\phi}_1^2 A_\mu A^\mu$$
$$- g v A_\mu \partial^\mu \phi_2 - g A_\mu \tilde{\phi}_1 \partial^\mu \phi_2 + g^2 v A_\mu A^\mu \tilde{\phi}_1$$
$$+ \frac{\lambda v^4}{4} - \lambda v^2 \tilde{\phi}_1^2 - \lambda v \tilde{\phi}_1^3 - \lambda v \tilde{\phi}_1 \phi_2^2$$
$$- \frac{\lambda}{4}(\tilde{\phi}_1^2 + \phi_2^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}(\partial_\mu \tilde{\phi}_1)^2 - \frac{1}{2}(2\lambda v^2)\tilde{\phi}_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2$$
$$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} g^2 v^2 A_\mu A^\mu - \underbrace{g v A_\mu \partial^\mu \phi_2}_{\text{mixing term}}$$

Instead of unitary gauge we use R_ξ gauge.

$$F = (\partial^\mu A_\mu + \xi M \phi_2)$$

we need $\frac{\delta F}{\delta \theta}$ inf. gauge transf.

$$\begin{cases} A_\mu \to A_\mu - \frac{1}{g}\partial_\mu \theta \\ \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \to \frac{e^{-i\theta}}{\sqrt{2}}(\phi_1 + i\phi_2) \end{cases}$$

$$\frac{1}{\sqrt{2}} e^{-i\theta}(\phi_1 + i\phi_2) \simeq \frac{(1-i\theta)}{\sqrt{2}}(\phi_1 + i\phi_2) = \frac{\phi_1 + i\phi_2}{\sqrt{2}} + \frac{\theta}{\sqrt{2}}(-i\phi_1 + \phi_2)$$

$$\delta\phi_1 = \theta\phi_2 \qquad \delta\phi_2 = -\theta\phi_1 = -\theta(v+\tilde{\phi}_1)$$

$$\delta\phi_1 = \delta\tilde{\phi}_1$$

$$\frac{\delta G(x)}{\delta\theta(y)} = -\frac{1}{g}\,\partial^2\delta(x-y) - \xi M\,(\tilde{\phi}_1+v)\,\delta(x-y)$$

ghost term : $\quad -\frac{1}{g}\,\bar{c}\,\partial^2 c - \xi M \tilde{\phi}_1\,\bar{c}c - \xi M v\,\bar{c}c$

gauge fixing :

$$-\frac{1}{2\xi}\left(\partial^\mu A_\mu + \xi M\phi_2\right)^2 = -\frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2 - \frac{1}{\cancel{\xi}}\,\partial^\mu A_\mu\, M\cancel{\xi}\,\phi_2$$
$$-\frac{1}{2\cancel{\xi}}\,\xi^{\cancel{2}} M^2\phi_2^{\,2}$$

$$\mathcal{L}^{(2)} = -\frac{1}{g}\,\bar{c}\,\partial^2 c - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}g^2v^2 A_\mu A^\mu$$
$$+\frac{1}{2}(\partial_\mu\phi_1)^2 - \frac{1}{2}(2\lambda v^2)\,\tilde{\phi}_1^{\,2} + \frac{1}{2}(\partial_\mu\phi_2)^2$$
$$\cancel{-gv A_\mu\partial^\mu\phi_2} + \cancel{M\partial^\mu\phi_2 A_\mu} - \frac{1}{2}\xi M^2\phi_2^{\,2} - \xi M v\,\bar{c}c$$

$$M = gv$$

$$M_A = gv \qquad m^2_{\tilde{\phi}_1} = 2\lambda v^2 \qquad m^2_{\phi_2} = \xi M^2 \qquad m_c^2 = \xi g^2 v^2$$

$\xi = 1$ is like 4 scalar of mass M (although A_0 has a $-$ sign)

$\boxed{\xi = 1}$

$$\mathcal{L}^{(2)} = \frac{1}{g}\left(\partial\bar{c}\,\partial c - \xi g^2 v^2 \bar{c} c\right) - \frac{1}{2}\partial^\mu A_\nu^a\,\partial_\mu A^{a\nu} +$$

$$+ \frac{1}{2} g^2 v^2 A_\mu A^\mu + \frac{1}{2}(\partial_\mu \tilde{\phi}_1)^2 - \frac{1}{2}(2\lambda v^2)\,\tilde{\phi}_1^2 +$$

$$+ \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}\xi g^2 v^2 \phi_2^2$$

$A^a_\nu \to M_A = g v$ $\leftarrow$ propagator $\sim 1/k^2$, $k \to \infty$.

$\phi_2 \to M_2 = \sqrt{\xi}\, g v$

$\phi_1 \to m = \sqrt{2\lambda v^2}$

$c \to m_c = \sqrt{\xi}\, g v$

BRST symmetry

U(1) case

$$\mathcal{L} = \underbrace{-\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\not{D}-m)\psi}_{\text{is invariant}} + \bar{c}\,\partial^2 c - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$

$$\left.\begin{aligned} \delta A_\mu &= \zeta\,\partial_\mu c \\ \delta\psi &= -i\zeta c\psi \end{aligned}\right\} \quad \text{like a gauge transf.}$$

ζ = fermionic parameter.

we still have

$$\bar{c}\,\partial^2 c - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$$

$$\delta\bar{c}\,\partial^2 c + \bar{c}\,\partial^2\delta c - \frac{1}{2\xi}\, 2\,\partial^\mu A_\mu\, \zeta\,\partial^2 c = 0$$

we also want $\delta^2 = 0 \Rightarrow \delta c = 0 \qquad \delta^2 A_\mu = \partial_\mu \delta c.$

$$\left(\delta\bar{c} - \frac{\zeta}{\xi}\partial^\mu A_\mu\right)\partial^2 c = 0$$

$$\delta\bar{c} = \frac{1}{\xi}\,\zeta\,\partial^\mu A_\mu$$

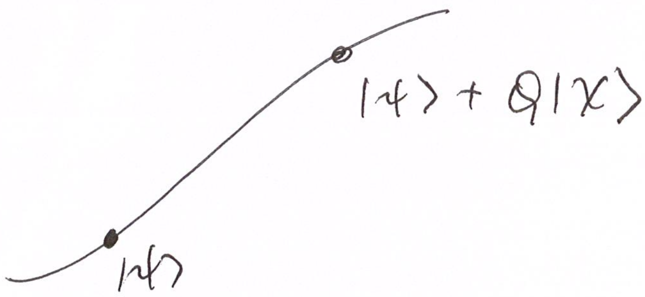
$\delta A_\mu = \partial_\mu c \qquad \delta\psi = -ic\psi \qquad \delta\bar{c} = \frac{1}{\xi}\zeta\,\partial^\mu A_\mu \qquad \delta c = 0$

$$\delta\delta A_\mu = \partial_\mu\delta c = 0$$

$$\delta\delta\psi = +ic\,\delta\psi = \underbrace{cc}_{0 \text{ by antisymmetry}}\psi$$

$$\delta\delta\bar{c} = \frac{1}{\xi}\,\partial^2 c \underset{\text{by e.o.m.}}{=0}$$

Q_{BRST} charge $\quad Q = Q^{\dagger} \quad Q^2 = 0$



physical states $\quad Q|\psi_{phys}\rangle = 0$

$$\langle \psi_{phys} | \phi_{phys} \rangle = \langle \psi_{phys} | \left(|\phi_{phys}\rangle + Q|\chi\rangle \right)$$

$$= \langle \psi_{phys} | \phi_{phys} \rangle + \underbrace{\langle \psi_{phys} | Q | \chi \rangle}_{0}$$

$Q|\chi\rangle$ is physical $\quad Q^2|\chi\rangle = 0$

but trivially.

We define equivalent classes $\quad |\psi\rangle \equiv |\psi\rangle + Q|\chi\rangle$

of physical states.

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=0}^{3} \left(a_k^\sigma \epsilon_\mu^\sigma(k)\, e^{-ikx} + (a_k^\sigma)^\dagger (\epsilon_\mu^\sigma)^* e^{ikx} \right)$$

$$\epsilon_\mu^{(1)} = (0, \hat{k}_1) \qquad \hat{k}_1 \cdot k = 0$$

$$\epsilon_\mu^{(2)} = (0, \hat{k}_2)$$

$$\epsilon_\mu^{(+)} = (k_0, \vec{k}) = k_\mu .$$

$$\epsilon_\mu^{(-)} = (k_0, -\vec{k})$$

$$A_\mu |0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=0}^{3} (\epsilon_\mu^{(\sigma)})^* e^{ikx} \, | 1_{k,\sigma} \rangle$$

$$Q A_\mu |0\rangle = \partial_\mu c |0\rangle = \int i k_\mu \, e^{ikx} \, | 1_{c,k} \rangle \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=0}^{3} (\epsilon_\mu^\sigma)^* e^{ikx} \, Q \, | 1_{k,\sigma} \rangle$$

$$\sum_{\sigma=0}^{3} (\epsilon_\mu^\sigma)^* \, Q \, | 1_{k,\sigma} \rangle = i k_\mu \, | 1_{c,k} \rangle$$

$$Q | 1_{k,1} \rangle = 0 \qquad Q | 1_{k,-} \rangle = 0$$

$$Q | 1_{k,2} \rangle = 0 \qquad Q | 1_{k,+} \rangle = i | 1_{c,k} \rangle$$

$|1_{k,+}\rangle$ unphysical.

$$\bar{c}|0\rangle = \int e^{ikx} \frac{d^3x}{(2\pi)^3\sqrt{2\omega_k}} \,|1_{\bar{c},k}\rangle$$

$$Q\bar{c}|0\rangle = \frac{1}{\xi}\,\partial^\mu A_\mu\,|0\rangle = \frac{1}{\xi}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{\sigma=0}^{3} k^\mu \epsilon_\mu^\sigma e^{ikx}|1_{k,\sigma}\rangle$$

$$= \frac{1}{\xi}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(k^\mu \epsilon_{\mu,-}\right) e^{ikx}\,|1_{k,-}\rangle$$

$|1_{k,-}\rangle$ is trivially physical.

$\epsilon_\mu^{(1)}$ and $\epsilon_\mu^{(2)}$ are physical.

BRST symmetry non-Abelian case.

We reintroduce auxiliary field B^a so that $Q^2=0$ without e.o.m.

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D}-m) + \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu + \partial^\mu \bar{c}^a D^{ac}_\mu c^c$$

$$\left.\begin{aligned}\delta A^a_\mu &= \zeta D^{ac}_\mu c^c \\ \delta\psi &= ig\zeta c^a T^a \psi\end{aligned}\right\} \text{"gauge transf" (similar)}$$

$$\delta c^a = -\frac{1}{2} g\zeta f^{abc} c^b c^c$$

$$\delta\bar{c}^a = \xi B^a$$

$$\delta B^a = 0$$

gauge inv. part is OK $\qquad (B^a)^2$ is inv.

we need

$$\delta\left(B^a\partial^\mu A^a_\mu + \underbrace{\cancel{\partial}\,\bar{c}^a\partial^\mu}_{\text{int. by parts}} D^{ac}_\mu c^a\right) = 0$$

$$\cancel{B^a\zeta\partial^\mu D^{ac}_\mu c^c} - \cancel{\zeta B^a\partial^\mu D^{ac}_\mu c^a} - \bar{c}^a\partial^\mu\delta(D^{ac}_\mu c^a) = 0$$

$$D^{ac}_\mu c^c = \partial_\mu c^a + g f^{abc} A^b_\mu c^c$$

$$\delta D^{ac}_\mu c^c = \partial_\mu\left(-\frac{1}{2}g\zeta f^{abc}c^bc^c\right) + g f^{abc}\zeta(\partial_\mu c^b + g f^{bde}A^d_\mu c^e)c^c$$
$$+ g f^{abc} A^b_\mu \left(-\frac{1}{2}\right) g\zeta f^{cef} c^e c^f =$$

$$= -g\int \cancel{f^{abc}\,\partial_\mu c^b c^c} + g\int \cancel{f^{abc}\,\partial_\mu c^b c^c} +$$

$$+ g^2 \int f^{\overset{cf}{abc}} f^{\overset{cb}{bde}} A^{\overset{b}{d}}_\mu c^e c^{\overset{f}{c}} - \frac{1}{2} g^2 \int f^{abc} f^{cef} A^b_\mu c^e c^f$$

$$= g^2 \int \left(f^{acf} f^{cbe} - \frac{1}{2} f^{abc} f^{cef} \right) A^b_\mu c^e c^f$$

$$[T^a, T^b]\, T^c + [T^c, T^a]\, T^b + [T^b, T^c]\, T^a = 0$$

$$f^{abe} f^{ecf} + f^{cae} f^{ebf} + f^{bce} f^{eaf} = 0$$

$$\left(-f^{afc} f^{cbe} - \frac{1}{2} f^{abc} f^{cef} \right) c^e c^f \qquad (a\, b\, c\, f)$$

$$-\frac{1}{2} \left(f^{afc} f^{cbe} - f^{aec} f^{cbf} + f^{abc} f^{cef} \right) c^e c^f$$

$$f^{\overset{abe}{afc}} f^{\overset{ecf}{cbe}} - f^{\overset{afe}{aec}} f^{\overset{ecb}{cbf}} + f^{\overset{ace}{abc}} f^{\overset{efb}{cef}}$$

$$f^{abe} f^{ecf} - f^{afe} f^{ecb} + f^{ace} f^{efb}$$

$$f^{eaf} f^{cbe} \qquad + f^{cae} f^{ebf} = 0 \quad \checkmark$$

$$+ f^{bce} f^{eaf}$$

Nolico $\delta(D^{ac}_\mu c^c) = 0$

$Q^2 = 0$? (Here we remove ξ)

$$\delta^2 A_\mu = 0 \qquad (\delta(D_\mu c) = 0)$$

$$\delta^2 \psi = \delta\,(ig\, c^a T^a \psi) = -ig\, c^a T^a (ig\, c^b T^b \psi) +$$

$$+ ig\left(-\tfrac{1}{2} g f^{abc} c^b c^c\right) T^a \psi$$

$$= g^2\, c^a c^b\, \frac{i}{2} \overset{cab}{f^{abc}}\, T^c \psi - \frac{ig^2}{2} f^{abc} \overset{a}{c^b} \overset{b}{c^c} \overset{c}{T^a} \psi$$

$$= 0 \checkmark$$

$$\delta^2 B^a = 0 \quad ; \quad \delta^2 \bar{c}^a = 0$$

$$\delta^2 c^a = -\tfrac{1}{2} g f^{abc} (\delta c^b c^c - c^b \delta c^c)$$

$$= -\frac{1}{2} g f^{abc} \left(-\frac{1}{2} g f^{bde} c^d c^e c^c + c^b \frac{1}{2} g f^{cde} c^d c^e\right)$$

$$= \cancel{\frac{g^2}{4} f^{abc} \left(f^{bde} - f^{cde}\right)}$$

$$= \frac{1}{4} g^2 \left(f^{abc} f^{bde} c^d c^e c^c - \overset{cb}{f^{abc}} \overset{b}{f^{cde}} c^b \overset{c}{c^d} c^e\right)$$

$$= \frac{1}{4} g^2 \left(f^{abc} f^{bde} - f^{acb} f^{bde} \right) c^c c^d c^e$$

$$= \frac{1}{4} g^2 \left(- f^{acb} f^{bde} - f^{acb} f^{bde} \right) c^c c^d c^e$$

$$= -\frac{g^2}{2} f^{acb} f^{bde} \underbrace{c^c c^d c^e}_{\text{totally antisymmetric.}}$$

$$\begin{array}{ccc} abe \;\; ecf & afe \;\; ebc & ace \;\; efb \end{array}$$

$$= -\frac{g^2}{6} \underbrace{\left(f^{acb} f^{bde} + f^{aeb} f^{bcd} + f^{adb} f^{bec} \right)}_{0} c^c c^d c^e$$

$$\begin{array}{ccc} \checkmark & eaf \;\; bce \;\; \checkmark & cae \;\; ebf \end{array}$$

✓

$$\boxed{Q^2 = 0}$$

Only dim ≤ 4 terms can appear in infinities since all terms in the action are dim ≤ 4. (power counting).

One can use BRST symmetry + ghost # conservation

+ anti-ghost translations to prove counterterms are as in the action