Semidefinite Programming (SDP)

Semi-definite Positive Matrix

$M \succeq 0$ if (M is either hermitian $M = M^{\dagger}$ or symm. $M = M^{t}$)

$$\sum_{i,j} c_i^{*} M_{ij} c_j \geq 0 \quad \forall c_i$$

$\Rightarrow$ • all eigenvalues are positive or 0 $\lambda_i \geq 0$

• $\det(M) \geq 0$ & all leading principle minors ≥ 0

SDP is
$$\begin{cases} \text{minimize} & \sum_{i=1}^{N} c_i x_i \\ \text{subject to} & X = \sum F_i x_i - F_0 \\ & X \succeq 0 \end{cases}$$

Coefficient matrices

Example 1

Given 3 random variables A, B, C

We have the correlation (covariance) matrix

$$\begin{pmatrix} 1 & \rho_{AB} & \rho_{AC} \\ \rho_{AB} & 1 & \rho_{BC} \\ \rho_{AC} & \rho_{BC} & 1 \end{pmatrix} \succeq 0$$

Suppose (by experiments) we find

$$-0.2 \leq \rho_{AB} \leq -0.1$$

$$0.4 \leq \rho_{BC} \leq 0.5$$

We want to find the upper and lower bound of

$$\boxed{\rho_{AC}}$$

How to build in inequalities

- Slack variables

$$\rho_{AB} + s_1 = -0.1$$

$$\rho_{AB} - s_2 = -0.2$$

$$\rho_{BC} + s_3 = 0.5$$

$$\rho_{BC} - s_4 = 0.4$$

- Enlarge the matrix (block diagonal)

$$\begin{pmatrix} 1 & \rho_{AB} & \rho_{AC} & 0 & 0 & 0 & 0 \\ \rho_{AB} & 1 & \rho_{BC} & 0 & 0 & 0 & 0 \\ \rho_{AC} & \rho_{BC} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_4 \end{pmatrix} \succeq 0$$

2d Yang-Mills theory $SU(N)$ $\quad \mathcal{L} = \frac{1}{4g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$

$$Z_{1P} = \int \mathcal{D}U\, e^{\frac{1}{2\lambda}\frac{1}{N}\mathrm{Tr}(U+U^\dagger)} \qquad \boxed{Z_{2D} = Z_{1P}^{N_P}}$$

$$\hat{P}e^{-ig\int_x^{x+a} A\,dy}$$

Where $U =$ 

$$W_n = \frac{1}{N}\langle \mathrm{Tr}(U^n)\rangle$$

$$\boxed{S_{eff} = -\frac{1}{\lambda}W_1 + \sum_{n=1}^{\infty}\frac{W_n^2}{n}}$$

$$Z = e^{-\overbrace{\beta F}^{S_{eff}}}$$

$$W_1 = \begin{cases} \frac{1}{2\lambda} & \lambda \geq 1 \quad \text{S.C.} \\ 1 - \frac{\lambda}{2} & \lambda \leq 1 \quad \text{W.C.} \end{cases}$$

Naively Minimizing S_{eff} gives

$$\frac{\partial S_{eff}}{\partial W_1} = 0 \quad \Rightarrow \quad W_1 = \frac{1}{2\lambda}$$

$$\frac{\partial S_{eff}}{\partial W_n} = 0 \quad \Rightarrow \quad W_n = 0 \quad n > 1$$

S.C. result!

<u>However</u> fails at W.C.

Why? doesn't take into account SU(N) properties that are lost when tracing

<u>Consider</u>

$$A = \sum_{i=0}^{L} c_i U^i$$

$$\Rightarrow \mathrm{Tr}(A^+ A) \geq 0 \quad \text{by definition}$$

$$\frac{1}{N}\mathrm{Tr}(A^+ A) = \cancel{\sum_{i,j} c_j^* \,\mathrm{Tr}\Big[\underbrace{(U^+)^j U^i}_{\rho_{ij}}\Big] c_i} \geq 0$$

$$\frac{1}{N}\mathrm{Tr}(A^+ A) = \sum_{i,j} c_j^* \frac{1}{N}\mathrm{Tr}\left((U^+)^j U^i\right) c_i \geq 0$$

$$\rho = \begin{pmatrix} 1 & W_1 & W_2 & W_3 & \cdots & W_L \\ W_1 & 1 & W_1 & W_2 & \ddots & \vdots \\ W_2 & W_1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & & \\ W_L & & & & & 1 \end{pmatrix}$$

minimize (S_{eff})

s.t. $\rho \succeq 0$

S_{eff} is quadratic and SDP only handles linear constraints ... or does it?

Consider the matrix

$$S = \begin{pmatrix} t & (w_1 - \frac{1}{2\lambda}) & \frac{w_2}{\sqrt{2}} & \frac{w_3}{\sqrt{3}} & \cdots \\ (w_1 - \frac{1}{2\lambda}) & 1 & 0 & 0 & \\ w_2/\sqrt{2} & 0 & 1 & 0 & \\ w_3/\sqrt{3} & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$$

$$\det(S) = t - \frac{1}{4\lambda^2} + \frac{w_1}{\lambda} - \sum_{n=1}^{L} \frac{w_n^2}{n} \geq 0$$

$$\boxed{t \geq S_{eff} + \frac{1}{2\lambda^2}}$$

now minimize t s.t. $S \succeq 0$ & $\rho \succeq 0$

Smallest Eigenvalue

Given some $M \succeq 0$

~~[illegible]~~ define

$M - t\mathbb{1}$ ~~[illegible]~~ as semi definite

maximize (t)

Gamma Matrices in d dimensions using pauli matrices

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

$$\boxed{\begin{gathered}\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \\ SO(1, d-1)\end{gathered}}$$

d = 2

$$\Gamma^0 = i\sigma_2 = \epsilon$$

$$\Gamma^1 = \sigma_1$$

$$\Gamma = i\Gamma^0\Gamma^1 = \sigma_3$$

d = 4

Kronocker Product

$$A \otimes B = \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \\ A_{n1}B & \cdots & A_{nn}B \end{pmatrix}$$

Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

or

$$\gamma^0 = i\sigma_2 \otimes \sigma_0 = \epsilon \otimes \sigma_0$$

$$\gamma^i = \sigma_1 \otimes \sigma_i = \sigma_1 \otimes \sigma_i$$

$$\gamma^0\gamma^0 = \epsilon^2 \otimes \sigma_0^2 = -\mathbb{1}_4$$

$$\Gamma = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= -i\epsilon\sigma_1\sigma_1\sigma_1 \otimes \sigma_0\sigma_1\sigma_2\sigma_3$$

$$= +i^4\,\sigma_3 \otimes \sigma_0 \quad = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$L = \tfrac{1}{2}(1+\Gamma) \qquad R = \tfrac{1}{2}(1-\Gamma)$$

Majorana Basis (Real)

$\epsilon \otimes \sigma_0$	$-$
$\sigma_3 \otimes \sigma_3$	$+$
$\sigma_1 \otimes \sigma_0$	$+$
$\sigma_3 \otimes \sigma_1$	$+$

$$\Gamma = -i\,\epsilon\,\sigma_3\sigma_1\sigma_3 \otimes \sigma_3\sigma_1$$

$$= -i^3\,\sigma_3 \qquad \otimes -i\sigma_2$$

$$= i^4\,\sigma_3 \otimes \sigma_2 = \sigma_3 \otimes \sigma_2$$

$SO(5,5)$

$$\Gamma^{10} = \epsilon \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \qquad -$$

$$\Gamma^{1} = \sigma_3 \otimes \epsilon \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \qquad -$$

$$\Gamma^{2} = \sigma_3 \otimes \sigma_3 \otimes \epsilon \otimes \sigma_0 \otimes \sigma_0 \qquad -$$

$$\Gamma^{3} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \epsilon \otimes \sigma_0 \qquad -$$

$$\Gamma^{4} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \epsilon \qquad -$$

$$\Gamma^{5} = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \qquad +$$

$$\Gamma^{6} = \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \qquad +$$

$$\Gamma^{7} = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \qquad +$$

$$\Gamma^{8} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \qquad +$$

$$\Gamma^{9} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \qquad +$$

$$\lambda = C \bar{\lambda}^T$$

$$\lambda = \pm \Gamma \lambda$$