Separation of variables in 3d Schrödinger eqn.

Spherical coordinates; $V(\vec{r}) = V(|\vec{r}|) = V(r)$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\,\psi = E\psi$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 s^2\theta\, d\varphi^2$$

$$\nabla^2\psi = \frac{1}{\sqrt{g}}\,\partial_\mu\left(g^{\mu\nu}\sqrt{g}\,\partial_\nu\psi\right)$$

$g_{rr} = 1 \quad g_{\theta\theta} = r^2$

$g_{\varphi\varphi} = r^2 s^2\theta$

$g_{r\theta} = g_{r\varphi} = g_{\theta\varphi} = 0$

$$g^{rr} = 1 \quad g^{\theta\theta} = \frac{1}{r^2} \quad g^{\varphi\varphi} = \frac{1}{r^2 s^2\theta}$$

$$g = \det g = r^4 s^2\theta \Rightarrow \sqrt{g} = r^2 s\theta$$

$$\nabla^2\psi = \frac{1}{r^2 \cancel{s\theta}}\,\partial_r\left(r^2 \cancel{s\theta}\,\partial_r\psi\right) + \frac{1}{r^2 s\theta}\,\partial_\theta\left(\frac{\cancel{r^2} s\theta}{\cancel{r^2}}\,\partial_\theta\psi\right) +$$

$$+ \frac{1}{r^2 \cancel{s\theta}}\,\partial_\varphi\left(\frac{\cancel{r^2}\cancel{s\theta}}{\cancel{r^2} s^2\theta}\,\partial_\varphi\psi\right)$$

$$= \frac{1}{r^2}\,\partial_r\left(r^2\partial_r\psi\right) + \frac{1}{r^2 s\theta}\,\partial_\theta\left(s\theta\,\partial_\theta\psi\right) + \frac{1}{r^2 s^2\theta}\,\partial_\varphi^2\psi$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\partial_r(r^2\partial_r\psi)+\frac{1}{r^2 s\theta}\partial_\theta(s\theta\,\partial_\theta\psi)+\frac{1}{r^2 s^2\theta}\partial_\varphi^2\psi\right]+$$

$$+V(r)\,\psi = E\psi$$

Propose: $\Psi = \tilde{R}(r)\,\Theta(\theta)\,\Phi(\varphi)$

$$-\frac{\hbar^2}{2m}\left[\frac{\Theta\Phi}{r^2}\partial_r(r^2\partial_r\tilde{R})+\frac{\tilde{R}\Phi}{r^2 s\theta}\partial_\theta(s\theta\,\partial_\theta\Theta)+\right.$$

$$\left.+\frac{\tilde{R}\Theta}{r^2 s^2\theta}\partial_\varphi^2\Phi\right]+V(r)\,\tilde{R}\Theta\Phi = E\tilde{R}\Theta\Phi$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2\tilde{R}}\partial_r(r^2\partial_r\tilde{R})+\frac{1}{r^2 s\theta\,\Theta}\partial_\theta(s\theta\,\partial_\theta\Theta)+\right.$$

$$\left.+\frac{1}{r^2 s^2\theta\,\Phi}\partial_\varphi^2\Phi\right]+V(r) = E$$

has to be constant.

$$\partial_\varphi^2\Phi = \mu\Phi \quad\rightarrow\quad \Phi = e^{\pm\sqrt{\mu}\,\varphi}$$

but $\varphi: 0\to 2\pi$ Φ should be periodic.

$$\boxed{\Phi = e^{im\varphi}} \quad m\in\mathbb{Z}.$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2\tilde{R}}\partial_r\left(r^2\partial_r\tilde{R}\right)+\frac{1}{r^2}\underbrace{\left\{\frac{\partial_\theta\left(s\theta\,\partial_\theta\Theta\right)}{s\theta\,\Theta}-\frac{m^2}{s^2\theta}\right\}}_{\text{constant}\,=\,\lambda.}\right]+V(r)=E$$

$$\frac{1}{s\theta}\partial_\theta\left(s\theta\,\partial_\theta\Theta\right)-\frac{m^2}{s^2\theta}\Theta-\lambda\Theta=0$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\partial_r\left(r^2\partial_r\tilde{R}\right)+\frac{\lambda}{r^2}\tilde{R}\right]+V(r)\tilde{R}=E\tilde{R}$$

$$\tilde{R}=\frac{1}{r}\chi \qquad \partial_r\tilde{R}=-\frac{1}{r^2}\chi+\frac{1}{r}\partial_r\chi \quad ; \quad r^2\partial_r\tilde{R}=-\chi+r\partial_r\chi$$

$$\partial_r\left(r^2\partial_r\tilde{R}\right)=-\cancel{\partial_r\chi}+\cancel{\partial_r\chi}+r\partial_r^2\chi=r\partial_r^2\chi$$

$$-\frac{\hbar^2}{2m}\frac{1}{r\cancel{r}}\cancel{r}\partial_r^2\chi-\frac{\hbar^2}{2m}\frac{\lambda}{r^2}\frac{1}{r}\chi+\frac{1}{r}V\chi=\frac{E}{r}\chi$$

$$\boxed{-\frac{\hbar^2}{2m}\partial_r^2\chi+\Big(\underbrace{V-\frac{\hbar^2}{2m}\frac{\lambda}{r^2}}_{\text{1 dim effective potential.}}\Big)\chi=E\chi}$$

$r \geq 0$

$$\boxed{\chi(0)=0}$$

$$\frac{1}{s\theta}\partial_\theta\left(s\theta\,\partial_\theta\,\Theta\right)-\frac{m^2}{\cancel{\theta}\,s^2\theta}\,\Theta-\lambda\,\Theta=0$$

$$x=\cos\theta$$

$$\partial_\theta=-s\theta\,\partial_x$$

$$\frac{1}{\cancel{s\theta}}(\cancel{+}\cancel{s\theta})\,\partial_x\left(\cancel{+}s^2\theta\,\partial_x\,\Theta\right)-\frac{m^2}{\cancel{\theta}\,s^2\theta}\,\Theta-\lambda\,\Theta=0$$

$$\partial_x\left((1-x^2)\,\partial_x\,\Theta\right)-\frac{m^2}{s^2\theta}\,\Theta-\lambda\,\Theta=0$$

$$\boxed{\partial_x\left((1-x^2)\,\partial_x\,\Theta\right)-\frac{m^2}{1-x^2}\,\Theta-\lambda\,\Theta=0}$$

$X: -1\to 1$

$X\to\pm 1$ $\quad (1-x^2)\,\partial_x^2\,\Theta-2x\,\partial_x\,\Theta-\frac{m^2}{1-x^2}\,\Theta\approx 0$

$$\Theta\sim(1-x^2)^\alpha$$

$$\partial_x\Theta=-2x\,\alpha\,(1-x^2)^{\alpha-1}$$

$$(1-x^2)\,\partial_x\,\Theta=-2\alpha x\,(1-x^2)^\alpha$$

$$\partial_x\quad=-2\alpha\,(1-x^2)^\alpha+4\alpha^2x^2\,(1-x^2)^{\alpha-1}$$

$$-2\alpha\,(1-x^2)^\alpha+4\alpha^2x^2(1-x^2)^{\alpha-1}-m^2\,(1-x^2)^{\alpha-1}\sim 0$$

$$4\alpha^2=m^2\quad\boxed{\alpha=|m|/2}$$

$$\Theta = (1-x^2)^{|m|/2} \; P(x)$$

$|m| \to m$ for the moment

$$\Theta = (1-x^2)^{m/2} \, P(x)$$

$$\partial_x \Theta = -2x \frac{m}{2} (1-x^2)^{m/2-1} P(x) + (1-x^2)^{m/2} P'(x)$$

$$(1-x^2)\partial_x \Theta = -\cancel{2} x \frac{m}{\cancel{2}} (1-x^2)^{m/2} P(x) + (1-x^2)^{m/2+1} P'(x)$$

$$\partial_x \left[(1-x^2)\partial_x \Theta\right] = -m (1-x^2)^{m/2} P + \cancel{2} x^2 m \frac{m}{\cancel{2}} (1-x^2)^{m/2-1} P(x)$$

$$-mx (1-x^2)^{m/2} P'(x) + \left(\frac{m}{2}+1\right) (1-x^2)^{m/2} P'(x) (-2x)$$

$$+ (1-x^2)^{m/2+1} P''(x)$$

$$= (1-x^2)^{m/2} \left[-mP + \frac{m^2x^2}{1-x^2} P - mxP' + \overset{-2x}{\left(\frac{m}{2}+1\right)} P' + \right.$$

$$\left. + (1-x^2) P'' \right]$$

$$(1-x^2)P'' - mP + \frac{m^2x^2}{1-x^2} P - mxP' - x(+m+2) P' - \frac{m^2}{1-x^2} P - \lambda P = 0$$

$$\boxed{(1-x^2)P'' - 2(m+1)xP' - mP - m^2 P - \lambda P = 0}$$

$$\frac{m^2(x^2-1)}{1-x^2} = -m^2$$

$$\boxed{(1-x^2)P'' - 2(m+1)xP' - (\lambda + m(m+1))P = 0}$$

$$P = \sum_n c_n x^n \quad ; \quad P' = \sum_n n c_n x^{n-1}$$

$$P'' = \sum_n n(n-1) c_n x^{n-2} = \sum_n (n+2)(n+1) c_{n+2} x^n$$

$$\sum_n (n+1)(n+2) c_{n+2} x^n - \sum_n n(n-1) c_n x^n -$$

$$-2(m+1) \sum_n n c_n x^n - (\lambda + m(m+1)) \sum_n c_n x^n = 0$$

$$(n+1)(n+2) c_{n+2} = \left[n(n-1) + 2n(m+1) + \lambda + m(m+1) \right] c_n$$

$$n^2 - n + 2nm + 2n + \lambda + m(m+1)$$

$$(n+m)^2 + n + m + \lambda$$

$$\boxed{c_{n+2} = \frac{(n+m)(n+m+1) + \lambda}{(n+1)(n+2)} c_n}$$

$n \to \infty \qquad c_{n+2} \simeq c_n \qquad c \sum x^n \to$ diverges at $x=1$.

Series terminates if $\quad \lambda = -\ell(\ell+1)$

$$\boxed{\ell = n_{max} + m}$$

$\Rightarrow P$: polynomial of order $n_{max} = \ell - m \geq 0$

$|m| \leq \ell \qquad \boxed{m : -\ell \ldots \ell}$

$$\boxed{\textcircled{4} = (1-x^2)^{|m|/2} \left\{ \boxed{P^{\ell}_{|m|}(x) = (S\Theta)^{|m|} P^{\ell}_{m}(\cos\Theta)} \right.}$$

Associated Legendre polynomials.

$$\boxed{Y_{\ell m}(\theta,\varphi) = A_{\ell m}\, e^{im\phi}\,(S\theta)^{|m|}\, P^{\ell}_{m}(\cos\theta)}$$

spherical harmonics.

Radial potential $V(r)$:

Solve 1-dim eqn.

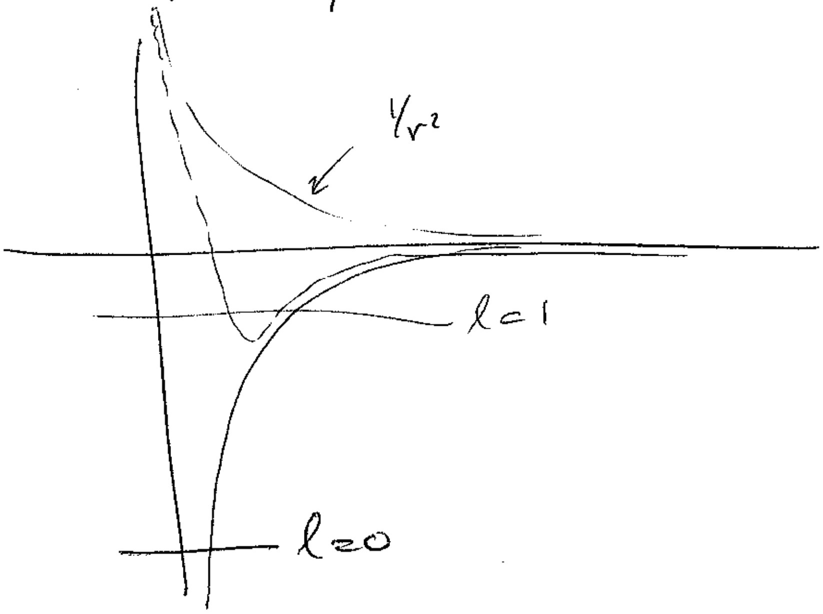
$$\boxed{-\frac{\hbar^2}{2m}\partial_r^2 \chi + \Big(V(r) + \underbrace{\frac{\hbar^2}{2mr^2}\ell(\ell+1)}_{\text{centrifugal barrier.}}\Big)\chi = E\,\chi}$$

$r \geq 0$

$\chi(0) = 0$

$$\psi = \frac{1}{r}\chi(r)\, Y_{\ell m}(\theta,\varphi)$$

Example : hydrogen atom $\quad V(r) = -\frac{e^2}{r}$



$$-\frac{\hbar^2}{2m}\partial_r^2\chi + \left(-\frac{e^2}{r} + \frac{\hbar^2}{2mr^2}\,l(l+1)\right)\chi = E\chi$$

$$-\partial_r^2\chi + \left(-\frac{2me^2}{\hbar^2 r} + \frac{l(l+1)}{r^2}\right)\chi = \frac{2mE}{\hbar^2}\chi$$

$$r = a\xi$$

$$-\partial_\xi^2\chi + a^2\left(-\frac{2me^2}{\hbar^2 a\xi} + \frac{l(l+1)}{a^2\xi^2}\right)\chi = \frac{2mEa^2}{\hbar^2}\chi$$

$$\frac{2me^2a}{\hbar^2} = 1 \qquad \boxed{a = \frac{\hbar^2}{2me^2}} \qquad \varepsilon = \frac{2mE}{\hbar^2}\frac{\hbar^4}{4m^2e^4} = \frac{-E\hbar^2}{2me^4}$$

$$\boxed{E = -\frac{2me^4}{\hbar^2}\varepsilon}$$

$$-\partial_\xi^2\chi + \left(\frac{l(l+1)}{\xi^2} - \frac{1}{\xi}\right)\chi = -\varepsilon\chi$$

bound state

units

$$\frac{\hbar^2}{me^2} = \frac{\hbar^2 c^2}{mc^2 e^2} = \frac{\cancel{MeV^2}\, fm^2}{\cancel{MeV}\, \cancel{MeV\, fm}} = \text{length.}$$

$a \to$ units of length.

$$\frac{me^4}{\hbar^2} = \frac{mc^2 e^4}{\hbar^2 c^2} = \frac{MeV\, \cancel{MeV^2 fm^2}}{\cancel{MeV^2 fm^2}} = MeV \quad \text{energy} \checkmark.$$

$$-\partial_\xi^2 \chi + \left(\frac{\ell(\ell+1)}{\xi^2} - \frac{1}{\xi}\right)\chi = -\varepsilon \chi$$

$\xi \to \infty$ $\quad -\partial_\xi^2 \chi = -\varepsilon \chi \;;\; \chi \sim e^{\pm\sqrt{\varepsilon}\, r}$

$$\chi \sim e^{-\sqrt{\varepsilon}\,\xi} \qquad \xi \to \infty$$

$\xi \to 0 \qquad \chi \sim \xi^\alpha$

$$-\alpha(\alpha-1)\xi^{\alpha-2} + \frac{\ell(\ell+1)}{\xi^2}\xi^\alpha = 0$$

$\alpha(\alpha-1) = \ell(\ell+1)$

$\alpha = \ell+1$ or $\boxed{\alpha = -\ell}$ diverges.

$$\chi = \xi^{\ell+1} e^{-\sqrt{\varepsilon}\,\xi} \cdot f(\xi) \qquad \xi \to 0 \quad \chi \to 0 \checkmark$$

take $\varepsilon = k^2$ $\quad \chi = e^{-k\xi} F(\xi)$

$F(\xi) = \xi^{l+1} f(\xi)$

$\chi' = -k e^{-k\xi} F + e^{-k\xi} F'$

$\chi'' = k^2 e^{-k\xi} F - 2k e^{-k\xi} F' + e^{-k\xi} F''$

$$\cancel{-k^2 F} + 2k F' - F'' + \frac{l(l+1)}{\xi^2} F - \frac{1}{\xi} F = \cancel{-k^2 F}$$

$$F'' - 2kF' - \frac{l(l+1)}{\xi^2} F + \frac{1}{\xi} F = 0$$

$F = \sum_n c_n \xi^n \qquad F' = \sum_n n c_n \xi^{n-1} \qquad F'' = \sum_n n(n-1) c_n \xi^{n-2}$

$$\sum_n (n+1)(n+2) c_{n+2} \xi^n - 2k \sum_n (n+1) c_{n+1} \xi^n -$$

$$- l(l+1) \sum_n c_{n+2} \xi^n + \sum_n c_{n+1} \xi^n = 0$$

$$\left[(n+1)(n+2) - l(l+1) \right] c_{n+2} = \left(-1 + 2k(n+1) \right) c_{n+1}$$

$n \to \infty \qquad n^2 c_{n+2} = 2kn\, c_{n+1} \qquad c_{n+2} \sim \frac{2k}{n} c_n .$

~~Cn~~ series terminates. $\quad 2k(n+1) = 1$

$k = \dfrac{1}{2(n+1)}$ for some n

We also need $n \geq l$ otherwise left hand side vanishes.

Solution is $\chi = e^{-k\xi}\, \xi^{l+1} \cdot P(\xi)$

polynomial,

Laguerre polynomial.

of order $\boxed{n+1}$ $\quad k = \dfrac{1}{2(n+1)}$

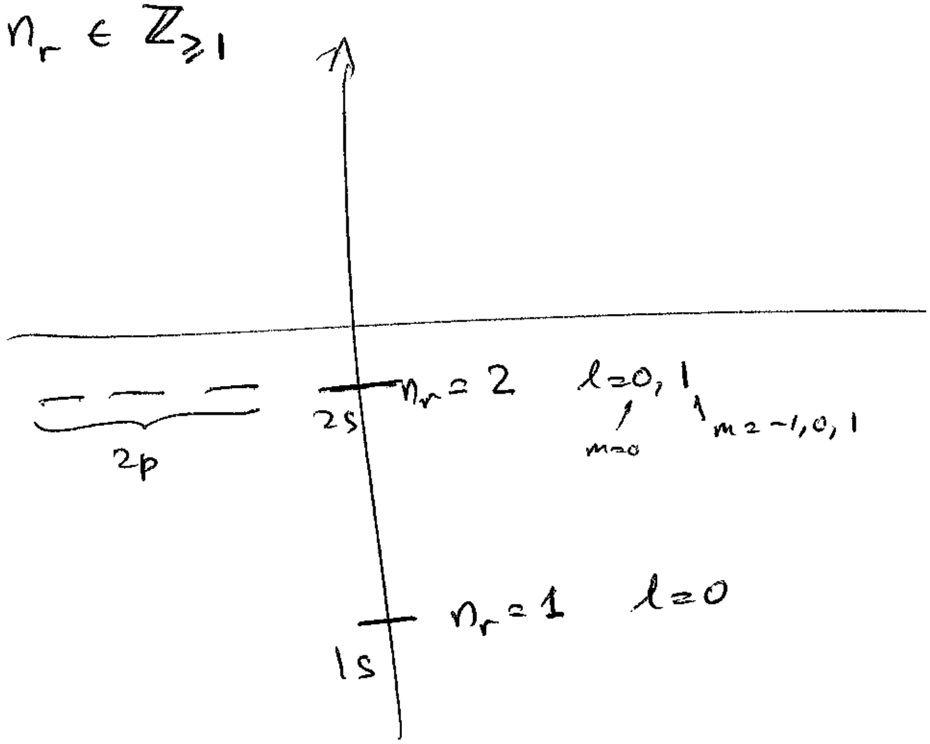
$$\varepsilon = \frac{1}{4(n+1)^2}$$

$$\boxed{E = -\frac{me^4}{2\hbar^2}\frac{1}{n_r^2}}$$

$n_r = 1, 2, \ldots \qquad n_r \in \mathbb{Z}_{\geq 1}$

$$\boxed{l \leq n_r - 1}$$

$$\boxed{|m| \leq l \leq n_r - 1}$$



s p d f
$l = 0$ 1 2 3

Hydrogen atom Algebraic SO(4) method Laplace Runge Lenz vector (1)

$$A_i = \frac{1}{2}\epsilon_{ijk}(p_j L_k + L_k p_j) - me^2\frac{r_i}{r}$$ ← hermitian

$$L_k p_j = p_j L_k + [L_k, p_j] = p_j L_k + i\hbar\,\epsilon_{kjl} p_l$$

$$A_i = \epsilon_{ijk} p_j L_k + \frac{i\hbar}{2}\underbrace{\epsilon_{ijk}}_{\substack{kij\\ -kji}}\underbrace{\epsilon_{kjl}}_{-2\delta_{il}} p_l - me^2\frac{r_i}{r}$$

$$\boxed{A_i = \epsilon_{ijk} p_j L_k - i\hbar p_i - me^2\frac{r_i}{r}}$$

$$[A_i, r_j] = \epsilon_{ij'k'}[p_{j'}L_{k'}, r_j] - i\hbar\underbrace{[p_i, r_j]}_{-i\hbar\delta_{ij}}$$

$$= \epsilon_{ij'k'} p_{j'}\, i\hbar\,\epsilon_{k'jl} r_l + \epsilon_{ij'k'}(-i\hbar)\delta_{j'j} L_{k'} - \hbar^2\delta_{ij}$$

$$\boxed{[A_i, r_j] = i\hbar\,\delta_{ij}(pr) - i\hbar\, p_j r_i - i\hbar\,\epsilon_{ijk} L_k - \hbar^2\delta_{ij}}$$

$$[A_i, p_j] = \epsilon_{ij'k'} p_{j'}[L_{k'}, p_j] - me^2\left[\frac{r_i}{r}, p_j\right]$$

$$= \epsilon_{ij'k'} p_{j'}\, i\hbar\,\epsilon_{k'jl} p_l - me^2\, i\hbar\left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3}\right)$$

$$\boxed{[A_i, p_j] = i\hbar(\delta_{ij}p^2 - p_j p_i) - i\hbar\frac{me^2}{r}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)}$$

$$\boxed{[A_i, L_j] = i\hbar\,\epsilon_{ijk} A_k}$$ A is a vector.

$$\left[A_i, \frac{1}{r}\right] = \epsilon_{ijk}\left[p_j, \frac{1}{r}\right]L_k - i\hbar\left[p_i, \frac{1}{r}\right] =$$

$$= \epsilon_{ijk}\,(+i\hbar)\left(+\frac{r_j}{r^3}\right)L_k + i\hbar\,(+i\hbar)\left(+\frac{r_i}{r^3}\right)$$

$$\boxed{\left[A_i, \frac{1}{r}\right] = i\hbar\,\epsilon_{ijk}\,\frac{r_j}{r^3}L_k + \frac{\hbar^2 r_i}{r^3}}$$

$$[A_i, A_{i'}] = \epsilon_{i'jk}\,[A_i, p_j L_k] - i\hbar\,[A_i, p_{i'}] - me^2\left[A_i, \frac{r_{i'}}{r}\right]$$

$$= \underset{ki'j}{\epsilon_{i'jk}}\;p_j\;i\hbar\underset{kli}{\epsilon_{ikl}}A_l + \epsilon_{i'jk}\Big[i\hbar\,(\delta_{ij}p^2 - p_i p_j) -$$

$$- i\hbar\frac{me^2}{r}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\Big]L_k - i\hbar\,i\hbar\left[\delta_{ii'}p^2 - p_i p_{i'} - \frac{me^2}{r}\left(\delta_{ii'} - \frac{r_i r_{i'}}{r^2}\right)\right]$$

$$-me^2\,r_{i'}\left(i\hbar\epsilon_{ijk}\frac{r_j}{r^3}L_k + \frac{\hbar^2 r_i}{r^3}\right) - me^2\,(i\hbar\,\delta_{ii'}\,(pr) -$$

$$-i\hbar\,p_{i'}r_i - i\hbar\,\epsilon_{ii'k}L_k - \hbar^2\delta_{ii'})\,\frac{1}{r}$$

$$= i\hbar\,p_i A_{i'} - i\hbar\,\delta_{i'i}\,p_j A_j + i\hbar\,\epsilon_{i'ik}\left(p^2 - \frac{me^2}{r}\right)L_k +$$

$$+ \epsilon_{i'jk}\,i\hbar\left(-p_i p_j + \frac{me^2}{r^3}r_i r_j\right)L_k + \hbar^2\left(\delta_{ii'}p^2 - p_i p_{i'} - \frac{me^2}{r}\left(\delta_{ii'} - \frac{r_i r_{i'}}{r^2}\right)\right)$$

$$- i\hbar\,me^2\,\epsilon_{ijk}\frac{r_{i'}r_j}{r^3}L_k - me^2\hbar^2\frac{r_{i'}r_i}{r^3} - i\hbar me^2\,\delta_{ii'}\,(pr)\frac{1}{r} +$$

$$+ me^2 i\hbar\,p_{i'}\frac{r_i}{r} + i\hbar\frac{me^2}{r}\epsilon_{ii'k}L_k + me^2\frac{\hbar^2}{r}\delta_{ii'}$$

$$= -i\hbar\,\epsilon_{ii'k}\left(p^2 - \frac{2me^2}{r}\right)L_k + \cancel{i\hbar\, p_i A_{i'}} - i\hbar\,\delta_{ii'}\,p_j A_j$$

$$-i\hbar\, p_i\left(\cancel{A_{i'}} + i\hbar p_{i'} + me^2\frac{r_{i'}}{r}\right) + \frac{i\hbar m e^2}{r^3}\,\epsilon_{i'jk}\, r_i r_j L_k + \hbar^2\left(\delta_{ii'}p^2 - p_i p_{i'}\right)$$

$$-\frac{me^2}{r}\hbar^2\left(\cancel{\delta_{ii'}} - \cancel{\frac{r_i r_{i'}}{r^2}}\right) - i\hbar m e^2\,\epsilon_{ijk}\frac{r_{i'} r_j}{r^3}L_k - \cancel{\frac{me^2\hbar^2}{r^3} r_i r_{i'}} -$$

$$-i\hbar m e^2\,\delta_{ii'}(pr)\frac{1}{r} + i\hbar m e^2\, p_{i'} r_i\frac{1}{r} + \cancel{\frac{me^2\hbar^2}{r}\delta_{ii'}}$$

$$= -2mi\hbar\,\epsilon_{ii'k}\left(\frac{p^2}{2m} - \frac{e^2}{r}\right)L_k - i\hbar\,\delta_{ii'}\left(\cancel{-i\hbar p^2} - \cancel{me^2 p_j\frac{r_j}{r}}\right) +$$

$$+\cancel{\hbar^2 p_i p_{i'}} - i\hbar m e^2\, p_i\frac{r_{i'}}{r} + \frac{i\hbar m e^2}{r^3}\left(\epsilon_{i'jk} r_i r_j L_k - \epsilon_{ijk} r_{i'} r_j L_k\right) +$$

$$+\cancel{\hbar^2\delta_{ii'}p^2} - \cancel{\hbar^2 p_i p_{i'}} - \cancel{i\hbar m e^2\delta_{ii'}(pr)\frac{1}{r}} + i\hbar m e^2\, p_{i'} r_i\frac{1}{r}$$

$$= -2mi\hbar\,\epsilon_{ii'k}HL_k - i\hbar m e^2\underbrace{\left(p_i r_{i'} - p_{i'} r_i\right)}_{-\epsilon_{ii'k}L_k}\frac{1}{r} + \frac{i\hbar m e^2}{r^3}\left(\epsilon_{i'jk} r_i r_j L_k - \epsilon_{ijk} r_{i'} r_j L_k\right)$$

$$p_i r_j - p_j r_i = -\underbrace{\epsilon_{ijk}\epsilon_{klm}}_{r_i p_j - r_j p_i} r_l p_m = -\epsilon_{ijk}L_k$$

$$\underset{k\neq j}{\epsilon_{ijk} r_j L_k} = \epsilon_{ijk} r_j \epsilon_{klm} r_l p_m = r_j r_i p_j - r^2 p_i = r_i(rp) - r^2 p_i$$

$$= -2mi\hbar\,\epsilon_{ii'k}HL_k + i\frac{\hbar m e^2}{r}\epsilon_{ii'k}L_k + \frac{i\hbar m e^2}{r^3}\Big(\cancel{r_i r_{i'}(rp)} - r_i r^2 p_{i'} -$$

$$-\cancel{r_{i'} r_i (rp)} + r_{i'} r^2 p_i\Big) = -2mi\hbar\,\epsilon_{ii'k}HL_k + i\frac{\hbar m e^2}{r}\epsilon_{ii'k}L_k - \frac{i\hbar m e^2}{r}(r_i p_{i'} - r_{i'} p_i)$$

$$= -2mi\hbar\,\epsilon_{ii'k}HL_k + \cancel{i\frac{\hbar m e^2}{r}\epsilon_{ii'k}L_k} - \cancel{i\frac{\hbar m e^2}{r}\epsilon_{ii'k}L_k}$$

$$\boxed{[A_i, A_j] = -2mi\hbar\,\epsilon_{ijk}HL_k}$$

$$\boxed{\begin{aligned} [L_i, L_j] &= i\hbar\, \varepsilon_{ijk} L_k \\ [L_i, A_j] &= i\hbar\, \varepsilon_{ijk} A_k \\ [A_i, A_j] &= -2m i\hbar\, \varepsilon_{ijk} H L_k \end{aligned}}$$

Also

$$A_i L_i = \varepsilon_{ijk} p_j L_k L_i - i\hbar p_i L_i - me^2 \frac{r_i L_i}{r}$$

$$= \frac{1}{2} \varepsilon_{ijk} p_j\, i\hbar\, \varepsilon_{kil} L_l - i\hbar \overset{0}{\cancel{p_i \varepsilon_{ijk} r_j p_k}} - \frac{me^2}{r} \overset{0}{\cancel{r_i \varepsilon_{ijk} r_j p_k}}$$

$$= \frac{i\hbar}{2} p_j L_j = 0$$

$$A_i A_i = \left(\varepsilon_{ijk} p_j L_k - i\hbar p_i - me^2 \frac{r_i}{r} \right) \left(\varepsilon_{ij'k'} p_{j'} L_{k'} - i\hbar p_i - me^2 \frac{r_i}{r} \right)$$

$$= \varepsilon_{ijk} \varepsilon_{ij'k'} p_j L_k p_{j'} L_{k'} - i\hbar\, \varepsilon_{ijk} p_j L_k p_i - me^2 \varepsilon_{ijk} p_j L_k \frac{r_i}{r}$$

$$- \hbar^2 p^2 + i\hbar me^2 p_i \frac{r_i}{r} - me^2 \varepsilon_{ij'k'} \frac{r_i}{r} p_{j'} L_{k'} + i\hbar me^2 \frac{r_i}{r} p_i + m^2 e^4$$

$$= p_j L_k p_j L_k - p_j \underbrace{L_k p_k}_{0} L_j - i\hbar\, \varepsilon_{ijk} p_j L_k p_i - me^2 \varepsilon_{ijk} p_j L_k \frac{r_i}{r} - \hbar^2 p^2$$

$$+ i\hbar me^2 \left(p_i \frac{r_i}{r} + \frac{r_i}{r} p_i \right) - me^2 \varepsilon_{ijk} \frac{r_i}{r} p_j L_k + m^2 e^4$$

$$= p^2 L^2 + \cancel{[p_j, L_k]\, p_j L_k} + p_j [L_k, p_j] L_k + i\hbar\, \varepsilon_{ijk} p_j [L_k, p_i] - \hbar^2 p^2 + m^2 e^4$$

$$+ i\hbar me^2 \left(p_i \frac{r_i}{r} + \frac{r_i}{r} p_i \right) - me^2 \varepsilon_{ijk} p_j L_k \frac{r_i}{r} - me^2 \varepsilon_{ijk} \frac{r_i}{r} p_j L_k$$

$$= p^2 L^2 + i\hbar \overset{0}{\cancel{\varepsilon_{kjl} p_j p_l}} L_k - \hbar^2 \underset{kij}{\varepsilon_{ijk}} \underset{2\delta_{jl}}{\varepsilon_{kil}} p_j p_l - \hbar^2 p^2 + m^2 e^4 +$$

$$+ i\hbar me^2 \left(p_i \frac{r_i}{r} + \frac{r_i}{r} p_i \right) - me^2 \varepsilon_{ijk} p_j L_k \frac{r_i}{r} - me^2 \varepsilon_{ijk} \frac{r_i}{r} p_j L_k$$

$$= 2mHL^2 + p^2\hbar^2 + m^2e^4 + i\hbar me^2\, i\hbar\left(\frac{\delta_{ii}}{r} - \frac{r_i r_i}{r^3}\right)$$

$$\frac{3}{r} - \frac{1}{r} = 2/r$$

$$= 2mHL^2 + p^2\hbar^2 + m^2e^4 - m\hbar^2e^2\frac{2}{r}$$

$$= 2mHL^2 + \left(p^2 - \frac{2me^2}{r}\right)\hbar^2 + m^2e^4$$

$$= 2mHL^2 + 2mH\hbar^2 + m^2e^4$$

$$\boxed{A_iA_i = 2mH(L^2+\hbar^2) + m^2e^4}$$

$$A_iL_i = \varepsilon_{ijk}\,p_j L_k L_i - i\hbar\,\overset{0}{(p_iL_i)} - me^2\,\overset{0}{\frac{r_iL_i}{r}}$$

$$= \frac{1}{2}\,\underset{kij}{\varepsilon_{ijk}}\,p_j\, i\,\underset{kil}{\varepsilon_{kil}}\,L_l = i\,p_jL_j = 0.$$

$$\boxed{A_iL_i = 0}$$

$$[A_i, H] = \left[A_i, \frac{p^2}{2m} - \frac{e^2}{r}\right] = \frac{1}{2m}\,p_j\,[A_i, p_j] + \frac{1}{2m}[A_i,p_j]p_j - e^2\left[A_i, \frac{1}{r}\right]$$

$$= \frac{1}{2m}\,p_j\, i\hbar\,(\delta_{ij}p^2 - p_jp_i) - \frac{i\hbar}{2m}me^2\,p_j\,\frac{1}{r}\left(\delta_{ij} - \frac{r_ir_j}{r^2}\right) +$$

$$+ \frac{1}{2m}\, i\hbar\,(\delta_{ij}p^2 - p_jp_i)\,p_j - \frac{i\hbar}{2m}me^2\frac{1}{r}\left(\delta_{ij} - \frac{r_ir_j}{r^2}\right)p_j - e^2\, i\hbar\,\varepsilon_{ijk}\,\frac{r_jL_k}{r^3} -$$

$$-e^2\hbar^2 r_i/r^3$$

$$[A_i, H] = \frac{i\hbar}{m}\cancel{(p_i p^2 - p^2 p_i)}^{\,0} - \frac{i\hbar}{2}e^2\left(p_i \frac{1}{r} + \frac{1}{r}p_i\right) +$$

$$+\frac{i\hbar}{2}e^2\left(p_j \frac{r_i r_j}{r^3} + \frac{r_i r_j}{r^3}p_j\right) - i\frac{e^2\hbar}{r^3}\left(r_i(rp) - r^2 p_i\right) - e^2\hbar^2\frac{r_i}{r^3}$$

$$= -i\frac{\hbar}{2}e^2\left(p_i\frac{1}{r} + \frac{1}{r}p_i\right) + \cancel{i\hbar e^2\frac{r_i}{r^3}(rp)} + \frac{i\hbar}{2}e^2\left[p_j, \frac{r_i r_j}{r^3}\right] -$$

$$\cancel{-\frac{ie^2\hbar}{r^3}r_i(rp)} + \frac{ie^2\hbar}{r}p_i - e^2\hbar^2\frac{r_i}{r^3}$$

$$= \cancel{-i\hbar e^2\frac{1}{r}p_i} - \frac{i\hbar}{2}e^2\left[p_i, \frac{1}{r}\right] + \frac{i\hbar}{2}e^2\left[p_j, \frac{r_i r_j}{r^3}\right] + \cancel{\frac{i\hbar e^2}{r}p_i} - e^2\hbar^2\frac{r_i}{r^3}$$

$$= +\frac{i\hbar}{2}e^2(-i\hbar)\left(+\frac{r_i}{r^3}\right) + \frac{i\hbar}{2}e^2(-i\hbar)\left(\frac{\delta_{ij}r_j}{r^3} + \frac{r_i\delta_{jj}}{r^3} - \frac{3}{r^4}r_i r_j\frac{r_j}{r}\right)$$

$$-e^2\hbar^2\frac{r_i}{r^3} = \cancel{\frac{\hbar^2 e^2}{2}\frac{r_i}{r^3}} + \cancel{\frac{\hbar^2 e^2}{2}\left(\frac{r_i}{r^3} + \frac{3r_i}{r^3} - \frac{3r_i r^2}{r^5}\right)} - \cancel{e^2\hbar^2\frac{r_i}{r^3}} = 0$$

$$[A_i, H] = 0$$

$$[L_i, H] = 0$$

Summary:

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$$

$$[A_i, L_j] = i\hbar \varepsilon_{ijk} A_k$$

$$[A_i, A_j] = -2mi\hbar \varepsilon_{ijk} H L_k$$

$$[H, A_i] = 0$$

$$[H, L_i] = 0$$

$$A_i L_i = 0 \qquad A_i A_i = 2mH(L^2 + \hbar^2) + m^2 e^4$$

Take eigenstates of energy $\mathcal{E}$.

$$l_i = L_i/\hbar \qquad a_i = \frac{A_i}{\hbar\sqrt{-2m\mathcal{E}}}$$

$$\Rightarrow \quad [l_i, l_j] = i\varepsilon_{ijk} l_k \qquad a_i l_i = 0$$

$$[a_i, l_j] = i\varepsilon_{ijk} a_k \qquad a_i a_i = \frac{A_i A_i}{\hbar^2(-2m\mathcal{E})} = -\left(\frac{L^2}{\hbar^2} + 1\right) + \frac{m^{\cancel{2}} e^4}{-2\cancel{m}\mathcal{E}\hbar^2}$$

$$[a_i, a_j] = i\varepsilon_{ijk} l_k$$

$$[H, l_i] = 0 \qquad a_i a_i = -1 - \frac{L^2}{\hbar^2} - \frac{me^4}{2\hbar^2\mathcal{E}}$$

$[H_n$

$$\left[\frac{a_i \pm l_i}{2}, \frac{a_j \pm l_j}{2}\right] = \frac{1}{4} i \varepsilon_{ijk} l_k + \frac{1}{4} i \varepsilon_{ijk} l_k \pm \frac{1}{4} i \varepsilon_{ijk} a_k \pm$$

$$\pm \frac{1}{4} i \varepsilon_{ijk} a_k$$

$$= \frac{1}{2} i \varepsilon_{ijk} (l_k \pm a_k)$$

$$\left[\frac{l_i \pm a_i}{2}, \frac{l_j \pm a_j}{2}\right] = i \varepsilon_{ijk} \frac{1}{2} (l_k \pm a_k)$$

$$I_i = \frac{l_i + a_i}{2} \qquad J_i = \frac{l_i - a_i}{2}$$

$$I_i^2 = \frac{\vec{l}^2}{4} + \frac{a^2}{4} + \frac{1}{4} \underbrace{(l_i a_i + a_i l_i)}_{0}$$

$$= \frac{1}{4} \left(-1 - \cancel{\vec{l}^2} - \frac{me^4}{2\hbar^2 \mathcal{E}} + \cancel{\vec{l}^2}\right) = \frac{1}{4} \left(-1 - \frac{me^4}{2\hbar^2 \mathcal{E}}\right)$$

$$J^2 = I^2$$

$SU(2) \times SU(2)$ but (j, j)

$$j(j+1) = \frac{1}{4} \left(-1 - \frac{me^4}{2\hbar^2 \mathcal{E}}\right) \Rightarrow \quad j^2 + 1 + \frac{1}{4} = -\frac{me^4}{8\hbar^2 \mathcal{E}}$$

$$\mathcal{E} (j + 1/2)^2 = -\frac{me^4}{8\hbar^2} \quad \Rightarrow \quad \boxed{\mathcal{E} = -\frac{me^4}{2\hbar^2 (2j+1)^2}}$$

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

degeneracy $(2j+1)^2$

$j=0 \quad \mathcal{E} = -\frac{me^4}{2\hbar^2}$ 1 state

$j=\frac{1}{2} \quad \mathcal{E} = -\frac{me^4}{2\hbar^2 2^2}$ 4 states

$$\boxed{\mathcal{E} = -\frac{me^4}{2\hbar^2 n_r^2} \ ; \ n_r^2 \text{ states}}$$