Angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$L_i = \epsilon_{ijk} x_j p_k$; sum over $j=1,2,3$, $k=1,2,3$ understood.

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ even permutation of } 123 \\ -1 & \text{" " odd " " "} \\ 0 & \text{if any two indices repeated are equal)} \end{cases}$$

$$[L_i, L_j] = \epsilon_{ipq}\epsilon_{jlm}\,[x_p p_q, x_l p_m]$$

$$= \epsilon_{ipq}\epsilon_{jlm}\left(p_q x_l \, i\hbar \, \delta_{pm} + x_p p_m (-i\hbar)\, \delta_{ql} \right)$$

$$= i\hbar\left(\epsilon_{i\underset{\substack{\downarrow\\ q}}{m}\underset{\substack{\downarrow\\ m}}{q}}\, \epsilon_{j\underset{\substack{\downarrow\\ p}}{l}\underset{\substack{\downarrow\\ q}}{m}}\, \underset{\substack{\downarrow\\ m}}{p_q}\, \underset{\substack{\downarrow\\ p}}{x_l} - \epsilon_{ipq}\epsilon_{jqm}\, x_p p_m \right)$$

$$= i\hbar\left(-\epsilon_{imq}\epsilon_{jpq} + \epsilon_{jmq}\epsilon_{ipq} \right) x_p p_m$$

$$\epsilon_{ijk}\epsilon_{ipm} = \delta_{jp}\delta_{km} - \delta_{jm}\delta_{kp}$$

$$[L_i, L_j] = i\hbar\left(-\cancel{\delta_{ij}\delta_{mp}} + \delta_{ip}\delta_{mj} + \cancel{\delta_{ji}\delta_{mp}} - \delta_{jp}\delta_{mi} \right) x_p p_m$$

$$= i\hbar\left(x_i p_j - x_j p_i \right) = i\hbar \underbrace{\epsilon_{ijk}\epsilon_{lmk}}_{\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} x_l p_m = i\hbar\, \epsilon_{ijk} L_k$$

$$\boxed{[L_i, L_j] = i\,\epsilon_{ijk}\,\hbar\, L_k}$$

$$l_i = \frac{1}{\hbar} L_i \quad \Rightarrow \quad \boxed{[l_i, l_j] = i\,\epsilon_{ijk} l_k}$$

②

Also $\quad [S_i, S_j] = i\hbar\,\epsilon_{ijk} S_k \quad$ (spin).

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2 \quad (= L_1^2 + L_2^2 + L_3^2)$$

$$\Big[l_j, \sum_i l_i^2\Big] = l_i\,[l_j, l_i] + [l_j, l_i]\,l_i$$

$$= i\hbar\, l_i\,\epsilon_{jik} l_k + i\hbar\,\epsilon_{jik} l_k l_i$$

$$= i\hbar \underbrace{\epsilon_{jik}}_{\text{anti symmetric}} \underbrace{(l_i l_k + l_k l_i)}_{\text{symmetric } i \leftrightarrow k} = 0$$

<u>Also</u>

$$[L_i, x_j] = \epsilon_{ilm}[x_l p_m, x_j] = (-i\hbar)\,\delta_{mj}\,\epsilon_{ilm} x_l$$

$$= i\hbar\,\epsilon_{ijk} x_k$$

$$[L_i, p_j] = \epsilon_{ilm}[x_l p_m, p_j] = i\hbar\,\delta_{lj}\,\epsilon_{ilm} p_m$$

$$= i\hbar\,\epsilon_{ijk} p_k$$

In general for a vector v_i; $\boxed{[L_i, v_j] = i\hbar\,\epsilon_{ijk} v_k}$

Define

$$l_\pm = l_1 \pm i l_2 \quad ; \quad l_+^\dagger = l_- \quad ; \quad l_{1,2} \text{ hermitian}$$

$$[l_\pm, l_3] = [l_1, l_3] \pm i\,[l_2, l_3] = -i l_2 \pm i\,(i l_1)$$

$$= -i l_2 \mp l_1 = \mp (l_1 \pm i l_2) = \mp l_\pm$$

$$\hat{l}_3 |l_z\rangle = l_z |l_z\rangle$$

$$l_+ l_3 |l_z\rangle = [l_+, l_3] |l_z\rangle + l_3 l_+ |l_z\rangle$$

$$l_3 \underbrace{l_+ |l_z\rangle} = [l_3, l_+] |l_z\rangle + l_+ l_3 |l_z\rangle$$

$$= + l_+ |l_z\rangle + l_z\, l_+ |l_z\rangle$$

$$= (l_z + 1) \underbrace{l_+ |l_z\rangle}$$

$[l_3, l_+] = l_+$ ← increases eigenvalue of l_3 by 1

$[l_3, l_-] = -l_-$ ← decreases " " " "

$$[l_+, l_-] = [l_1 + i l_2,\ l_1 - i l_2] = -i\,i\,l_3 + i(-i) l_3 = 2 l_3$$

$$l_+ l_- = (l_1 + i l_2)(l_1 - i l_2) = l_1^2 + l_2^2 + i\underbrace{(l_2 l_1 - l_1 l_2)}_{-i l_3} = l_1^2 + l_2^2 + l_3$$

$$l_- l_+ = (l_1 - i l_2)(l_1 + i l_2) = l_1^2 + l_2^2 + i[l_1, l_2] = l_1^2 + l_2^2 - l_3$$

$$\hat{l}^2 = \frac{1}{2}(l_+ l_- + l_- l_+) + l_3^2$$

$$\hat{l}^2 = l_+ l_- + l_3^2 - l_3$$

$$\hat{l}^2 = l_- l_+ + l_3^2 + l_3$$

useful expressions for $\hat{l}^2$

Summary

$$[l_+, l_3] = l_+ \qquad\qquad l_+ = (l_-)^\dagger$$

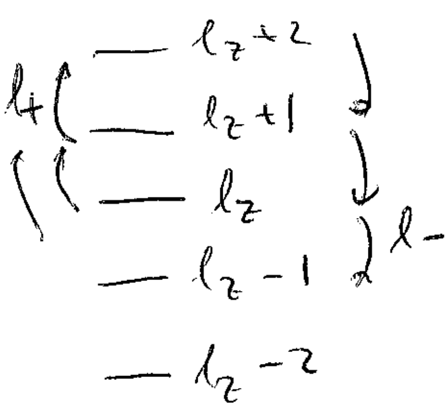
$$[l_-, l_3] = -l_-$$

$$[l_+, l_-] = 2l_3$$

Representations

$[\hat{l}^2, l_3] = 0$ we can diagonalize $\hat{l}^2, l_3$.

Take $|\psi\rangle$ s.t. $\boxed{\begin{array}{l}\hat{l}^2|\psi\rangle = \lambda|\psi\rangle \\ l_3|\psi\rangle = l_z|\psi\rangle\end{array}}$



$$\langle\psi|\hat{l}^2|\psi\rangle = \langle\psi|l_- l_+|\psi\rangle + l_z^2 + l_z$$

$$= \| l_+|\psi\rangle\|^2 + l_z^2 + l_z = \lambda$$

$$\Rightarrow \| l_+|\psi\rangle\|^2 = \lambda - l_z^2 - l_z \geq 0 \Rightarrow l_z \text{ is bounded from below and above.}$$

(1)

There is a maximum value of l_z if λ is fixed.

But l_+ increases l_z $\Rightarrow$

$$l_+ | l_z^{max} \rangle = 0$$

$$\langle l_z^{max} | \hat{l}^2 | l_z^{max} \rangle = \langle l_z^{max} | l_- \underbrace{l_+}_{0} + l_3^2 + l_3 | l_z^{max} \rangle$$

$$= (l_z^{max})^2 + l_z^{max} = \lambda$$

Let's call $l_z^{max} = l \Rightarrow \boxed{\lambda = l(l+1)}$

$\hat{l}^2 | l, l_z \rangle = l(l+1) | l, l_z \rangle$ For $l_z \leq l$.

Also there is l_z^{min} / $l_- | l_z^{min} \rangle = 0$

$$\langle l_z^{min} | \hat{l}^2 | l_z^{min} \rangle = \langle l_z^{min} | l_+ \underbrace{l_-}_{0} + l_3^2 - l_3 | l_z^{min} \rangle$$

$$= (l_z^{min})^2 - l_z^{min} = \lambda = l(l+1)$$

$l_z^{min}(l_z^{min} - 1) = l(l+1) \Rightarrow l_z^{min} = -l$ ✓

$l+1$ ← not possible

$l_z^{min} < l_z^{max}$

— $+l$

⋮

— $-l+1$

— $-l$

$$|l\rangle = (l_+)^{2l} |-l\rangle$$

$\Rightarrow 2l$ has to be integer $l = 0, 1, 2, \ldots$

But also $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

Eigenstates of $\hat{l}^2$ and l_z are given by two integers

l, l_z ; $|l_z| \leq l$.

$$\boxed{\begin{aligned} \hat{l}^2 |l, l_z\rangle &= l(l+1) |l, l_z\rangle \\ l_z |l, l_z\rangle &= l_z |l, l_z\rangle \end{aligned}}$$

How about $l_\pm$?

$$l_+ | l, l_z\rangle = C_{l_z} \, | l, l_z + 1\rangle$$

$$\| l_+ |l, l_z\rangle \|^2 = l(l+1) - l_z(l_z+1) = |C_{l_z}|^2$$

From ① before

$$C_{l_z} = \sqrt{l(l+1) - l_z(l_z+1)}$$

$$\| l_- | l, l_z\rangle \|^2 = l(l+1) - l_z(l_z-1)$$

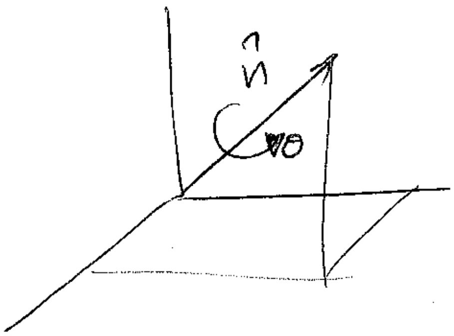
$$\Rightarrow \boxed{\begin{aligned} l_+ |l, l_z\rangle &= \sqrt{l(l+1) - l_z(l_z+1)} \, |l, l_z+1\rangle \\ l_- |l, l_z\rangle &= \sqrt{l(l+1) - l_z(l_z-1)} \, |l, l_z-1\rangle \end{aligned}}$$

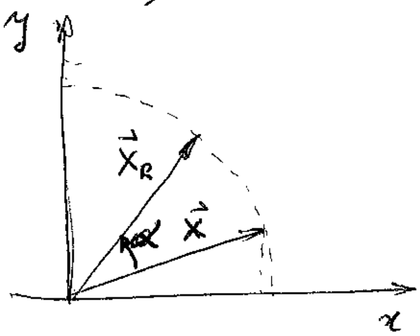
Rotations $\rightarrow SO(3)$ non-abelian group

$$R_{\hat{n},\theta} \, |\vec{x}\rangle = |R^{(\hat{n},\theta)}_{ij} x_j\rangle$$

$R_{\hat{n},\theta}$: operator

$R^{(\hat{n},\theta)}_{ij}$: orthogonal matrix ($3\times 3$ $R^t R = \mathbb{1}$)





~~$X = (rc\varphi, r$~~

$$\vec{X} = (rs\theta c\varphi,\ rs\theta s\varphi,\ rc\theta)$$

$$\vec{X}_R = (rs\theta\, c(\varphi+\alpha),\ rs\theta\, s(\varphi+\alpha),\ rc\theta)$$

$$c\varphi c\alpha - s\varphi s\alpha \qquad s\varphi c\alpha + s\alpha c\varphi$$

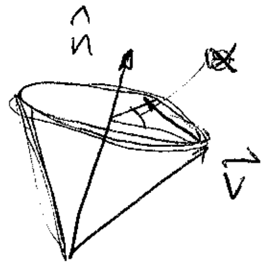
$$\vec{X}_R = (xc\alpha - ys\alpha,\ yc\alpha + xs\alpha,\ z)$$

$$\vec{X}_R = \begin{pmatrix} x_R \\ y_R \\ z_R \end{pmatrix} = \underbrace{\begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R^{(\hat{z},\alpha)}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-i\alpha \frac{L_z}{\hbar}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\partial_\alpha R^{\hat{z},\alpha} = -i \frac{L_z}{\hbar} R^{\hat{z},\alpha}$$

Also $\alpha \to 0$ $\quad R \simeq \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} + \alpha \begin{pmatrix} 0&-1&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}$

$$\partial_\alpha R = \begin{pmatrix} -s\alpha & -c\alpha & 0 \\ c\alpha & -s\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow L_z = i\hbar \begin{pmatrix} 0&-1&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}$$



$$\vec{V}_R = a\vec{V} + \tilde{b}\,\hat{n} + c(\hat{n}\times\vec{V})$$

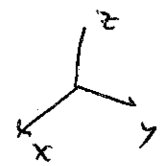
$$\vec{V}_R = a\vec{V} + b(\vec{V}\cdot\hat{n})\,\hat{n} + c(\hat{n}\times\vec{V}) \quad \leftarrow \text{linear in } \vec{V}$$

$$\vec{V}_R\cdot\hat{n} = \vec{V}\cdot\vec{n} = a\vec{V}\cdot\vec{n} + b(\vec{V}\cdot\vec{n}) \Rightarrow a+b=1$$

$$\vec{V}_R = a\vec{V} + (1-a)(\vec{V}\cdot\hat{n})\,\hat{n} + c(\hat{n}\times\vec{V})$$

If $\hat{n} = \hat{z}$

$$\hat{V}_R = a\vec{V} + (1-a)V_z\,\hat{z} + c(\hat{z}\times\vec{V})$$

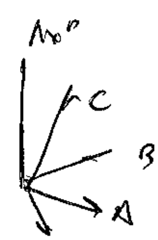


$$V = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$$

$$\hat{V}_R = aV_x\,\hat{x} + aV_y\,\hat{y} + \cancel{aV_z\hat{z}} + (1-\cancel{a})V_z\,\hat{z} + c(V_x\hat{y} - V_y\hat{x})$$

$$= (\underset{c\alpha}{a}V_x - \underset{s\alpha}{c}V_y)\,\hat{x} + (\underset{c\alpha}{a}V_y + \underset{s\alpha}{c}V_x)\,\hat{y} + V_z\hat{z}$$

$$\boxed{\vec{V}_R = c\alpha\,\vec{V} + (1-c\alpha)(\vec{V}\hat{n})\hat{n} + s\alpha(\hat{n}\times\hat{V})}$$



$$C\times(A\times B) = (BC)A - (AC)B$$

$$\partial_\alpha \vec{V}_R = -s\alpha\vec{V} + s\alpha(\vec{V}\hat{n})\hat{n} + c\alpha(\hat{n}\times\hat{V})$$

$$\hat{n}\times\vec{V}_R = c\alpha\,\hat{n}\times\vec{V} + s\alpha\,\underset{C}{\hat{n}}\times(\underset{A}{\hat{n}}\times\underset{B}{\vec{V}}) = c\alpha\,\hat{n}\times\vec{V} + s\alpha(nv)\hat{n} - s\alpha\underbrace{(nn)}_{1}\vec{V} = \partial_\alpha\vec{V}_R$$

$\epsilon_{ijk} n_j V_k$

3

$$\vec V_{R_i} = R_{ij}^{\hat n,\alpha} V_j = \left(c\alpha\,\delta_{ij} + (1-c\alpha)\, n_i n_j + s\alpha\,\epsilon_{ikj} n_k \right) V_j$$

$$R_{ij}^{\alpha,\hat n} = c\alpha\,\delta_{ij} + (1-c\alpha)\, n_i n_j + s\alpha\,\epsilon_{ikj} n_k$$

$$\partial_\alpha R_{ij}^{\alpha,\hat n} = -s\alpha\,\delta_{ij} + s\alpha\, n_i n_j + c\alpha\,\epsilon_{ikj} n_k$$

$$= -\frac{i}{\hbar}(\vec L\cdot\hat n)\, R$$

$$R^{-1}\partial_\alpha R = -\frac{i}{\hbar}(\vec L\cdot\hat n)$$

$$R^{-1} = R_{-\alpha}$$

$$\left(R^{-1}\partial_\alpha R\right)_{il} = \left(c\alpha\,\delta_{ij} + (1-c\alpha)\, n_i n_j - s\alpha\,\epsilon_{ikj} n_k\right)\left(-s\alpha\,\delta_{jl} + s\alpha\, n_j n_l + c\alpha\,\epsilon_{jpl} n_p\right)$$

$$= -s\alpha c\alpha\,\delta_{il} + s\alpha c\alpha\, n_i n_l + c^2\alpha\,\epsilon_{ipl} n_p \cancel{- s\alpha(1-c\alpha)\, n_i n_l} \cancel{+ s\alpha(1-c\alpha)\, n_i n_l}$$

$$+ s^2\alpha\,\epsilon_{ikl} n_k - s\alpha c\alpha\,\epsilon_{ikj} n_k\, \epsilon_{jpl} n_p$$

$$(\delta_{ip}\delta_{kl} - \delta_{il}\delta_{kp})$$

$$= \cancel{-s\alpha c\alpha\,\delta_{il}} \cancel{+ s\alpha c\alpha\, n_i n_l} + c^2\alpha\,\epsilon_{ipl} n_p + s^2\alpha\,\epsilon_{ikl} n_k -$$

$$\cancel{-s\alpha c\alpha\, n_i n_l} \cancel{+ s\alpha c\alpha\,\delta_{il}} = \epsilon_{ikl} n_k$$

$$-\frac{i}{\hbar}(\vec L\cdot\vec n)_{il} = \epsilon_{ikl} n_k \Rightarrow L^{(k)}_{il}\, n^{(k)} = i\hbar\,\epsilon_{ikl} n_k$$ faster

$$\boxed{L^{(k)}_{il} = -i\hbar\,\epsilon_{kil}}$$

$$(\partial_\alpha \vec V_R)_i = \epsilon_{ijk} n_j V^R_k = (-i\hbar)\frac{i}{\hbar}(-n_j\,\epsilon_{jik})\, V^R_k \checkmark$$

(4)

$$L^z_{il} = -i\hbar\,\epsilon_{zil} = -i\hbar\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L^x_{il} = -i\hbar\,\epsilon_{xil} = -i\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$L^y_{il} = -i\hbar\,\epsilon_{yil} = -i\hbar\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$l_x = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad l_y = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad l_z = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[l_x, l_y] = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = i\,l_z \quad \checkmark$$

correspond to states with $\boxed{l = 1}$

Another representation. → spin 1/2

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Also obey $[S_i, S_j] = i\hbar\epsilon_{ijk} S_k$.

We see that angular momentum can be identified with infinitesimal rotations.

Indeed:

$$L_z = x p_y - y p_x = -i\hbar x \partial_y + i\hbar y \partial_x = i\hbar (y\partial_x - x\partial_y)$$

$$\langle x | R_{z,\alpha} | \psi \rangle = \psi \langle R_{z,-\alpha} x | \psi \rangle = \psi(R_{z,-\alpha} \cdot x)$$

$$= \psi(x c\alpha + y s\alpha, y c\alpha - x s\alpha, z) =$$

$$\underset{\alpha \to 0}{=} \psi(x + \alpha y, y - \alpha x, z) = \psi + y\alpha\partial_x \psi - \alpha x \partial_y \psi + \ldots$$

$$= \psi + \alpha (y\partial_x - x\partial_y)\psi = \psi - \frac{i}{\hbar}\alpha L_z \psi$$

$$= \langle x|\psi\rangle - \frac{i\alpha}{\hbar} \langle x | L_z | \psi \rangle$$

$$= \langle x | e^{-\frac{i\alpha}{\hbar} L_z} | \psi \rangle_{\alpha \to 0}.$$

In spherical coordinates $L_z = -i\hbar \frac{\partial}{\partial \varphi}$ ← translates φ.

requires l integer
half-integer is only for spin.

$$Y_{lm}(\theta, \varphi) = \langle \theta | \varphi | l m \rangle$$

eigenstates of L_z and $\hat{L}^2$.

because $[\hat{L}^2, H]$ so energy eigenstates are eigenstates of $\hat{L}^2$ for generic $V(r)$.

and $m = -l \ldots l$.

Using basis of eigenstates of $\hat{l}^2$, l_z we have.

$$\langle l, l'_z | R^{\hat{\theta}, \hat{n}} | l\, l_z \rangle = \mathcal{D}^{(l)}_{l'_z l_z}(\theta, \hat{n}) \quad \leftarrow \text{notation.}$$

same since $[R, \hat{L}^2] = 0$

Useful parameterization: Euler angles (α, β, γ) (instead of $\hat{n}, \theta$)

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

Much simpler since

$$\langle l\, l'_z | R_z(\alpha) R_y(\beta) R_z(\gamma) | l\, l_z \rangle =$$

$$= \langle l l'_z | e^{-i L_z \frac{\alpha}{\hbar}} e^{-i L_y \frac{\beta}{\hbar}} e^{-i L_z \frac{\gamma}{\hbar}} | l l_z \rangle =$$

$$= e^{-i l'_z \alpha - i \gamma l_z} \langle l l'_z | e^{-i L_y \beta / \hbar} | l l_z \rangle$$

$$= e^{-i \alpha l'_z - i \gamma l_z} d^{l}_{l'_z l_z}(\beta)$$

we only need this one.

$\boxed{\text{Spin } 1/2}$

$$d^{1/2}_{s_z' s_z} = \langle s_z' | e^{-\frac{i}{\hbar}\beta S_y} | s_z \rangle$$

$$= \langle s_z' | e^{-\frac{i}{2}\beta \sigma_y} | s_z \rangle$$

$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2} = \sum_k \frac{x^{2k}}{(2k)!}$$

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2} = \sum_k \frac{x^{2k+1}}{(2k+1)!}$$

$$\operatorname{sh}(ix) = i\frac{e^{ix} - e^{-ix}}{2i} = i \sin x$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{-\frac{i}{2}\beta\sigma_y} = \sum_{n=0}^{\infty} \left(-\frac{i}{2}\beta\right)^n \frac{1}{n!}\sigma_y^n = \sum_{k=0}^{\infty}\left(-\frac{i}{2}\beta\right)^{2k}\frac{\mathbb{1}}{(2k)!} + \sum_{k=0}^{\infty}\left(-\frac{i}{2}\beta\right)^{2k+1}\frac{\sigma_y}{(2k+1)!}$$

$$= \operatorname{ch}\left(\frac{i\beta}{2}\right) - \operatorname{sh}\left(\frac{i}{2}\beta\right)\sigma_y = \cos\beta/2\, \mathbb{1} - i \sin\frac{\beta}{2}\sigma_y$$

$$= \begin{pmatrix} c\beta/2 & -s\beta/2 \\ s\beta/2 & c\beta/2 \end{pmatrix}$$

$$\boxed{d^{1/2} = \begin{pmatrix} c\beta/2 & -s\beta/2 \\ s\beta/2 & c\beta/2 \end{pmatrix}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, s\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\, s\theta\,(c\phi + i s\phi) = -\frac{1}{r}\sqrt{\frac{3}{8\pi}}\,(x+iy) = \sqrt{\frac{3}{8\pi}}\,\frac{x_+}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\, c\theta = \sqrt{\frac{3}{8\pi}}\,\sqrt{2}\,\frac{z}{r} = \sqrt{\frac{3}{8\pi}}\,\frac{x_0}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}}\, s\theta\,(c\phi - i s\phi) = \frac{1}{r}\sqrt{\frac{3}{8\pi}}\,(x-iy) = \sqrt{\frac{3}{8\pi}}\,\frac{x_-}{r}$$

$$x_+ = -(x+iy) \qquad x_0 = \sqrt{2}\, z \qquad x_- = x - iy$$

$$\boxed{\begin{aligned} x_+ + x_- &= -2iy \\ x_+ - x_- &= -2x \\ x &= \frac{-x_+ + x_-}{2} \\ y &= \frac{i}{2}(x_+ + x_-) \end{aligned}}$$

change of basis

$$\begin{pmatrix} x_+ \\ x_0 \\ x_- \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix}}_{S} \begin{pmatrix} x \\ y \\ z \end{pmatrix} ; \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -1/2 & 0 & +1/2 \\ i/2 & 0 & i/2 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}}_{S^{-1}} \begin{pmatrix} x_+ \\ x_0 \\ x_- \end{pmatrix}$$

$$\begin{pmatrix} x_+ \\ x_0 \\ x_- \end{pmatrix}_R = \begin{pmatrix} S \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_R = \begin{pmatrix} S \end{pmatrix} R^{\beta, \hat{y}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} S \end{pmatrix} R^{\beta\hat{y}} S^{-1}}_{d} \begin{pmatrix} x_+ \\ x_0 \\ x_- \end{pmatrix}$$

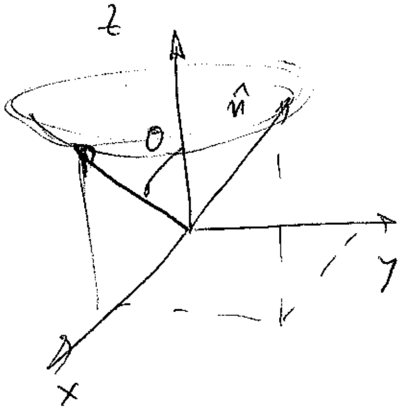
$$d^{(1)} = \begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \begin{pmatrix} -1/2 & 0 & +1/2 \\ i/2 & 0 & i/2 \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}c\beta & \frac{s\beta}{\sqrt{2}} & +\frac{1}{2}c\beta \\ i/2 & 0 & i/2 \\ \frac{s\beta}{2} & \frac{c\beta}{\sqrt{2}} & -s\beta/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c\beta + \frac{1}{2} & -\frac{s\beta}{\sqrt{2}} & -\frac{1}{2}c\beta + \frac{1}{2} \\ \frac{1}{\sqrt{2}}s\beta & c\beta & -\frac{1}{\sqrt{2}}s\beta \\ -\frac{1}{2}c\beta + \frac{1}{2} & \frac{s\beta}{\sqrt{2}} & \frac{1}{2}c\beta + \frac{1}{2} \end{pmatrix} \checkmark$$

Rotational matrices and spherical harmonics.

$$Y_{em} = \langle \theta\varphi | lm \rangle$$

$$|\theta\varphi\rangle = |\hat{n}\rangle = R_{\varphi,\hat{z}} R_{\hat{\theta},y} |\hat{z}\rangle \quad ; \quad \langle\theta\varphi| = \langle\hat{z}| R^{-1}_{\theta,y} R^{-1}_{\varphi,\hat{z}}$$



$$Y_{em} = \langle\hat{z}| R^{-1}_{\hat{\theta},\hat{y}} R^{-1}_{\hat{\varphi},\hat{z}} |lm\rangle$$

$$Y^{*}_{em}(\theta,\varphi) = \langle lm| R_{\varphi,\hat{z}} R_{\theta,\hat{y}} |\hat{z}\rangle$$

$$= e^{-im\varphi} \sum_{m'} \langle lm| R_{\theta,\hat{y}} |lm'\rangle \langle lm'|\hat{z}\rangle$$

$$= e^{-im\varphi} \sum_{m'} d^{(l)}_{mm'}(\theta) \underbrace{Y^{*}_{em'}(\theta=0,\tilde{\varphi})}_{0 \text{ except if } m'=0}$$

$$= e^{-im\varphi} d^{l}_{m0}(\theta) \underbrace{Y^{*}_{e0}(\theta=0)}_{\sqrt{\frac{2l+1}{4\pi}}}$$

$$\boxed{Y^{*}_{em}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, e^{-im\varphi}\, d^{l}_{m0}(\theta)}$$