Hydrogen atom with Klein-Gordon.

$\partial_\mu \to \partial_\mu + iqA_\mu$

$i\partial_\mu \to i\partial_\mu - qA_\mu$

$\varepsilon^2 - \vec{p}^{\,2} = m^2$; $c=1$ $\hbar=1$

$\psi \sim e^{-i\omega t + i\vec{k}\vec{x}}$ free particle $\to e^{-i\omega t}\psi(\vec{x})$

$E \to i\partial_t$ $\quad \vec{p} = -i\partial_{\vec{x}} = -i\vec{\nabla}$

$$(i\partial_t)^2\psi - (i\nabla)^2\psi = m^2\psi$$

$$i\partial_t - qA_0 \to E + \frac{Ze^2}{r}$$

$$\left(E + \frac{Ze^2}{r}\right)^2\psi + \Delta\psi = m^2\psi \quad ; \; E = m + E_b$$

$$(E^2 - m^2)\psi = -\Delta\psi - \frac{2Ze^2}{r}E\psi - \frac{Z^2e^4}{r^2}\psi$$

$$-\frac{1}{2m}\Delta\psi - \frac{Ze^2E}{mr}\psi - \frac{Z^2e^4}{2mr^2}\psi = \frac{E^2-m^2}{2m}\psi$$

~~[illegible]~~ check: $E_b \ll m$ $\quad \frac{E^2-m^2}{2m} \simeq E_b\psi$; $\quad \frac{\frac{Z^2e^4}{2\cancel{m}r^2}}{\frac{Ze^2}{\cancel{m}r}E} \sim \frac{e^2}{2r}\frac{1}{E} \sim \frac{13.6\,eV}{511\,keV}$ small.

$H \approx p^2/2m - \frac{Ze^2}{r}$ ✓

$$-\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2}\right)\psi - \frac{Ze^2E}{mr}\psi - \frac{Z^2e^4}{2mr^2}\psi = \frac{E^2-m^2}{2m}\psi$$

$$\boxed{-\frac{1}{2m}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\partial_r\psi\right) + \frac{l(l+1) - Z^2e^4}{2mr^2}\psi - \frac{Ze^2E}{mr}\psi = \frac{E^2-m^2}{2m}\psi}$$

$$\left.\begin{aligned}\hat{\ell}(\hat{\ell}+1) &= \ell(\ell+1) - Z^2e^4\\ \hat{Z}\hat{e}^2 &= \frac{Ze^2E}{m}\\ \hat{E} &= \frac{E^2-m^2}{2m}\end{aligned}\right\} \rightarrow -\frac{1}{2m}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\partial_r\psi\right)+\frac{\hat{\ell}(\hat{\ell}+1)}{2mr^2}\psi - \frac{\hat{Z}\hat{e}^2}{r}\psi = \hat{E}\psi$$

$$\hat{E} = -\frac{\hat{Z}^2\hat{e}^4 m}{2(n_r+\hat{\ell}+1)^2} \qquad n_r = 0,1,2\ldots$$

$$\frac{E^2-m^2}{\cancel{2m}} = -\frac{Z^2e^4E^2}{\cancel{m^2}}\,\frac{\cancel{m}}{\cancel{2}(n_r+\hat{\ell}+1)^2}$$

$$E^2 = \frac{m^2}{1+\frac{Z^2e^4}{(n_r+\hat{\ell}+1)^2}} \quad ; \quad \hat{\ell}^2+\hat{\ell}-\ell(\ell+1)+Z^2e^4 = 0$$

$$\hat{\ell} = \frac{-1\pm\sqrt{1+4\ell^2+4-4Z^2e^4}}{2}$$

$$\hat{\ell} = -\frac{1}{2}\pm\sqrt{(\ell+\tfrac{1}{2})^2-Z^2e^4}$$

Keep + (both are the same in terms of $\hat{\ell}(\hat{\ell}+1)$)

$$\hat{\ell} = -\frac{1}{2}+\sqrt{(\ell+\tfrac{1}{2})^2-Z^2e^4}$$

<u>Restore units:</u>

$$\hat{\ell} = -\frac{1}{2}+\sqrt{(\ell+\tfrac{1}{2})^2-Z^2\alpha^2} \qquad \alpha = e^2/\hbar c$$

$$E = \frac{mc^2}{\sqrt{1+\frac{Z^2\alpha^2}{(n_r+\hat{\ell}+1)^2}}}$$

problems: $\ell = 0$ $\quad \hat{\ell}$ imaginary part. $\rightarrow$ E ~~complex~~

if $\quad Z^2\alpha^2 > 1/4 \quad \boxed{Z > \frac{137}{2}}$ no

~~(low energy consequence~~ "(low energy consequence)"

small Z:

$$\hat{\ell} \simeq -\frac{1}{2} + \left(\ell+\frac{1}{2}\right)\left(1 - \frac{Z^2\alpha^2}{2(\ell+\frac{1}{2})^2}\right) = \ell - \frac{Z^2\alpha^2}{(2\ell+1)} = \ell - \delta_\ell$$

$$\delta_\ell = \frac{Z^2\alpha^2}{(2\ell+1)}$$

$$E \simeq mc^2\left(1 - \frac{1}{2}\frac{Z^2\alpha^2}{(n-\delta_\ell)^2} + \frac{3}{8}\frac{Z^4\alpha^4}{(n-\delta_\ell)^4}\right)$$

$$\simeq mc^2\left(1 - \frac{1}{2}\frac{Z^2\alpha^2}{n^2}\left(1+\frac{2\delta_\ell}{n}\right) + \frac{3}{8}\frac{Z^4\alpha^4}{n^4}\right)$$

$$E_b = -\frac{mc^2}{2}\frac{Z^2\alpha^2}{n^2} - \frac{Z^2\alpha^2}{n^3} mc^2 \frac{Z^2\alpha^2}{(2\ell+1)} + \frac{3}{8}\frac{mc^2 Z^4\alpha^4}{n^4}$$

$$E_b = -\frac{mZ^2e^4}{2n^2\hbar^2} + mc^2 Z^4\alpha^4\left(\frac{3}{8n^4} - \frac{1}{n^3(2\ell+1)}\right)$$

should be $\underline{2j+1}$

does not agree with experiment