$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$\psi_n(\xi) = H_n(\xi)\, e^{-\frac{1}{2}\xi^2} A_n$$

Hermite polynomial.

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$E = (n + \tfrac{1}{2})\hbar\omega$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega}\right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega}\right)$$

$$\boxed{H = \hbar\omega(a^\dagger a + \tfrac{1}{2})}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a)$$

$[a, a^\dagger] = 1$ $\boxed{\hat{N}^\dagger = \hat{N}}$ 1

$\hat{N} = a^\dagger a$

$[a^\dagger a, a] = -a$

$[a^\dagger a, a^\dagger] = a^\dagger$

$$\boxed{\begin{aligned} &[\hat{N}, a] = -a \\ &[\hat{N}, a^\dagger] = a^\dagger \\ &[a, a^\dagger] = 1 \end{aligned}}$$

$\leftarrow$ no finite dim rep.

$\mathrm{Tr}\,[a, a^\dagger] = 0 \neq \mathrm{Tr}\,\mathbb{1}$.

$$\hat{N}|\lambda\rangle = \lambda|\lambda\rangle \qquad \lambda \in \mathbb{R}.$$

$$\hat{N} a^\dagger|\lambda\rangle = a^\dagger \hat{N}|\lambda\rangle + [\hat{N}, a^\dagger]|\lambda\rangle$$

$$= \lambda a^\dagger|\lambda\rangle + a^\dagger|\lambda\rangle = (\lambda+1) a^\dagger|\lambda\rangle$$

$$\hat{N} a|\lambda\rangle = (\lambda-1) a|\lambda\rangle$$

$$\ldots \lambda-2, \lambda-1, \lambda, \lambda+1, \lambda+2, \ldots$$

$\mathcal{E} = \hbar\omega(\lambda + 1/2)$ $\qquad \lambda$ bounded from below

λ_{min}

$$a|\lambda_{min}\rangle = 0$$

$$\hat{N}|\lambda_{min}\rangle = 0 \Rightarrow \boxed{\lambda_{min} = 0}$$

$$|0\rangle, |1\rangle, |2\rangle, \ldots |n\rangle, \ldots$$

$\boxed{n \in \cancel{\mathbb{Z}}_{\geq 0}}$ $\boxed{n \in \mathbb{Z}_{\geq 0}}$

$$\boxed{\mathcal{E} = \hbar\omega(n+1/2)}$$

How about $\underline{\langle x|n\rangle}$?

$$|n\rangle = \alpha_n (a^\dagger)^n |0\rangle$$

$$\langle n|n\rangle = |\alpha_n|^2 \langle 0| a^n (a^\dagger)^n |0\rangle$$

$$a(a^\dagger)^n = (aa^\dagger - a^\dagger a)(a^\dagger)^{n-1} + a^\dagger a (a^\dagger)^{n-1}$$

$$= (a^\dagger)^{n-1} + a^\dagger a (a^\dagger)^{n-1} = \cdots$$

$$= n(a^\dagger)^{n-1} + (a^\dagger)^n a$$

$$a(a^\dagger)^n |0\rangle = n(a^\dagger)^{n-1}$$

$$a^2 (a^\dagger)^n |0\rangle = n(n-1)(a^\dagger)^{n-2}|0\rangle$$

$$a^n (a^\dagger)^n |0\rangle = n!\, |0\rangle$$

$$\langle n|n\rangle = n!\, |\alpha_n|^2 \qquad \alpha_n = \frac{1}{\sqrt{n!}}$$

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle}$$

Also:

$$a^\dagger|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^{n+1}|0\rangle$$

$$= \frac{\sqrt{n+1}}{\sqrt{(n+1)!}} (a^\dagger)^{n+1}|0\rangle$$

$$= \sqrt{n+1}\, |n+1\rangle$$

$$a|n\rangle = \frac{1}{\sqrt{n!}} a(a^\dagger)^n|0\rangle =$$

$$= \frac{n}{\sqrt{n!}} (a^\dagger)^{n-1}|0\rangle =$$

$$= \sqrt{n}\, |n-1\rangle$$

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}} \langle x|(a^\dagger)^n|0\rangle =$$

$$= \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{n/2} \langle x|\underbrace{\left(x-\frac{ip}{m\omega}\right)}_{}^n|0\rangle$$

$$x - \frac{i(-i\hbar\partial_x)}{m\omega} = x - \frac{\hbar}{m\omega}\partial_x$$

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{n/2}\left(x-\frac{\hbar}{m\omega}\partial_x\right)^n \langle x|0\rangle$$

$\langle x|0\rangle = ?$

$$\langle x|a|0\rangle = 0 = \sqrt{\frac{m\omega}{2\hbar}}\,\langle x|x+\frac{i\overset{-i\hbar\partial_x}{p}}{m\omega}|0\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\,\langle x|\,x+\frac{\hbar\partial_x}{m\omega}\,|0\rangle =$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{\hbar}{m\omega}\partial_x\right)\langle x|0\rangle = 0$$

$$\left(x+\frac{\hbar}{m\omega}\partial_x\right)\psi_0(x) = 0 \qquad x\psi_0(x) = -\frac{\hbar}{m\omega}\partial_x\psi_0(x)$$

$$\partial_x \ln\psi_0 = -\frac{m\omega}{\hbar}x \qquad \ln\psi_0 = \hat{A} - \frac{m\omega}{2\hbar}x^2; \quad \boxed{\psi_0 = Ae^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}}$$

$$\psi_n(x) = \frac{A}{\sqrt{n!}} \left(\frac{m\omega}{2\hbar}\right)^{\frac{n}{2}} \left(x - \frac{\hbar}{m\omega}\partial_x\right)^n e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$ (4)

$$\int |\psi_0|^2 = A^2 \int e^{-\frac{m\omega}{\hbar}x^2} = A^2 \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{n!}} \left(\frac{m\omega}{2\hbar}\right)^{\frac{n}{2}} \left(x - \frac{\hbar}{m\omega}\partial_x\right)^n e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$$= A_n H_n(x) e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$\uparrow$ polynomial of order n. $\Rightarrow$ Hermite.

eigenstates of a

$$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle} \qquad \alpha \in \mathbb{C} \qquad (a \text{ is not hermitian})$$

$$|\alpha\rangle = \sum_n c_n |n\rangle$$

$$a|\alpha\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle = \sum_{n=0}^{\infty} c_n \alpha |n\rangle$$

$$= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\, |n\rangle = \sum_{n=0}^{\infty} c_n \alpha |n\rangle$$

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n$$

$$c_1 = \frac{\alpha}{\sqrt{1}} c_0 \qquad c_2 = \frac{\alpha^2}{\sqrt{2}\sqrt{1}} c_0 \quad \dots \quad c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle = c_0 e^{\alpha a^\dagger} |0\rangle$$

$$\langle\alpha|\alpha\rangle = \sum_{n=0}^{\infty} |c_n|^2 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{\alpha\alpha^*} = 1$$

$$c_0 = e^{-\frac{1}{2}\alpha\alpha^*} \qquad \boxed{|\alpha\rangle = e^{-\frac{1}{2}\alpha\alpha^*} e^{\alpha a^\dagger} |0\rangle}$$

$$\langle x|\alpha\rangle = ?$$

$$\langle x|\, a|\alpha\rangle = \alpha\langle x|\alpha\rangle$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\partial_x\right)\psi_\alpha(x) = \alpha\,\psi_\alpha(x)$$

$$\frac{\hbar}{m\omega}\partial_x\psi_\alpha(x) = \left(\sqrt{\frac{2\hbar}{m\omega}}\,\alpha - x\right)\psi_\alpha(x)$$

$$\partial_x\psi_\alpha = \frac{m\omega}{\hbar}\left(\sqrt{\frac{2\hbar}{m\omega}}\,\alpha - x\right)\psi_\alpha(x)$$

$$\partial_x \ln\psi_\alpha = \frac{m\omega}{\hbar}\left(\sqrt{\frac{2\hbar}{m\omega}}\,\alpha - x\right)$$

$$\ln\psi_\alpha = \tilde{A} + \frac{m\omega}{\hbar}\left(\sqrt{\frac{2\hbar}{m\omega}}\,\alpha x - \frac{x^2}{2}\right)$$

$$= \tilde{A} - \frac{m\omega}{2\hbar}(x - X_0)^2 + \frac{m\omega}{2\hbar}X_0^2$$

$$+\frac{m\omega}{2\hbar}\cancel{2}xX_0 \qquad \frac{m\omega}{\hbar}X_0 = \sqrt{\frac{m\omega}{\hbar}}\sqrt{2}\,\alpha$$

$$X_0 = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2}\,\alpha$$

$$\psi_\alpha = A\, e^{-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\alpha\right)^2}$$

$$\langle x|\alpha\rangle = A\, e^{-\frac{m\omega}{2\hbar}\left(x-\sqrt{\frac{2\hbar}{m\omega}}\,\alpha\right)^2} \quad ; \quad \alpha \in \mathbb{C}$$

$$|\langle x|\alpha\rangle|^2 = |A|^2\, e^{-\frac{m\omega}{2\hbar}\left[\left(x-\sqrt{\frac{2\hbar}{m\omega}}\,\alpha\right)^2 + \left(x-\sqrt{\frac{2\hbar}{m\omega}}\,\bar{\alpha}\right)^2\right]}$$

$$x^2 - 2\sqrt{\frac{2\hbar}{m\omega}}\,\alpha x + \frac{2\hbar}{m\omega}\alpha^2 + x^2 - 2\sqrt{\frac{2\hbar}{m\omega}}\,\bar{\alpha} x + \frac{2\hbar}{m\omega}\bar{\alpha}^2$$

$$2x^2 - 2\sqrt{\frac{2\hbar}{m\omega}}\,(\alpha+\bar{\alpha})x + \frac{2\hbar}{m\omega}(\alpha^2+\bar{\alpha}^2)$$

$$\alpha = \alpha_1 + i\alpha_2$$

$$|\langle x|\alpha\rangle|^2 = |A|^2\, e^{-\frac{m\omega}{2\hbar}\left[2x^2 - 4\sqrt{\frac{2\hbar}{m\omega}}\,\alpha_1 x + \frac{2\hbar}{m\omega}(\alpha_1^2+\alpha_2^2)\right]}$$

$$= |A|^2\, e^{-\alpha_2^2}\, e^{-\frac{m\omega}{\hbar}\left[x^2 - 2\sqrt{\frac{2\hbar}{m\omega}}\,\alpha_1 x + \frac{2\hbar}{m\omega}\alpha_1^2\right] + \alpha_1^2}$$

$$= |A|^2\, e^{\alpha_1^2-\alpha_2^2}\, e^{-\frac{m\omega}{\hbar}\left(x-\sqrt{\frac{2\hbar}{m\omega}}\,\alpha_1\right)^2}$$

Normalized. $\times\sqrt{\frac{\pi\hbar}{m\omega}} = 1 \Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\alpha_1^2}{2}+\frac{\alpha_2^2}{2}}$

$$|\langle x|\alpha\rangle|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega}{\hbar}\left(x-\sqrt{\frac{2\hbar}{m\omega}}\,\alpha_1\right)^2}$$

$$\langle x|\alpha\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{\alpha_2^2-\alpha_1^2}{2}}\, e^{-\frac{m\omega}{2\hbar}\left(x-\sqrt{\frac{2\hbar}{m\omega}}\,\alpha\right)^2}$$

$\Rightarrow$ up to phase.

Another (algebraic) way to get $\langle x|\alpha\rangle$.

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A[A,B]]-\frac{1}{12}[B[B,A]]+\cdots}$$

if $[A,B] = \text{number}\cdot\mathbb{1}$

$$\Rightarrow e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

$$|\alpha\rangle = e^{-\frac{1}{2}\alpha\alpha^*} e^{\alpha a^\dagger}|0\rangle$$

$$\langle x|\alpha\rangle = e^{-\frac{1}{2}\alpha\alpha^*}\langle x|e^{\alpha a^\dagger}|0\rangle$$

recall $\quad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)$

$$e^{\alpha a}|0\rangle = |0\rangle \Rightarrow \langle x|\alpha\rangle = e^{-\frac{1}{2}\alpha\alpha^*}\langle x|e^{\alpha a^\dagger}e^{\alpha a}|0\rangle$$

$$= e^{-\frac{1}{2}\alpha\alpha^*}\langle x|e^{\alpha(a+a^\dagger)+\frac{1}{2}\alpha^2}|0\rangle =$$

$$= e^{-\frac{1}{2}\alpha^2-\frac{1}{2}\alpha\alpha^*}\langle x|e^{\alpha\sqrt{\frac{2m\omega}{\hbar}}\hat{x}}|0\rangle =$$

$$= e^{-\frac{1}{2}\alpha^2-\frac{1}{2}\alpha\alpha^*} e^{\alpha\sqrt{\frac{2m\omega}{\hbar}}x} \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{\alpha^2}{2}-\frac{\alpha\alpha^*}{2}} e^{\alpha\sqrt{\frac{2m\omega}{\hbar}}x} \cdot e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}$$

$$|\langle x|\alpha\rangle|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}\; e^{-\frac{\alpha^2}{2}-\frac{\alpha^{*2}}{2}-\alpha\alpha^*}\; e^{\sqrt{\frac{2m\omega}{\hbar}}\,x(\alpha+\bar{\alpha})}\; e^{-\frac{m\omega}{\hbar}x^2} \quad (9)$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\; e^{-\frac{m\omega}{\hbar}(x-x_0)^2}$$

$$e^{-\frac{m\omega}{\hbar}x^2 + \frac{2m\omega}{\hbar}x x_0 - \frac{m\omega}{\hbar}x_0^2}$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}}\,\frac{(\alpha+\bar{\alpha})}{2} \quad \checkmark$$

agrees

$$\frac{2m\omega}{\hbar}x_0 = \sqrt{\frac{2m\omega}{\hbar}}\,(\alpha+\bar{\alpha})$$

$$x_0 = \sqrt{\frac{\hbar}{2m\omega}}\,(\alpha+\bar{\alpha})$$

$$\cancel{\frac{m\omega}{\hbar}}\;\cancel{\frac{\hbar}{2m\omega}}\,(\alpha+\bar{\alpha})^2$$

$$= \frac{1}{2}(\alpha^2+2\alpha\bar{\alpha}+\bar{\alpha}^2)$$

$$a|\alpha\rangle = \alpha|\alpha\rangle \rightarrow \langle\alpha|a^\dagger = \langle\alpha|\alpha^*$$

$$\langle\alpha|x|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(\alpha+\bar{\alpha})$$

from espression $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^\dagger)$

$-i\alpha_i - i\alpha_i$

$$\langle\alpha|p|\alpha\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,(\bar{\alpha}-\alpha)$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^\dagger - a)$$

$$\boxed{\begin{aligned}\langle\alpha|x|\alpha\rangle &= \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}(\alpha)\\ \langle\alpha|p|\alpha\rangle &= \sqrt{2m\hbar\omega}\,\mathrm{Im}(\alpha)\end{aligned}}$$

$$|\psi(t)\rangle = e^{-\frac{i\omega t}{2}} e^{-i\omega t a^\dagger a} |\alpha_0\rangle$$

$$= A(t)\, e^{-\frac{i\omega t}{2}} |e^{-i\omega t}\alpha_0\rangle$$

$$= A(t)\, e^{-\frac{i\omega t}{2}} |\alpha(t)\rangle$$

$$\alpha(t) = e^{-i\omega t}\alpha_0$$

<u>$A(t) = ?$</u>

$$\partial_t |\alpha(t)\rangle = \partial_t \left(e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle \right)$$

$|\alpha|^2 = |\alpha_0|^2$ time indep

$$\partial_t |\alpha(t)\rangle = e^{-\frac{1}{2}|\alpha|^2} \dot{\alpha} a^\dagger e^{\alpha a^\dagger}|0\rangle = -i\omega\,\alpha(t)\, a^\dagger |\alpha(t)\rangle$$

$$\partial_t |\psi(t)\rangle = \left(\frac{\dot{A}}{A} - \frac{i\omega}{2} - i\omega\alpha a^\dagger \right) |\psi(t)\rangle$$

$$-\frac{iH}{\hbar}|\psi(t)\rangle = -i\omega\left(a^\dagger a + \tfrac{1}{2}\right)|\psi(t)\rangle = -i\omega\left(\alpha a^\dagger + \tfrac{1}{2}\right)|\psi\rangle$$

$$\Rightarrow \boxed{\dot{A} = 0}$$

$$|\psi(t)\rangle = A\, e^{-\frac{i\omega t}{2}}\, |\alpha(t)\rangle$$

$|A|^2 = 1$ by normalization. $t=0 \to A=1$

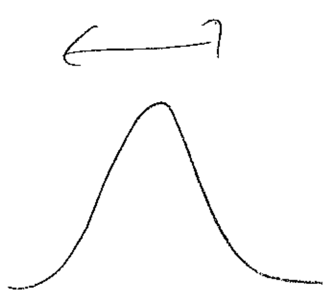
A time indep.

$$\boxed{|\psi(t)\rangle = e^{-\frac{i\omega t}{2}}\, |\alpha(t)\rangle}$$

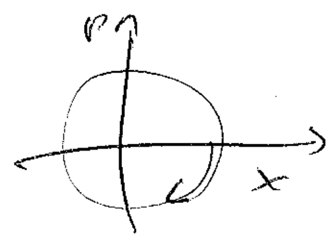
Take $\alpha_0 \in \mathbb{R}$.

$$\alpha = e^{-i\omega t}\alpha_0 = (\cos\omega t - i\sin\omega t)\,\alpha_0$$

$$\mathrm{Re}\,\alpha = \cos\omega t\,\alpha_0 \qquad \mathrm{Im}\,\alpha = -\alpha_0 \sin\omega t$$



oscillates



Recall $\langle x\rangle = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\alpha = \sqrt{\frac{2\hbar}{m\omega}}\,\alpha_0\cos\omega t$

$$\langle p\rangle = \sqrt{2m\hbar\omega}\,\mathrm{Im}\,\alpha = \sqrt{2m\hbar\omega}\,(-\alpha_0)\sin\omega t$$