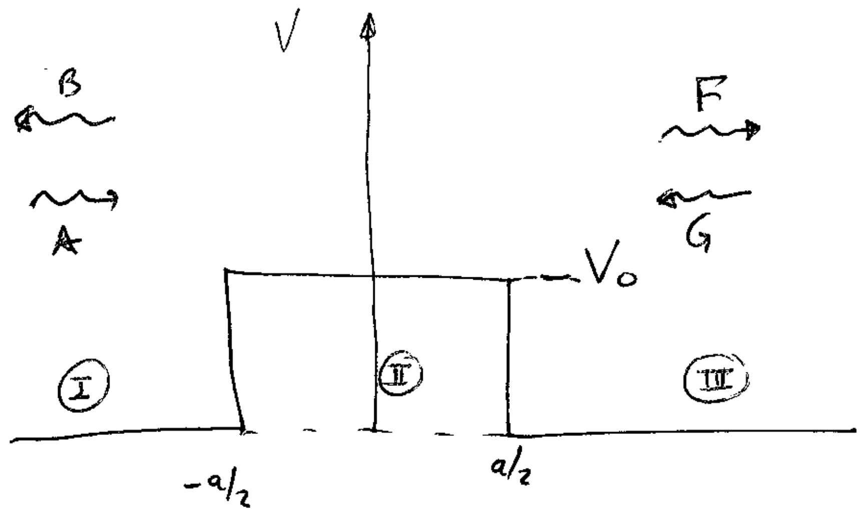

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$K = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\psi_{I} = Ae^{ikx} + Be^{-ikx}$$
$$\psi_{II} = Ce^{iKx} + De^{-iKx}$$
$$\psi_{III} = Fe^{ikx} + Ge^{-ikx}$$

$-a/2$

$$Ae^{-ik\frac{a}{2}} + Be^{+ik\frac{a}{2}} = Ce^{-iK\frac{a}{2}} + De^{iK\frac{a}{2}}$$
$$ikAe^{-ik\frac{a}{2}} - ikBe^{ik\frac{a}{2}} = iKCe^{-iK\frac{a}{2}} - iKDe^{iK\frac{a}{2}}$$

$a/2$

$$Ce^{iK\frac{a}{2}} + De^{-iK\frac{a}{2}} = Fe^{ik\frac{a}{2}} + Ge^{-ik\frac{a}{2}}$$
$$iKCe^{iK\frac{a}{2}} - iKDe^{-iK\frac{a}{2}} = ikFe^{ik\frac{a}{2}} - ikGe^{-ik\frac{a}{2}}$$

2

$$\begin{pmatrix} e^{-iK\frac{a}{2}} & e^{iK\frac{a}{2}} \\ iK e^{-iK\frac{a}{2}} & -iK e^{iK\frac{a}{2}} \end{pmatrix}\begin{pmatrix} C \\ D\end{pmatrix} = \begin{pmatrix} e^{-ik\frac{a}{2}} & e^{ik\frac{a}{2}} \\ ik e^{-ik\frac{a}{2}} & -ik e^{ik\frac{a}{2}}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$$

$\det = -iK - iK = -2iK$

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$$

inverse $= \dfrac{i}{2K}\begin{pmatrix} -iK e^{iK\frac{a}{2}} & -e^{iK\frac{a}{2}} \\ -iK e^{-iK\frac{a}{2}} & e^{-iK\frac{a}{2}}\end{pmatrix}$

$$\begin{pmatrix} C \\ D\end{pmatrix} = \frac{i}{2K}\begin{pmatrix} -iK e^{iK\frac{a}{2}} & -e^{iK\frac{a}{2}} \\ -iK e^{-iK\frac{a}{2}} & e^{-iK\frac{a}{2}}\end{pmatrix}\begin{pmatrix} e^{-ik\frac{a}{2}} & e^{ik\frac{a}{2}} \\ ik e^{-ik\frac{a}{2}} & -ik e^{ik\frac{a}{2}}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$$

$$\underbrace{\begin{pmatrix} e^{ika/2} & e^{-ika/2} \\ ik e^{ika/2} & -ik e^{-ika/2}\end{pmatrix}}\begin{pmatrix} F \\ G\end{pmatrix} = \begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iK e^{iK\frac{a}{2}} & -iK e^{-iK\frac{a}{2}}\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix}$$

$\det = -2ik$

$$\begin{pmatrix} F \\ G\end{pmatrix} = \frac{i}{2k}\begin{pmatrix} -ik e^{-ika/2} & -e^{-ik\frac{a}{2}} \\ -ik e^{ik\frac{a}{2}} & +e^{ik\frac{a}{2}}\end{pmatrix}\begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iK e^{iK\frac{a}{2}} & -iK e^{-iK\frac{a}{2}}\end{pmatrix}\cdot\frac{i}{2K}\begin{pmatrix} -iK e^{iK\frac{a}{2}} & -e^{iK\frac{a}{2}} \\ -iK e^{-iK\frac{a}{2}} & e^{-iK\frac{a}{2}}\end{pmatrix}$$

$$\cdot\begin{pmatrix} e^{-ik\frac{a}{2}} & e^{ik\frac{a}{2}} \\ ik e^{-ik\frac{a}{2}} & -ik e^{ik\frac{a}{2}}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$$

$$\begin{pmatrix} F \\ G\end{pmatrix} = -\frac{1}{4kK}\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} -iK e^{iKa} - iK e^{-iKa} & -e^{iKa} + e^{-iKa} \\ K^2 e^{iKa} - K^2 e^{-iKa} & -iK e^{iKa} - iK e^{-iKa}\end{pmatrix}\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{1}{4kK} \begin{pmatrix} -ik\, e^{-ika/2} & -e^{-\frac{ika}{2}} \\ -ik\, e^{\frac{ika}{2}} & e^{\frac{ika}{2}} \end{pmatrix} \begin{pmatrix} -2iKcKa & -2iska \\ 2iK^2sKa & -2iKcKa \end{pmatrix} \begin{pmatrix} e^{-\frac{ika}{2}} & e^{\frac{ika}{2}} \\ ike^{-\frac{ika}{2}} & -ike^{\frac{ika}{2}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \quad 3$$

$(-)$

$$= \frac{i}{2kK} \begin{pmatrix} -ik\, e^{-\frac{ika}{2}} KcKa + K^2 sKa\, e^{-\frac{ika}{2}} & -ikska\, e^{-\frac{ika}{2}} - Kckae^{-\frac{ika}{2}} \\ -ikKcKa\, e^{\frac{ika}{2}} - K^2 sKa\, e^{\frac{ika}{2}} & -ikska\, e^{\frac{ika}{2}} + Ke^{\frac{ika}{2}} cKa \end{pmatrix} \begin{pmatrix} e^{-\frac{ika}{2}} & e^{\frac{ika}{2}} \\ ike^{-\frac{ika}{2}} & -ike^{\frac{ika}{2}} \end{pmatrix}$$

$$= \frac{i}{2kK} \begin{pmatrix} e^{-ika}\left[-ikKcKa + k^2 sKa + k^2 sKa - ikKcKa\right] & -i\cancel{kKcKa} + K^2 sKa - k^2 sKa + \cancel{ikKcKa} \\ -i\cancel{kKcKa} - K^2 sKa + k^2 sKa + i\cancel{kKcKa} & e^{ika}\left[-ikKcKa - K^2 sKa - k^2 sKa - ikKcKa\right] \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\boxed{\begin{pmatrix} F \\ G \end{pmatrix} = \frac{i}{2kK} \begin{pmatrix} e^{-ika}\left[(k^2+K^2)sKa - 2ikK\, cKa\right] & (K^2-k^2)\, sKa \\ -(K^2-k^2)\, sKa & -e^{ika}\left[(k^2+K^2)\, sKa + 2ikK\, cKa\right] \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = T(E) \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ B \end{pmatrix} = S \begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix}$$

$F = t_{11} A + t_{12} B$ $\qquad$ $B = \frac{1}{t_{22}} G - \frac{t_{21}}{t_{22}} A$

$G = t_{21} A + t_{22} B$ $\qquad$ $F = t_{11} A + \frac{t_{12}}{t_{22}} G - \frac{t_{12} t_{21}}{t_{22}} A$

$$F = \frac{1}{t_{22}} (t_{11} t_{22} - t_{12} t_{21}) A + \frac{t_{12}}{t_{22}} G$$

$$B = -\frac{t_{21}}{t_{22}} A + \frac{1}{t_{22}} G$$

$$\det T(E) = t_{11}t_{22} - t_{12}t_{21} = \frac{(-1)}{4k^2K^2}\left[-\left[(k^2+K^2)^2 s^2Ka + 4k^2K^2 c^2Ka\right] + (K^2-k^2)^2 s^2Ka\right]$$

$$= -\frac{1}{4k^2K^2}\left[-4k^2K^2c^2Ka - \left[(k^2+K^2)^2 - (K^2-k^2)^2\right] s^2Ka\right]$$

$$\left.\begin{matrix} k^4+K^4+2k^2K^2 \\ -(k^4+K^4-2k^2K^2) \end{matrix}\right\} = 4k^2K^2$$

$$= -\frac{1}{4k^2K^2}\left[-4k^2K^2c^2Ka - 4k^2K^2s^2Ka\right] = 1.$$

$$\begin{pmatrix} F \\ B \end{pmatrix} = \underbrace{\frac{1}{t_{22}}\begin{pmatrix} 1 & t_{12} \\ -t_{21} & 1 \end{pmatrix}}_{S}\begin{pmatrix} A \\ G \end{pmatrix}$$

$$S = \frac{2kK}{i}\,\frac{(-)e^{-ika}}{\left[(k^2+K^2)sKa + 2ikKcKa\right]}\begin{pmatrix} 1 & \frac{i(K^2-k^2)}{2kK}sKa \\ \frac{i(K^2-k^2)}{2kK}sKa & 1 \end{pmatrix}$$

$$\boxed{S = \frac{2ikK\,e^{-ika}}{\left[(k^2+K^2)\,sKa + 2ikK\,cKa\right]}\begin{pmatrix} 1 & \frac{i(K^2-k^2)}{2kK}sKa \\ \frac{i(K^2-k^2)}{2kK}sKa & 1 \end{pmatrix}}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad K = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

if $E > V_0$ K is real ; if $0 < E < V_0$ then K purely imaginary.

Reflection and transmission coeff.

$G=0 \Rightarrow$ no incoming wave from the right.

$$T = \frac{|F|^2}{|A|^2} \quad \leftarrow \text{outgoing}$$

↖ incoming

$$R = \frac{|B|^2}{|A|^2} \quad \leftarrow \text{reflected.}$$

$$\begin{pmatrix} F \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$F = A S_{11} \rightarrow T = |S_{11}|^2$

$B = A S_{21} \rightarrow R = |S_{21}|^2$

$\boxed{\text{For } E > V_0 \text{ ; } K \text{ real}}$

$$T = \frac{4k^2K^2}{\left[(k^2+K^2)^2 s^2Ka + 4k^2K^2 \underbrace{c^2Ka}_{1-s^2Ka}\right]} = \frac{1}{1 + \frac{(k^2-K^2)^2}{4k^2K^2} s^2Ka}$$

$$k^2 - K^2 = \frac{2mE}{\hbar^2} - \frac{2m(E-V_0)}{\hbar^2} = \frac{2mV_0}{\hbar^2} = k_0^2$$

$$T = \frac{1}{1 + \frac{k_0^4}{4k^2K^2} s^2Ka}$$

$$R = |S_{21}|^2 = \frac{(K^2-k^2)^2 s^2Ka/4k^2K^2}{1 + \frac{k_0^4}{4k^2K^2} s^2Ka} = \frac{k_0^4 s^2Ka/4k^2K^2}{1 + \frac{k_0^4}{4k^2K^2} s^2Ka} = \frac{k_0^2}{4k^2K^2} \frac{s^2Ka}{\left(1 + \frac{k_0^4}{4k^2K^2} s^2Ka\right)}$$

$\underline{T + R = 1}$

For $0 < E < V_0$ $\rightarrow K$ purely imaginary.

$K = i\eta \qquad \eta \in \mathbb{R}. \qquad \sin(Ka) = i\,\mathrm{Sh}(\eta a) \qquad \cos(Ka) = \mathrm{ch}\,\eta a$

$$S = \frac{-2k\eta\, e^{-ika}}{\left[(k^2-\eta^2)\, i\,\mathrm{Sh}(\eta a) - 2k\eta\,\mathrm{ch}\,\eta a\right]} \begin{pmatrix} 1 & \frac{i(-\eta^2-k^2)}{\cancel{2}\eta k}\cancel{i}\,\mathrm{Sh}\,\eta a \\ -\frac{i(\eta^2+k^2)}{2\eta k}\,\mathrm{Sh}\,\eta a & 1 \end{pmatrix}$$

$$|s_u|^2 = \frac{4k^2\eta^2}{(k^2-\eta^2)^2\,\mathrm{Sh}^2\eta a + 4k^2\eta^2\,\underbrace{\mathrm{ch}^2\eta a}_{1+\mathrm{Sh}^2\eta a}} = \frac{1}{1 + \frac{(k^2+\eta^2)^2}{4k^2\eta^2}\,\mathrm{Sh}^2\eta a}$$

$$\eta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} \qquad k^2+\eta^2 = k_0^2$$

$$\boxed{T = \frac{1}{1 + \frac{k_0^4}{4k^2\eta^2}\,\mathrm{Sh}^2\eta a} \qquad R = \frac{\frac{k_0^4}{4\eta^2k^2}\,\mathrm{Sh}^2\eta a}{1 + \frac{k_0^4}{4k^2\eta^2}\,\mathrm{Sh}^2\eta a}}$$

Symmetries of S-matrix

∘) parity $\quad V(x) = V(-x)$

if $\psi(x)$ solution then $\psi(-x)$ is also a solution.

$$(A, B, F, G) \longrightarrow (G, F, B, A) \qquad \boxed{\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} F \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix} ; \quad \begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} G \\ A \end{pmatrix}$$

$$\sigma_1 \begin{pmatrix} F \\ B \end{pmatrix} = \begin{pmatrix} B \\ F \end{pmatrix} \qquad \sigma_1 \underbrace{\begin{pmatrix} F \\ B \end{pmatrix}}_{S\binom{A}{G}} = S \sigma_1 \begin{pmatrix} A \\ G \end{pmatrix} \Rightarrow \boxed{\sigma_1 S = S \sigma_1}$$

$$\boxed{S = \sigma_1 S \sigma_1} \text{ parity} \qquad \sigma_1 S \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} S_{21} & S_{22} \\ S_{11} & S_{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{pmatrix}$$

$$\boxed{S_{11} = S_{22}}$$

$$\boxed{S_{12} = S_{21}}$$

checks ✓.

∘) Time reversal. $\quad V(x) = V^*(x)$

if $\psi(x)$ solution also $\psi^*(x)$ is solution

$$(A\, B\, F\, G) \rightarrow (B^* A^* G^* F^*)$$

$$\begin{pmatrix} F \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix} \qquad \begin{pmatrix} G^* \\ A^* \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} B^* \\ F^* \end{pmatrix}$$

$$\begin{pmatrix} B^* \\ F^* \end{pmatrix} = S^{-1} \begin{pmatrix} G^* \\ A^* \end{pmatrix}$$

$$\sigma_1 \begin{pmatrix} F^* \\ B^* \end{pmatrix} = S^{-1} \sigma_1 \begin{pmatrix} A^* \\ G^* \end{pmatrix}$$

$$\sigma_1 S^* \begin{pmatrix} A^* \\ G^* \end{pmatrix} = S^{-1} \sigma_1 \begin{pmatrix} A^* \\ G^* \end{pmatrix}$$

$$\Rightarrow \boxed{\sigma_1 S^* \sigma_1 = S^{-1}}$$ time-reversal. (check later).

•) Unitarity (prob. conservation).

$$|A|^2 - |B|^2 = |F|^2 - |G|^2 \Rightarrow |A|^2 + |G|^2 = |F|^2 + |B|^2$$

S preserves norm.

$$S S^\dagger = 1$$

Assuming unitarity $S^{-1} = S^\dagger$

Time reversal $\rightarrow$ $\sigma_1 S^* \sigma_1 = S^\dagger = (S^t)^* \Rightarrow \boxed{\sigma_1 S \sigma_1 = S^t}$

$$\begin{pmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix} \Rightarrow \boxed{S_{11} = S_{22}}$$ time reversal verified. ✓.

We still have to check unitarity:

$$S S^{+} = 1$$

$\boxed{E > V_0}$

$$S S^{+} = \frac{4k^2K^2}{\left[(k^2+K^2)^2 s^2Ka + 4k^2K^2 \underbrace{c^2Ka}_{1-s^2Ka}\right]} \begin{pmatrix} 1 & i\frac{(K^2-k^2)}{2kK} sKa \\ i\frac{(K^2-k^2)}{2kK} sKa & 1 \end{pmatrix} \begin{pmatrix} 1 & -i\frac{(K^2-k^2)}{2kK} sKa \\ -i\frac{(K^2-k^2)}{2kK} sKa & 1 \end{pmatrix}$$

$$= \frac{4k^2K^2}{\left[4k^2K^2 + (k^2-K^2)^2 s^2Ka\right]} \begin{pmatrix} 1 + \frac{(K^2-k^2)^2}{4k^2K^2} s^2Ka & 0 \\ 0 & 1 + \frac{(K^2-k)^2}{4k^2K^2} s^2Ka \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark .$$

if $\boxed{E < V_0}$

$$S S^{+} = \frac{1}{1 + \frac{k_0^4}{4k^2\eta^2} sh^2\eta a} \begin{pmatrix} 1 & -\frac{ik_0^2}{2\eta k} sh\,\eta a \\ -\frac{ik_0^2 sh\,\eta a}{2\eta k} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{ik_0^2}{2\eta k} sh\,\eta a \\ \frac{ik_0^2 sh\,\eta a}{2\eta k} & 1 \end{pmatrix}$$

$$= \frac{1}{1 + \frac{k_0^4}{4k^2\eta^2} sh^2\eta a} \begin{pmatrix} 1 + \frac{k_0^4}{4\eta^2k^2} sh^2\eta a & 0 \\ 0 & 1 + \frac{k_0^4}{4\eta^2k^2} sh^2\eta a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Plotting $T(E)$

$0 < E < V_0$ $$T = \frac{1}{1 + \frac{k_0^4}{4k^2\eta^2}\,\mathrm{sh}^2\eta a}$$

$V_0 < E < \infty$ $$T = \frac{1}{1 + \frac{k_0^4}{4k^2K^2}\,s^2 K a.}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \eta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} \qquad K = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \qquad k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$$

Resonances $\underline{T=1}$ $\Rightarrow$ $\boxed{Ka = n\pi}$ except $n=0$

$\mathrm{sh}\,\eta a \neq 0$ except for $\eta = 0$ i.e. $E = V_0$ $\underline{n=0}$.

$\boxed{E=0}$ $\eta = k_0$ $$T = \frac{1}{1 + \underbrace{\frac{k_0^4}{4k^2k_0^2}}_{k\to 0}\,\mathrm{sh}^2 k_0 a} = 0$$

$T(E=0) = 0$

$$T(E=V_0) = \frac{1}{1 + \frac{k_0^4}{4k^2K^2}K^2a^2} = \frac{1}{1 + \frac{1}{4}k_0^2a^2}$$

$$\frac{k^2}{k_0^2} = \frac{E}{V_0} \qquad \frac{k^2}{k_0^2} = \frac{E-V_0}{V_0} = \frac{E}{V_0} - 1$$

$$T = \frac{1}{1 + \frac{1}{4\frac{E}{V_0}\left(\frac{E}{V_0}-1\right)} S^2\left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\, a\right)}$$

$$\varepsilon = E/V_0 \qquad Ka = \underbrace{\sqrt{\frac{2ma^2V_0}{\hbar^2}}}_{a_0}\sqrt{(\varepsilon-1)} = a_0\sqrt{\varepsilon-1}$$

$$\boxed{T(\varepsilon) = \frac{1}{1 + \frac{1}{4\varepsilon(\varepsilon-1)}\sin^2\left(a_0\sqrt{\varepsilon-1}\right)}}$$

We only need to choose $a_0 = \underbrace{\sqrt{\frac{2ma^2V_0}{\hbar^2}}}_{= k_0 a}$ to make a plot.

$$T(0) = 0 \qquad T(\varepsilon=1) = \frac{1}{1+\frac{1}{4}k_0^2a^2} = \frac{1}{1+a_0^2/4}$$

$a_0 = 20$ has nice peaks

Problem 2

i) $|k\rangle = \sum e^{ikj} |j\rangle = \sum a_j |j\rangle$

$\langle k | n^{(i)} | k \rangle = |a_i|^2 = 1$

Normalizing: (length N)

$$\boxed{\langle k | n^{(i)} | k \rangle = \frac{1}{N}}$$

ii)

$C_{ij} = \langle \psi | n^{(i)} n^{(j)} | \psi \rangle = |a_{ij}|^2$

2 particle states.

$C_{ij} = 4 \cos^2 \left(\frac{\theta}{2} + q\,|i_2 - i_1| \right)$ unbound state.

$C_{ij} = e^{-2\eta |i_2 - i_1|}$ for bound state.

Problem 3

$$-\frac{\hbar^2}{2m}\partial_x^2\psi + \lambda x^4\psi = E\psi$$

$$x = a\xi \rightarrow -\frac{\hbar^2}{2ma^2}\partial_\xi^2\psi + \lambda a^4\xi^4\psi = E\psi$$

$$-\partial_\xi^2\psi + \underbrace{\frac{2m\lambda a^6}{\hbar^2}}_{1}\xi^4\psi = \underbrace{\frac{2ma^2E}{\hbar^2}}_{\varepsilon}\psi$$

$$\boxed{a^6 = \frac{\hbar^2}{2m\lambda}} \qquad \boxed{E = \frac{\hbar^2}{2ma^2}\varepsilon}$$

$$[\lambda] = \frac{\text{MeV}}{L^4} \qquad a^6 = \frac{\hbar^2c^2}{2mc^2\lambda} \rightarrow \frac{\cancel{\text{MeV}}^2\,\text{fm}^2}{\cancel{\text{MeV}}\ \frac{\cancel{\text{MeV}}}{\text{fm}^4}} \sim \text{fm}^6 \checkmark$$

$$E = \frac{\hbar^2c^2}{2mc^2a^2}\varepsilon \rightarrow \frac{\text{MeV}^{\cancel{2}}\,\cancel{\text{fm}^2}}{\cancel{\text{MeV}}\,\cancel{\text{fm}^2}} = \text{MeV} \checkmark$$

We need to solve $\boxed{-\partial_\xi^2\psi + \xi^4\psi = \varepsilon\psi}$

ground state $\varepsilon = 1.06036$

other states 7.4557 303.9 (even states)

16.2618

$$\psi = A e^{-\alpha x^2} = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2}$$

$$\int |\psi|^2 = \int |A|^2 e^{-2\alpha x^2} = |A|^2 \sqrt{\frac{\pi}{2\alpha}}$$

$$\langle\psi|H|\psi\rangle = \int \psi^* \left(-\frac{\hbar^2}{2m}\partial_x^2 + \lambda x^4\right)\psi$$

$$= a\int_{-\infty}^{\infty} \sqrt{\frac{2\alpha}{\pi}}\, e^{-\alpha a^2\xi^2}\left(-\frac{\hbar^2}{2ma^2}\partial_\xi^2 + \lambda a^4\xi^4\right) e^{-\alpha a^2 \xi^2}\, d\xi$$

$x = a\xi$

$$= \partial_\xi^2 \left(e^{-\alpha a^2\xi^2}\right) = \partial_\xi\left(-2\alpha a^2 \xi\, e^{-\alpha a^2\xi^2}\right) =$$

$$= \left(-2\alpha a^2 e^{-\alpha a^2 \xi^2} + 4\alpha^2 a^4 \xi^2 e^{-\alpha a^2\xi^2}\right)$$

$$\langle\psi|H|\psi\rangle = a\sqrt{\frac{2\alpha}{\pi}}\int_{-\infty}^{\infty} d\xi\, e^{-2\alpha a^2\xi^2}\left(-\frac{\hbar^2}{2ma^2}\left(-2\alpha a^2 + 4\alpha^2 a^4\xi^2\right) + \lambda a^4 \xi^4\right)$$

$\frac{2m\lambda a^6}{\hbar^2} = 1.$

$$\frac{2ma^2}{\hbar^2}\langle H\rangle = a\sqrt{\frac{2\alpha}{\pi}}\int d\xi\, e^{-2\alpha a^2\xi^2}\left(2\alpha a^2 - 4\alpha^2 a^4 \xi^2 + \xi^4\right)$$

$u = \sqrt{2\alpha a^2}\,\xi$ $\Big|$ $= a\sqrt{\frac{2\alpha}{\pi}}\frac{1}{\sqrt{2\alpha a^2}}\int_{-\infty}^{\infty} du\, e^{-u^2}\left(2\alpha a^2 - \frac{4\alpha^2 a^4 u^2}{2\alpha a^2} + \frac{u^4}{4\alpha^2 a^4}\right)$

(3)

$$\int_{-\infty}^{\infty} e^{-bx^2} = \sqrt{\pi}\, b^{-1/2} \quad ; \quad -\partial_b \to \quad \int_{-\infty}^{\infty} x^2 e^{-bx^2} = \frac{1}{2}\sqrt{\pi}\, b^{-3/2}$$

$$-\partial_b \to \quad \int_{-\infty}^{\infty} x^4 e^{-bx^2} = \frac{3}{4}\sqrt{\pi}\, b^{-5/2}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \; ; \quad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}\sqrt{\pi} \; ; \quad \int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \frac{3}{4}\sqrt{\pi}$$

$$\frac{2ma^2}{\hbar^2}\langle H\rangle = \frac{\sqrt{\pi}}{\sqrt{\pi}}\left[2\alpha a^2 - 2\alpha a^2 \frac{1}{2} + \frac{3}{4}\,\frac{1}{4\alpha^2 a^4}\right]$$

$$= \alpha a^2 + \frac{3}{16}\,\frac{1}{\alpha^2 a^4}$$

$$\mathcal{E}(\alpha) = \alpha a^2 + \frac{3}{16}\,\frac{1}{\alpha^2 a^4}$$

$$\frac{\partial \mathcal{E}}{\partial \alpha} = a^2 - \frac{3}{8\alpha^3 a^4} = 0 \qquad \frac{3}{8\alpha^3} = a^6 \Rightarrow \boxed{\alpha = \frac{3^{1/3}}{2a^2}}$$

$$\mathcal{E}(\alpha) = \frac{3^{1/3}}{2} + \frac{3}{16}\,\frac{1}{\frac{3^{2/3}}{4}} = \frac{3^{1/3}}{2} + \frac{3^{1/3}}{4} = \frac{3\times 3^{1/3}}{4} = \frac{3^{4/3}}{4} \simeq \boxed{1.082}$$

$\boxed{\mathcal{E}_{min} = 1.082}$ to be compared with $\boxed{1.06036}$