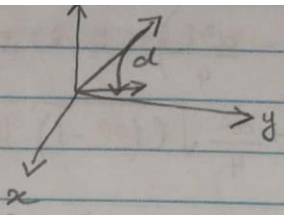
Problem 1-4: by Daniel, John

1. $|l, l_z = l\rangle$



$$\hat{R} = e^{-\frac{i}{\hbar}\alpha l_y}$$

$$\simeq 1 - \frac{i}{\hbar}\alpha l_y + \frac{1}{2}\left(\frac{i}{\hbar}\alpha\right)^2 l_y^2 + \cdots = 1 - \frac{i}{\hbar}\alpha l_y - \frac{1}{\hbar^2}\frac{\alpha^2}{2} l_y^2 + \cdots$$

In terms of raising & lowering operators, $l_y = \dfrac{l_+ - l_-}{2i}$

$$\therefore \hat{R} \simeq 1 - \frac{\alpha}{2\hbar}(l_+ - l_-) + \frac{\alpha^2}{8\hbar^2}(l_+^2 + l_-^2 - l_+ l_- - l_- l_+) + \cdots$$

we ignore terms of order higher than α^2

$$\therefore \hat{R}|l, l_z = l\rangle = |l, l_z = l\rangle - \frac{\alpha}{2\hbar}(l_+ - l_-)|l, l_z = l\rangle$$

$$+ \frac{\alpha^2}{8\hbar^2}(l_+^2 + l_-^2 - l_+ l_- - l_- l_+)|l, l_z = l\rangle$$

$$\rightarrow l_+|l, l_z = l\rangle = 0$$

$$\therefore \hat{R}|l, l_z = l\rangle = |l, l_z = l\rangle + \frac{\alpha}{2\hbar}\hat{l}_-|l, l_z = l\rangle + \frac{\alpha^2}{8\hbar^2}(\hat{l}_-^2 - l_+ l_-)|l, l_z = l\rangle$$

$$l_\pm|l\, l_z\rangle = \sqrt{l(l+1) - l_z(l_z \pm 1)}\,\hbar\,|l, l_z \pm 1\rangle$$

equ. 3.5.39 in the book.

← multiplied by $\hbar^2$. The professor uses $\hat{L} = \hbar\hat{l}$

$$\therefore \hat{R}|l, l_z = l\rangle = |l, l_z = l\rangle + \frac{\alpha}{2\hbar}\hbar\sqrt{l(l+1) - l(l-1)}\,|l, l_z = l-1\rangle$$

$$+ \frac{\alpha^2}{8\hbar^2}\left(\hbar^2\sqrt{l(l+1) - l(l-1)}\sqrt{l(l+1) - (l-1)(l-2)}\right)|l, l_z = l-2\rangle$$

$$- \frac{\alpha^2}{8\hbar^2}\left(\hbar^2\sqrt{l(l+1) - l(l-1)}\sqrt{l(l+1) - (l-1)(l)}\right)|l, l_z = l\rangle$$

$$= |l, l_z = l\rangle + \frac{\alpha}{2}\sqrt{2l}\,|l, l_z = l-1\rangle + \frac{\alpha^2}{8}\sqrt{2l}\sqrt{4l - 2}\,|l, l_z = l-2\rangle$$

$$- \frac{\alpha^2}{8}\sqrt{2l}\sqrt{2l}\,|l, l_z = l\rangle$$

$$= \left(1 - \frac{\alpha^2}{8}\cdot 2l\right)|l, l_z = l\rangle + \frac{\alpha\sqrt{2l}}{2}|l, l_z = l-1\rangle + \frac{\alpha^2}{4}\sqrt{l(2l-1)}\,|l, l_z = l-2\rangle$$

$$\therefore \hat{R}|\ell, \ell_z = \ell\rangle = \left(1 - \frac{\alpha^2 \ell}{4}\right)|\ell, \ell_z = \ell\rangle + \frac{\alpha}{2}\sqrt{2\ell}\,|\ell, \ell_z = \ell - 1\rangle$$
$$+ \frac{\alpha^2}{4}\sqrt{\ell(2\ell - 1)}\,|\ell, \ell_z = \ell - 2\rangle$$

$\therefore$ The probability of measuring $\ell_z = \ell$ in the new, rotated state, is:

$$\left|\langle \ell, \ell_z = \ell|\hat{R}|\ell, \ell_z = \ell\rangle\right|^2 = \left(1 - \frac{\alpha^2}{4}\ell\right)^2$$
$$= 1 - \frac{\alpha^2}{2}\ell + \frac{\alpha^4}{16}\ell^2$$

$$P \simeq 1 - \frac{\alpha^2}{2}\ell \quad \text{as } \mathcal{O}(\alpha^4) \simeq 0$$

The probability of measuring $\ell_z = \ell$ in the rotated state
$\rightarrow$ We can see as $\alpha \rightarrow 0$, $P \rightarrow 1$ (as we expect)

$$\Psi(\underline{r}) = (x + y + 3z)f(r)$$

a) Note, the $\ell = 1$ spherical harmonics:

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{r\cos\theta}{r} = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

$$\therefore z = r\sqrt{\frac{4\pi}{3}}\,Y_1^0$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \quad \& \quad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\, e^{-i\phi}\sin\theta$$

$$\therefore Y_1^{-1} - Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin\theta\,\underbrace{\left(e^{-i\phi} + e^{i\phi}\right)}_{2\cos\phi} = \sqrt{\frac{3}{2\pi}}\sin\theta\cos\phi$$
$$= \frac{1}{r}\sqrt{\frac{3}{2\pi}}\,\underbrace{r\sin\theta\cos\phi}_{x} = \sqrt{\frac{3}{2\pi}}\frac{x}{r}$$

$$\therefore x = r\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^1\right)$$

$$\& \quad Y_1^{-1} + Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\left(e^{i\phi} - e^{-i\phi}\right) = -\sqrt{\frac{3}{8\pi}}\sin\theta \cdot 2i\sin\phi$$

$$\therefore Y_1^{-1} + Y_1^{1} = -2i\sqrt{\frac{3}{8\pi}}\sin\theta\sin\phi = -i\sqrt{\frac{3}{2\pi}}\,\overbrace{\frac{r\sin\theta\sin\phi}{r}}^{y}$$

$$\therefore y = ir\sqrt{\frac{2\pi}{3}}\left(Y_1^{1} + Y_1^{-1}\right)$$

Therefore, we can re-write $\psi(r)$ as:

$$\psi(\underline{r}) = \left(r\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{1}\right) + ir\sqrt{\frac{2\pi}{3}}\left(Y_1^{1} + Y_1^{-1}\right) + 3r\sqrt{\frac{4\pi}{3}}\,Y_1^{0}\right)f(r)$$

$$\psi(\underline{r}) = \sqrt{\frac{2\pi}{3}}\,r f(r)\left[\left(Y_1^{-1} - Y_1^{1}\right) + i\left(Y_1^{1} + Y_1^{-1}\right) + 3\sqrt{2}\,Y_1^{0}\right]$$

$\Rightarrow$ Therefore $\psi(\underline{r})$ is an eigenfunction of L^2 with the eigenvalue of $\hbar^2 l(l+1)$ where $\psi(\underline{r})$ is a linear combination of $l=1$ eigenstates $\therefore$ eigenvalues of $2\hbar^2$

b) $$\psi(\underline{r}) = \overbrace{\chi(r)}^{=rf(r)}\sqrt{\frac{2\pi}{3}}\left[(1+i)Y_1^{-1} + (i-1)Y_1^{1} + 3\sqrt{2}\,Y_1^{0}\right]$$

$$\therefore P_l = \frac{|c_l|^2}{\sum_{l=-1}^{+1}|c_l|^2} = \frac{|c_l|^2}{\frac{2\pi}{3}(2+2+18)} = \frac{|c_l|^2}{\frac{4\pi}{3}(11)}$$

$$\therefore P_{l_z=1} = \frac{|c_{l_z=1}|^2}{\frac{4\pi}{3}(11)} = \frac{\frac{2\pi}{3}\cdot 2}{\frac{4\pi}{3}(11)} = \frac{1}{11}$$

$$P_{l_z=0} = \frac{\frac{2\pi}{3}\cdot 3^2\cdot 2}{\frac{4\pi}{3}(11)} = \frac{9}{11}$$

$$\sum_{l_z=-1}^{+1} P_{l_z} = 1 \checkmark$$

$$P_{l_z=-1} = \frac{|c_{l_z=-1}|^2}{\frac{4\pi}{3}(11)} = \frac{2\pi/3\cdot 2}{\frac{4\pi}{3}(11)} = \frac{1}{11}$$

c) If Ψ is an energy eigenfunction, then we can write this as:

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\Psi = E\Psi$$

The Laplacian operator can then be written as:

$$\nabla^2 = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2}\left[\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right)\right]$$

$$= \frac{1}{r^2}\left[\partial_r\left(r^2\partial_r\right) - \frac{\underline{L}^2}{\hbar^2}\right]$$

$\therefore$ The Schrödinger equation becomes:

$$\left(\frac{-\hbar^2}{2mr^2}\partial_r\left(r^2\partial_r\right) + \frac{\underline{L}^2}{2mr^2} + V(\underline{r})\right)\Psi(\underline{r}) = E\Psi(\underline{r})$$

$\Rightarrow$ Here, $\underline{L}^2\Psi = \ell(\ell+1)\hbar^2\Psi = 2\hbar^2\Psi$ as $\ell = 1$

$$\therefore \left(\frac{-\hbar^2}{2mr^2}\partial_r\left(r^2\partial_r\right) + \frac{\hbar^2}{mr^2} + V(\underline{r})\right)\underbrace{\Psi(\underline{r})}_{= rf(r) = R(r)} = E\Psi(\underline{r})$$

$$\therefore V(\underline{r})\Psi(\underline{r}) = \left[E - \frac{\hbar^2}{mr^2}\right]\Psi(\underline{r}) + \frac{\hbar^2}{2mr^2}\partial_r\left(r^2\partial_r\Psi\right)$$

$$\boxed{V(\underline{r}) = \left[E - \frac{\hbar^2}{mr^2}\right] + \frac{\hbar^2}{2mr^2}\frac{1}{\Psi(\underline{r})}\left[2r\frac{\partial\Psi}{\partial r} + r^2\frac{\partial^2\Psi}{\partial r^2}\right]}$$

Using this, we could find $V(\underline{r})$

3. $|l=2, l_z=0\rangle$ is rotated by β about the y-axis

$$\Rightarrow e^{-iL_y\beta/\hbar}\,|l=2, l_z=0\rangle$$

The probability of measuring $l_z = l_z$ is:

$$P = \left|\underbrace{\langle l=2, l_z=l_z|\, e^{-iL_y\beta/\hbar}\, |l=2, l_z=0\rangle}\right|^2$$

$$D^{(2)}_{m0}(\alpha, \beta, \gamma=0) = \sqrt{\frac{4\pi}{2l+1}}\, Y_l^{m*}(\beta,\alpha) \quad \left(\text{Sakurai Eq. } 3.6.52\right)$$

$$\Rightarrow D^{(2)}_{2,0}(\alpha=0,\beta,\gamma=0) = \sqrt{\frac{4\pi}{5}}\, Y_2^{2*}(\beta,0) = \sqrt{\frac{4\pi}{5}}\sqrt{\frac{15}{32\pi}}\sin^2\beta$$

$$= \sqrt{\frac{3}{8}}\sin^2\beta$$

$$\Rightarrow D^{(2)}_{1,0}(\alpha=0,\beta,\gamma=0) = \sqrt{\frac{4\pi}{5}}\, Y_2^{1*}(\beta,0) = \sqrt{\frac{4\pi}{5}}\left[-\sqrt{\frac{15}{8\pi}}(\sin\beta\cos\beta)\right]$$

$$= -\sqrt{\frac{3}{2}}(\sin\beta\cos\beta)$$

$$\Rightarrow D^{(2)}_{0,0}(\alpha=0,\beta,\gamma=0) = \sqrt{\frac{4\pi}{5}}\, Y_2^{0*}(\beta,0) = \sqrt{\frac{4\pi}{5}}\sqrt{\frac{5}{16\pi}}(3\cos^2\beta-1)$$

$$= \frac{1}{2}(3\cos^2\beta-1)$$

$$\Rightarrow D^{(2)}_{-1,0}(\alpha=0,\beta,\gamma=0) = -D^{(2)}_{1,0}(\alpha=0,\beta,\gamma=0)$$

$$\Rightarrow D^{(2)}_{-2,0}(\alpha=0,\beta,\gamma=0) = D^{(2)}_{2,0}(\alpha=0,\beta,\gamma=0)$$

$$\therefore \boxed{\begin{aligned} P(l_z=\pm 2) &= |D^{(2)}_{\pm 2,0}(\alpha=0,\beta,\gamma=0)|^2 = \frac{3}{8}\sin^4\beta \\ P(l_z=\pm 1) &= |D^{(2)}_{\pm 1,0}(\alpha=0,\beta,\gamma=0)|^2 = \frac{3}{2}\sin^2\beta\cos^2\beta \\ P(l_z=0) &= |D^{(2)}_{0,0}(\alpha=0,\beta,\gamma=0)|^2 = \frac{1}{4}(3\cos^2\beta-1)^2 \end{aligned}}$$

$\uparrow\downarrow \quad \uparrow\rightarrow \quad \uparrow\uparrow$

4. $l_1 = 1$ & $l_2 = 1$ $\therefore$ $l_T = 0, 1, 2$

$\Rightarrow$ There are $(2l_1+1)\times(2l_2+1)$ possible states $\therefore$ 9 possible states

$\rightarrow$ Firstly, the only way to get $|l_T = 2, l_z = 2\rangle$ is if $l_{1z} = +1$ & $l_{2z} = +1$

Therefore: $|l_T = 2, l_z = 2\rangle = |l_{1z} = 1, l_{2z} = 1\rangle$ (1)

$\rightarrow$ Similarly, the only way to get $|l_T = 2, l_z = -2\rangle$ is with $l_{1z} = -1$ & $l_{2z} = -1$

Therefore: $|l_T = 2, l_z = -2\rangle \; |l_{1z} = -1, l_{2z} = -1\rangle$ (2)

$\rightarrow$ From the $|l_T = 2, l_z = 2\rangle$ to find the other l_z states.

$\rightarrow$ As before: $l_- |l_T, l_z\rangle = \hbar\sqrt{l(l+1) - l_z(l_z - 1)}\, |l_T, l_z - 1\rangle$

$$\therefore \; l_- |l_T = 2, l_z = 2\rangle = (l_{1-} + l_{2-}) |l_{1z} = 1, l_{2z} = 1\rangle$$

$$\hbar\sqrt{(6-2)}\, |l_T = 2, l_z = 1\rangle = \hbar\sqrt{(2-0)}\, |l_{1z} = 0, 1\rangle + \hbar\sqrt{(2-0)}\, |l_{1z} = 1, l_{2z} = 0\rangle$$

$$2\hbar |2, 1\rangle = \sqrt{2}\hbar |0, 1\rangle + \sqrt{2}\hbar |1, 0\rangle$$

(3) $$\therefore |l_T = 2, l_z = 1\rangle = \frac{1}{\sqrt{2}} |l_{1z} = 0, l_{2z} = 1\rangle + \frac{1}{\sqrt{2}} |l_{1z} = 1, l_{2z} = 0\rangle$$

Again: $$l_- |l_T = 2, l_z = 1\rangle = (l_{1-} + l_{2-}) \frac{1}{\sqrt{2}} |l_{1z} = 0, l_{2z} = 1\rangle + (l_{1-} + l_{2-}) \frac{1}{\sqrt{2}} |l_{1z} = 1, l_{2z} = 0\rangle$$

$$= \hbar\sqrt{6-0}\, |l_T = 2, l_z = 0\rangle = \frac{\hbar}{\sqrt{2}}\sqrt{2}\, |l_{1z} = -1, l_{2z} = +1\rangle + \frac{\hbar}{\sqrt{2}}\sqrt{2}\, |l_{1z} = 0, l_{2z} = 0\rangle + \frac{\hbar}{\sqrt{2}}\sqrt{2}\, |l_{1z} = 0, l_{2z} = 0\rangle + \frac{\hbar}{\sqrt{2}}\sqrt{2}\, |l_{1z} = 1, l_{2z} = -1\rangle$$

(4) $$\therefore |l_T = 2, l_z = 0\rangle = \frac{1}{\sqrt{6}} \Big(|l_{1z} = -1, l_{2z} = +1\rangle + 2|l_{1z} = 0, l_{2z} = 0\rangle + |l_{1z} = 1, l_{2z} = -1\rangle \Big)$$

Again: $l_- |l_T = 2, l_z = 0\rangle = (l_{1-} + l_{2-}) \frac{1}{\sqrt{6}} \left(|-1,+1\rangle + |+1,-1\rangle + 2|0,0\rangle \right)$

→ Similarly we find:

(5) $\therefore \boxed{|l_T = 2, l_z = -1\rangle = \frac{1}{\sqrt{2}} \left(|l_{1z} = -1, l_{2z} = 0\rangle + |l_{1z} = 0, l_{2z} = -1\rangle \right)}$

Now, to get $|l_T = 1, l_z = 1\rangle$, we can use orthogonality:

$\therefore |l_T = 1, l_z = 1\rangle = A|l_{1z} = 1, l_{2z} = 0\rangle + B|l_{1z} = 0, l_{2z} = 1\rangle$

& $|A|^2 + |B|^2 = 1$

→ Also, from orthogonality: $\langle l_T = 2, l_z = 1 | l_T = 1, l_z = 1\rangle = 0$

$\therefore 0 = \frac{1}{\sqrt{2}} (A + B) \Rightarrow A = -B$

⇒ & so $A = \frac{1}{\sqrt{2}}$ & $B = -\frac{1}{\sqrt{2}}$ ∴ we have:

(6) $\boxed{|l_T = 1, l_z = 1\rangle = \frac{1}{\sqrt{2}} \left(|l_{1z} = 1, l_{2z} = 0\rangle - |l_{1z} = 0, l_{2z} = 1\rangle \right)}$

We can again use the lowering operator to get the other $l_T = 1$ states:

$l_- |l_T = 1, l_z = 1\rangle = (l_{1-} + l_{2-}) \frac{1}{\sqrt{2}} \left(|l_{1z} = 1, l_{2z} = 0\rangle - |l_{1z} = 0, l_{2z} = 1\rangle \right)$

$\hbar \sqrt{2 - 0}\, |l_T, l_z = 0\rangle =$

$= \frac{\hbar}{\sqrt{2}} \cdot \sqrt{2} \left[|0,0\rangle + |1,-1\rangle - |-1,+1\rangle - |0,0\rangle \right]$

(7) $\therefore \boxed{|l_T = 1, l_z = 0\rangle = \frac{1}{\sqrt{2}} \left(|l_{1z} = 1, l_{2z} = -1\rangle - |l_{1z} = -1, l_{2z} = +1\rangle \right)}$

Again, use the lowering operator:

$l_- |l_T = 1, l_z = 0\rangle = (l_{1-} + l_{2-}) \frac{1}{\sqrt{2}} \left(|+1,-1\rangle - |-1,+1\rangle \right)$

Again: $\boxed{|l_T = 1, l_z = -1\rangle = \frac{1}{\sqrt{2}} \left(|l_{1z} = 0, l_{2z} = -1\rangle - |l_{1z} = -1, l_{2z} = 0\rangle \right)}$

(8)

Finally we need to find the $|l_T=0, l_z=0\rangle$ state. To do this, we know this state is orthogonal to the other 8 eigenstates & that it is a linear sum of $|1,-1\rangle, |-,1\rangle$ and $|0,0\rangle$ states:

$$|l_T=0, l_z=0\rangle = A|l_{1z}=1, l_{2z}=-1\rangle + B|l_{1z}=-1, l_{2z}=+1\rangle + C|l_{1z}=0, l_{2z}=0\rangle$$

& $|A|^2+|B|^2+|C|^2=1$

→ Also from orthogonality:

$$\Rightarrow \langle 2,0|0,0\rangle = 0 = \frac{A}{\sqrt{6}} + \frac{B}{\sqrt{6}} + \frac{2C}{\sqrt{6}}$$

$$\Rightarrow \langle 1,0|0,0\rangle = 0 = \frac{1}{\sqrt{2}}(A-B) \Rightarrow A=B$$

$$\& \therefore \frac{A}{\sqrt{6}} + \frac{A}{\sqrt{6}} + \frac{2C}{\sqrt{6}} = 0 \Rightarrow A=-C$$

$$\therefore |A|^2+|A|^2+|A|^2=1 \quad \therefore A = \frac{1}{\sqrt{3}} = B \ \& \ C = \frac{-1}{\sqrt{3}}$$

Therefore, we can write $|l_T=0, l_z=0\rangle$ as:

$$|l_T=0, l_z=0\rangle = \frac{1}{\sqrt{3}}\left(|l_{1z}=1, l_{2z}=-1\rangle + |l_{1z}=-1, l_{2z}=+1\rangle - |l_{1z}=0, l_{2z}=0\rangle\right)$$

(9) $$|l_T=0, l_z=0\rangle = \frac{1}{\sqrt{3}}\left(|l_{1z}=1, l_{2z}=-1\rangle + |l_{1z}=-1, l_{2z}=+1\rangle - |l_{1z}=0, l_{2z}=0\rangle\right)$$

↑ These are the 9 $|l_T, l_z\rangle$ eigenstates in terms of the basis states $|l_{1z}, l_{2z}\rangle$.

Problem 1-4: by Feng, Yicheng

PHYS 660 HW5 Yicheng Feng 0030193826

1. a) Quantum rotate operator: $\hat{D}(\vec{n},\phi) = \exp\left(-i\phi \frac{\vec{n}\cdot\hat{\vec{J}}}{\hbar}\right)$

Here, $\vec{J}$ is the space angular momentum operator $\hat{\vec{L}}$

$\vec{n}$ is along y-axis, $\vec{n} = \hat{y}$

$\phi = \alpha \ll 1$

$\therefore \hat{D}(\hat{y},\phi_\alpha) = \exp\left(-i\alpha\frac{\hat{L}_y}{\hbar}\right) \approx 1 - i\alpha\frac{\hat{L}_y}{\hbar} - \frac{1}{2}\alpha^2\frac{\hat{L}_y^2}{\hbar^2} + \cdots \quad (\because \alpha \ll 1)$

$\because \hat{L}_j = \hbar\hat{l}_j \quad \therefore \hat{D}(\hat{y},\alpha) = 1 - i\alpha\hat{l}_y - \frac{1}{2}\alpha^2\hat{l}_y^2 + \cdots$

$\because \hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y \quad \therefore \hat{l}_y = \frac{1}{2i}(\hat{l}_+ - \hat{l}_-)$

$\hat{l}_\pm|l,l_z\rangle = \sqrt{l(l+1) - l_z(l_z\pm1)}\,|l,l_z\pm1\rangle$

The state after rotation:

$\hat{D}(\hat{y},\alpha)|l,l_z=l\rangle \doteq (1 - i\alpha\hat{l}_y)|l,l_z=l\rangle = |l,l_z=l\rangle - i\alpha\frac{1}{2i}(\hat{l}_+ - \hat{l}_-)|l,l_z=l\rangle$

(first 2 terms) $= |l,l_z=l\rangle + \frac{\alpha}{2}\sqrt{l(l+1)-l(l-1)}\,|l,l_z=l-1\rangle$

$= |l,l_z=l\rangle + \frac{\alpha}{2}\sqrt{2l}\,|l,l_z=l-1\rangle$

third term:

$-\frac{1}{2}\alpha^2\hat{l}_y^2|l,l_z=l\rangle = \frac{1}{8}\alpha^2(\hat{l}_+ - \hat{l}_-)^2|l,l_z=l\rangle = \frac{1}{8}\alpha^2(\hat{l}_+ - \hat{l}_-)\left(-\sqrt{l(l+1)-l(l-1)}\,|l,l_z=l-1\rangle\right)$

$= -\frac{1}{8}\alpha^2\sqrt{2l}\,(\hat{l}_+ - \hat{l}_-)|l,l_z=l-1\rangle = -\frac{1}{8}\alpha^2\sqrt{2l}\Big(\sqrt{l(l+1)-(l-1)(l-1+1)}\,|l,l_z=l\rangle$
$-\sqrt{l(l+1)-(l-1)(l-1-1)}\,|l,l_z=l-2\rangle\Big)$

$= -\frac{1}{8}\alpha^2\sqrt{2l}\left(\sqrt{2l}\,|l,l_z=l\rangle - \sqrt{4l-2}\,|l,l_z=l-2\rangle\right)$

$\therefore$ keep to α^2, the rotated state:

$\cancel{\hat{D}(\hat{y},\alpha)} \approx |l,l_z=l\rangle + \frac{\alpha}{2}\sqrt{2l}\,|l,l_z=l-1\rangle - \frac{1}{4}\alpha^2 l\,|l,l_z=l\rangle + \frac{1}{4}\alpha^2\sqrt{l(2l-1)}\,|l,l_z=l-2\rangle$

$\hat{D}(\hat{y},\alpha)|l,l_z=l\rangle = (1 - \frac{1}{4}\alpha^2 l)|l,l_z=l\rangle + \frac{1}{2}\sqrt{2l}\,\alpha|l,l_z=l-1\rangle + \frac{1}{4}\alpha^2\sqrt{l(2l-1)}\,|l,l_z=l-2\rangle$

$(l \geq 1)$

(if $l=0$ $\hat{D}(\hat{y},\alpha)|l,l_z=l\rangle = |l,l_z=l\rangle$)

probability of obtaining $l_z = l$:

$P = |\langle l,l_z=l|\hat{D}(\hat{y},\alpha)|l,l_z=l\rangle|^2 \approx (1 - \frac{1}{4}\alpha^2 l^2)^2 = 1 - \frac{1}{2}\alpha^2 l^2 + \frac{1}{16}\alpha^4 l^4$

$\approx 1 - \frac{1}{2}\alpha^2 l^2$

2. a) Write $\psi(\vec{r})$ by spherical coordinates:

$\psi(\vec{r}) = (x+y+3z)f(r) = (r\sin\theta\cos\varphi + r\sin\theta\sin\varphi + 3r\cos\theta)f(r)$

$= (\sin\theta\cos\varphi + \sin\theta\sin\varphi + 3\cos\theta)\,rf(r)$

Here, we only have the first order terms of $\sin\theta$, $\cos\theta$, $\sin\varphi$, and $\cos\varphi$.

so, we write the relative eigenfunctions of L^2, L_z (spherical harmonics)

$Y_{1,-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\varphi} \qquad Y_{1,1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{i\varphi}$

$Y_{1,0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$

$\therefore \psi(\vec{r}) = \Big[\sqrt{\frac{2\pi}{3}}\left(Y_{1,-1}(\theta,\varphi) \mp Y_{1,1}(\theta,\varphi)\right) + \sqrt{\frac{2\pi}{3}}(+i)\left(Y_{1,-1}(\theta,\varphi) + Y_{1,1}(\theta,\varphi)\right)$
$2\sqrt{3\pi}\,Y_{1,0}(\theta,\varphi)\Big]\,rf(r)$

$\therefore \psi(\vec{r})$ is an eigenfunction of L^2, $l=1$

b) rewrite the expression of $\psi(\vec{r})$ in a)

$$\psi(\vec{r}) = \left[\sqrt{\frac{2\pi}{3}}(1+i)Y_{1,-1}(\theta,\varphi) + \sqrt{\frac{2\pi}{3}}(-1+i)Y_{1,1}(\theta,\varphi) + 2\sqrt{3\pi}\,Y_{1,0}(\theta,\varphi)\right] r f(r)$$

$$\because \iint \psi^*(\vec{r})\psi(\vec{r})\, r^2\, dr\, d\Omega = 1$$

$$\therefore \left(\frac{2\pi}{3}\times 2 + \frac{2\pi}{3}\times 2 + 4\times 3\pi\right)\int_0^\infty r^4 f^2(r)\, dr = 1$$

$$\therefore \int_0^\infty r^4 f^2(r)\, dr = \frac{3}{44\pi}$$

$$\therefore \psi'(\theta,\varphi) = \sqrt{\frac{3}{44\pi}}\left[\sqrt{\frac{2\pi}{3}}(1+i)Y_{1,-1}(\theta,\varphi) + \sqrt{\frac{2\pi}{3}}(-1+i)Y_{1,1}(\theta,\varphi) + 2\sqrt{3\pi}\,Y_{1,0}(\theta,\varphi)\right]$$

$$= \sqrt{\frac{1}{22}}(1+i)Y_{1,-1}(\theta,\varphi) + \sqrt{\frac{1}{22}}(-1+i)Y_{1,1}(\theta,\varphi) + \frac{3}{\sqrt{11}}Y_{1,0}(\theta,\varphi)$$

$$\therefore P(l_z=1) = \left|\int Y^*_{1,1}(\theta,\varphi)\,\psi'(\theta,\varphi)\, d\Omega\right|^2 = \frac{1}{11}$$

$$P(l_z=0) = \left|\int Y^*_{1,0}(\theta,\varphi)\,\psi'(\theta,\varphi)\, d\Omega\right|^2 = \frac{9}{11}$$

$$P(l_z=-1) = \left|\int Y^*_{1,-1}(\theta,\varphi)\,\psi'(\theta,\varphi)\, d\Omega\right|^2 = \frac{1}{11}$$

c) $\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$ $H\psi = E\psi$ not $HR = ER$

$$\because \hat{H}\psi(\vec{r}) = E\psi(\vec{r}) \quad \therefore -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\psi(\vec{r})$$

$$\therefore V(\vec{r}) = E + \frac{\hbar^2}{2m}\frac{1}{\psi(\vec{r})}\nabla^2\psi(\vec{r})$$

$$\frac{\partial^2}{\partial x^2}\psi(\vec{r}) = \frac{\partial^2}{\partial x^2}\left[(x+y+3z)f(r)\right] = \frac{\partial}{\partial x}\left(f(r) + (x+y+3z)\frac{df}{dr}\frac{x}{r}\right)$$

$$= \frac{df}{dr}\frac{x}{r} + \frac{df}{dr}\frac{x}{r} + (x+y+3z)\left(\frac{d^2f}{dr^2}\frac{x^2}{r^2} + \frac{df}{dr}\frac{1}{r} - \frac{df}{dr}\frac{x^2}{r^3}\right)$$

$$\therefore \nabla^2\psi(\vec{r}) = \frac{2}{r}\frac{df}{dr}(x+y+3z) + (x+y+3z)\left(\frac{d^2f}{dr^2}\frac{r^2}{r^2} + \frac{df}{dr}\frac{3}{r} - \frac{df}{dr}\frac{r^2}{r^3}\right)$$

$$= (x+y+3z)\left(\frac{d^2f}{dr^2} + \frac{4}{r}\frac{df}{dr}\right)$$

$$\therefore V(r) = E + \frac{\hbar^2}{2m}\frac{1}{f(r)}\left(\frac{d^2f(r)}{dr^2} + \frac{4}{r}\frac{df(r)}{dr}\right)$$

3. a) Rotation operator: $\hat{D}(\hat{y},\beta)$ $(\alpha=0,\ \gamma=0)$, we use the components of D-matrix.

$\hat{D}(\hat{y},\beta)|l=2, l_z=0\rangle = \sum_m d^{(2)}_{m,0}|2,m\rangle$ and we know from the form: $d^l_{m,0} = \sqrt{\frac{4\pi}{2l+1}}Y^m_l e^{-im\phi}$

here, $d^{(2)}_{m,0}$, Y^m_l are functions of θ,ϕ, $d^{(2)}_{m,0}(\theta,\phi)$ $Y^m_l(\theta,\phi)$

now, $\phi = \alpha = 0$ $\theta = \beta$

$$\therefore d^{(2)}_{m,0}(\beta,0) = \sqrt{\frac{4\pi}{5}}Y^m_2(\beta,0)$$

We know $Y^0_2(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$ $Y^1_2(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$ $Y^2_2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$

$Y^{-m}_l = (-1)^m Y^{m*}_l$

$$\therefore d^{(2)}_{0,0}(\beta,0) = \frac{3}{2}\cos^2\beta - \frac{1}{2} \qquad d^{(2)}_{1,0} = -\sqrt{\frac{3}{2}}\sin\beta\cos\beta \qquad d^{(2)}_{2,0} = \sqrt{\frac{3}{8}}\sin^2\beta$$

$$d^{(2)}_{-1,0} = \sqrt{\frac{3}{2}}\sin\beta\cos\beta \qquad d^{(2)}_{-2,0} = \sqrt{\frac{3}{8}}\sin^2\beta$$

The rotated state:

$$\cancel{\hat{D}(\hat{y},\beta)|2,0\rangle = \sqrt{\frac{3}{2}}\sin\beta\cos\beta\,|2,-1\rangle}$$

$$\hat{D}(\hat{y},\beta)|2,0\rangle = \sqrt{\frac{3}{2}}\sin\beta\cos\beta\,|2,-1\rangle + \sqrt{\frac{3}{8}}\sin^2\beta\,|2,-2\rangle + \left(\frac{3}{2}\cos^2\beta - \frac{1}{2}\right)|2,0\rangle$$

$$-\sqrt{\frac{3}{2}}\sin\beta\cos\beta\,|2,1\rangle + \sqrt{\frac{3}{8}}\sin^2\beta\,|2,2\rangle$$

$\therefore P(l_z=-2) = |\langle 2,-2|\hat{D}(\hat{y},\beta)|2,0\rangle|^2 = \frac{3}{8}\sin^4\beta$

$P(l_z=-1) = |\langle 2,-1|\hat{D}(\hat{y},\beta)|2,0\rangle|^2 = \frac{3}{2}\sin^2\beta\cos^2\beta$

$P(l_z=0) = |\langle 2,0|\hat{D}(\hat{y},\beta)|2,0\rangle|^2 = (\frac{3}{2}\cos^2\beta-\frac{1}{2})^2 = (1-\frac{3}{2}\sin^2\beta)^2$

$P(l_z=1) = |\langle 2,1|\hat{D}(\hat{y},\beta)|2,0\rangle|^2 = \frac{3}{2}\sin^2\beta\cos^2\beta$

$P(l_z=2) = |\langle 2,2|\hat{D}(\hat{y},\beta)|2,0\rangle|^2 = \frac{3}{8}\sin^4\beta$

4. a) to tell the difference between $|l_1,l_2;l_T,l_z\rangle$ and $|l_1,l_2;l_{1z},l_{2z}\rangle$ notations.

We write $|l_1,l_2;l_T,l_z\rangle$ as $|l_1,l_2;\underline{l_T,l_z}\rangle$

$\because \hat{l}_z = \hat{l}_{z_1}+\hat{l}_{z_2}$ $\therefore \hat{l}_z|1,1;1,1\rangle = (1+1)|1,1;1,1\rangle = 2|1,1;1,1\rangle$ $\therefore l_z=2$.

$\because l_T \geq l_z$ $\therefore$ now $l_T=2$ only $\therefore$ up to a global phase $|1,1;1,1\rangle = |1,1;\underline{2,2}\rangle$

$\therefore |1,1;\underline{2,1}\rangle = \frac{1}{\sqrt{2\times(2+1)-2\times(2-1)}}\hat{l}_-|1,1;\underline{2,2}\rangle = \frac{1}{2}(\hat{l}_{1-}+\hat{l}_{2-})|1,1;1,1\rangle$

$= \frac{1}{2}\left(\sqrt{1\times(1+1)-1\times(1-1)}\,|1,1;0,1\rangle + \sqrt{1\times(1+1)-1\times(1-1)}\,|1,1;1,0\rangle\right)$

$= \frac{\sqrt{2}}{2}\left(|1,1;0,1\rangle + |1,1;1,0\rangle\right)$

$|1,1;\underline{2,0}\rangle = \frac{1}{\sqrt{2\times(2+1)-1\times(1-1)}}\hat{l}_-|1,1;\underline{2,1}\rangle = \frac{1}{2\sqrt{3}}\left(\sqrt{2}|1,1;-1,1\rangle + 2\sqrt{2}|1,1;0,0\rangle + \sqrt{2}|1,1;1,-1\rangle\right)$

$|1,1;\underline{2,-1}\rangle = \frac{1}{\sqrt{2\times(2+1)-0\times(0-1)}}\hat{l}_-|1,1;\underline{2,0}\rangle = \frac{1}{6}\left(\sqrt{2}|1,1;-1,0\rangle + 2\sqrt{2}|1,1;-1,0\rangle + 2\sqrt{2}|1,1;0,-1\rangle + \sqrt{2}|1,1;0,-1\rangle\right)$

$= \frac{\sqrt{2}}{2}\left(|1,1;-1,0\rangle + |1,1;0,-1\rangle\right)$

$|1,1;\underline{2,-2}\rangle = \frac{1}{\sqrt{2\times(2+1)-(-1)\times(-1-1)}}\hat{l}_-|1,1;\underline{2,-1}\rangle = |1,1;-1,-1\rangle$

For $l_z=1$, possible combinations of l_{1z} l_{2z} are $(l_{1z}=0,\ l_{2z}=1)$ and $(l_{1z}=1,\ l_{2z}=0)$

$\therefore |1,1;\underline{1,1}\rangle = \alpha|1,1;0,1\rangle + \beta|1,1;1,0\rangle$

$\because 0 = \hat{l}_+|1,1;\underline{1,1}\rangle = (\hat{l}_{1+}+\hat{l}_{2+})(\alpha|1,1;0,1\rangle + \beta|1,1;1,0\rangle) = \sqrt{2}(\alpha+\beta)|1,1;1,1\rangle$

$\therefore \alpha+\beta=0$. $\therefore$ up to a global phase $|1,1;\underline{1,1}\rangle = \frac{1}{\sqrt{2}}|1,1;1,0\rangle - \frac{1}{\sqrt{2}}|1,1;0,1\rangle$

$\therefore |1,1;\underline{1,0}\rangle = \frac{1}{\sqrt{1\times(1+1)-1\times(1-1)}}\hat{l}_-|1,1;\underline{1,1}\rangle = \frac{1}{\sqrt{2}}(\hat{l}_{1-}+\hat{l}_{2-})\left(\frac{1}{\sqrt{2}}|1,1;1,0\rangle - \frac{1}{\sqrt{2}}|1,1;0,1\rangle\right)$

$= \frac{1}{2}\left(\sqrt{2}|1,1;1,-1\rangle + \sqrt{2}|1,1;0,0\rangle - \sqrt{2}|1,1;0,0\rangle - \sqrt{2}|1,1;-1,1\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|1,1;1,-1\rangle - |1,1;-1,1\rangle\right)$

$|1,1;\underline{1,-1}\rangle = \frac{1}{\sqrt{1\times(1+1)-0\times(0-1)}}\hat{l}_-|1,1;\underline{1,0}\rangle = \frac{1}{2}(\hat{l}_{1-}+\hat{l}_{2-})\left(|1,1;1,-1\rangle - |1,1;-1,1\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|1,1;0,-1\rangle - |1,1;-1,0\rangle\right)$

For $l_z=0$, possible (l_{1z},l_{2z})s are. $(0,0)$ $(1,-1)$ $(-1,1)$

$\therefore |1,1;\underline{0,0}\rangle = \lambda|1,1;0,0\rangle + \mu|1,1;1,-1\rangle + \xi|1,1;-1,1\rangle$

$\because 0 = \hat{l}_+|1,1;\underline{0,0}\rangle = (\hat{l}_{1+}+\hat{l}_{2+})(\lambda|1,1;0,0\rangle + \mu|1,1;1,-1\rangle + \xi|1,1;-1,1\rangle)$

$= (\sqrt{2}\lambda|1,1;1,0\rangle + \sqrt{2}\lambda|1,1;0,1\rangle + \sqrt{2}\mu|1,1;1,0\rangle + \sqrt{2}\xi|1,1;0,1\rangle)$

$\therefore \lambda+\mu=0$ $\lambda+\xi=0$.

$\therefore$ up to a global phase, $|1,1;\underline{0,0}\rangle = \sqrt{\frac{1}{3}}|1,1;1,-1\rangle - \sqrt{\frac{1}{3}}|1,1;0,0\rangle + \sqrt{\frac{1}{3}}|1,1;-1,1\rangle$