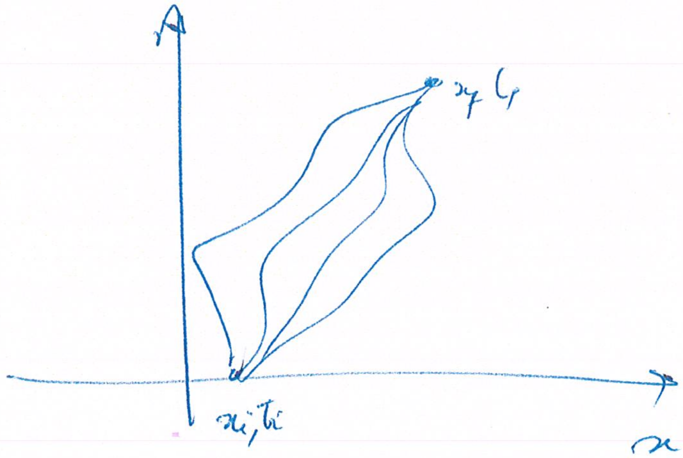
Path integrals in QM.



$\langle x_f, t_f | x_i, t_i \rangle =$

$= \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle$

$= K(x_f t_f ; x_i t_i)$

$$K(x_f t_f ; x_i t_i) = \int [\mathcal{D} x(t)]\, e^{iS[x(t)]}$$

$\psi(x_f, t_f) = \int dx_i\; K(x_f t_f ; x_i t_i)\, \psi(x_i, t_i)$

$K(x_f, t_f ; x_i t_i) = \int dx\; K(x_f t_f, x t)\, K(x t ; x_i t_i)$

$t_f > t > t_i$

example free particle

$$K(x_f t_f ; x_i t_i) = \int \frac{dp}{2\pi} \langle x_f | e^{-\frac{ip^2}{2m}\frac{(t_f - t_i)}{\hbar}} | p \rangle \langle p | x_i \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-\frac{ip^2}{2m}\frac{(t_f - t_i)}{\hbar}}\, e^{\frac{ip(x_f - x_i)}{\hbar}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi \hbar}{\frac{i}{2m}(t_f - t_i)}}\; e^{-\frac{(x_f - x_i)^2}{4\frac{i}{2m}(t_f - t_i)\hbar}} = \frac{\sqrt{m}}{\sqrt{2\pi}} \frac{e^{-\frac{i\pi}{4}}}{\sqrt{(t_f - t_i)}}\, e^{\frac{mi(x_f - x_i)^2}{2(t_f - t_i)\hbar}}$$

$$\boxed{K(x_f t_f ; x_i t_i) = e^{-\frac{i\pi}{4}} \sqrt{\frac{m\hbar}{2\pi(t_f - t_i)}}\; e^{\frac{im}{2}\frac{(x_f - x_i)^2}{(t_f - t_i)\hbar}}}$$

$f(x)$

$t_f - t_i \quad \frac{e^{-i\pi/4}}{\sqrt{\pi\sigma}} e^{ix^2/\sigma} \to \delta(x)$ by Fourier $\int_{-\infty}^{\infty} dx\, e^{ix^2/\sigma} f(x) = \int_{-\infty}^{\infty} dx\, e^{\frac{ix^2}{\sigma}} \int_{-\infty}^{\infty} dk\, e^{ikx} \tilde{f}(k)$

$= \sqrt{\pi\sigma} \int dk\, \tilde{f}(k)\, e^{\frac{ik^2\sigma}{4}} = \sqrt{\pi\sigma} \int dk\, \tilde{f}(k)$

h.o

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$x(t) = e^{-iHt} x e^{iHt} \qquad p(t) = e^{-iHt} p e^{iHt}$$

$$[x(t), p(t)] = i \quad ; \quad H = \frac{1}{2m} p^2(t) + \frac{1}{2} m\omega^2 x^2(t)$$

$$\partial_t x(t) = -iH x(t) + x(t) iH = -i[H, x(t)] =$$

$$= -\frac{i}{2m} p(-i) = -\frac{1}{m} p(t)$$

$$\partial_t p(t) = -i\, m\omega^2\, i\, x(t) = m\omega^2 x(t)$$

$$\partial_t^2 x(t) = -\frac{1}{m} m\omega^2 x(t) = -\omega^2 x(t)$$

$$(\partial_t^2 + \omega^2)\, x(t) = 0 \qquad x(t) = \cos\omega t\, \hat{A} + \sin\omega t\, \hat{B}$$

$$\dot{x}(t) = -\omega \sin\omega t\, \hat{A} + \omega \cos\omega t\, \hat{B} = -\frac{1}{m} p(t)$$

$$\boxed{x(t) = \hat{x}_s \cos\omega t - \frac{1}{m\omega} \sin\omega t\, \hat{p}_s}$$

$$X(t)\underbrace{e^{-iHt}|x_i\rangle}_{|\psi\rangle} = x_i \underbrace{e^{-iHt}|x_i\rangle}_{|\psi\rangle}$$

$$\langle x_f|\psi\rangle = \psi(x_f)$$

$$\cos\omega t\, x_f\, \psi + \frac{i}{m\omega}\sin\omega t\, \partial_{x_f}\psi = x_i \psi$$

$$\frac{\partial_{x_f}\psi}{\psi} = -i\frac{m\omega}{s\omega t}(x_i - x_f\cos\omega t)$$

$$\partial_{x_f}\ln\psi = -\frac{im\omega}{s\omega t}(x_i - x_f\cos\omega t)$$

$$\psi = e^{-\frac{im\omega}{s\omega t}\left(x_i x_f - \frac{1}{2}x_f^2\cos\omega t\right)} \cdot A(t, x_i)$$

Property $\langle x_f|e^{-iHt}|x_i\rangle^* = \langle x_i|e^{iHt}|x_f\rangle$

$$K(x_f,t;x_i,0)^* = K(x_i,-t;x_f,0)$$

$\psi^*(-t)$: same as $x_i \leftrightarrow x_f$

$$= e^{-\frac{im\omega}{s\omega t}\left(x_i x_f - \frac{1}{2}x_f^2 c\omega t\right)}$$

$$\psi = A(t)\, e^{-\frac{im\omega}{s\omega t}\left(x_i x_f - \frac{1}{2}(x_f^2 + x_i^2)c\omega t\right)}$$

$$\boxed{A^*(-t) = A(t)}$$

$$\partial_t \psi = -iH\psi = -i\left(-\frac{1}{2m}\partial_x^2\psi + \frac{1}{2}m\omega^2x^2\psi\right)$$

$$\psi = Ae^{\chi} \qquad \partial_x\psi = A\partial_x\chi e^{\chi}$$

$$\dot{A}e^{\chi} + A\dot{\chi}e^{\chi} = \frac{i}{2m}A\chi''e^{\chi} + \frac{i}{2m}A\chi'^2e^{\chi} - \frac{i}{2}m\omega^2x^2Ae^{\chi}$$

$$\frac{\dot{A}}{A} + \left(\frac{im\omega^2}{s^2}c\, x_iX_f - \frac{im\omega^2}{2}(X_f^2+x_i^2)\frac{1}{s^2\omega t}\right) =$$

$$= \frac{i}{2}\left(\frac{im\omega}{s\omega t}c\omega t\right) + \frac{i}{2m}\left(-\frac{im\omega}{s\omega t}(x_i - X_f c\omega t)\right)^2 - \frac{i}{2}m\omega^2X_f^2$$

$$\frac{\dot{A}}{A} = -\frac{i\omega}{2s\omega t}c\omega t - \frac{im\omega^2c}{s^2}x_iX_f + \frac{im\omega^2}{2}(X_i^2+X_f^2)\frac{1}{s^2\omega t} +$$

$$+ \frac{i\omega^2m}{2s^2\omega t}\left(x_i^2 + X_f^2c^2\omega t - 2X_iX_fc\omega t\right) - \frac{i}{2}m\omega^2X_f^2$$

$$= \frac{i\omega c\omega t}{s\omega t} + \frac{im\omega^2X_f^2}{2s^2\omega t}(1 - c^2\omega t) - \frac{i}{2}m\omega^2X_f^2$$

$$\frac{\dot{A}}{A} = -\frac{1}{2}\omega \frac{c\omega t}{s\omega t}$$

$$\partial_t \ln A = -\frac{1}{2} \ln s\omega t$$

$$A = \frac{A_0}{\sqrt{s\omega t}}$$

$$\psi = \frac{A_0}{\sqrt{s\omega t}} e^{-\frac{i m\omega}{s\omega t}\left(x_i x_f - \frac{1}{2} c\omega t (x_i^2 + x_f^2)\right)}$$

$t \to 0 \quad \psi \to \delta(x_i - x_f)$

$$\psi \simeq \frac{A_0}{\sqrt{\omega t}} e^{+\frac{i m\omega}{\omega t}\frac{1}{2}(x_i - x_f)^2}$$

$$\int \psi = \frac{A_0}{\sqrt{\omega t}} \sqrt{\frac{2\pi t}{-im}} = A_0 \sqrt{\frac{2\pi}{-im\omega}} = 1$$

$$A_0 = \sqrt{\frac{-im\omega}{2\pi}} = e^{-\frac{i\pi}{4}} \sqrt{\frac{m\omega}{2\pi}}$$

$$K(x_f t_f; x_i t_i) = e^{-\frac{i\pi}{4}} \sqrt{\frac{m\omega}{2\pi \sin(t_f - t_i)}}\, e^{-\frac{im\omega}{\sin(\omega(t_f - t_i))}\left(x_i x_f - \frac{1}{2}\cos(t_f - t_i)(x_i^2 + x_f^2)\right)}$$

Free particle

$$\int dx_1 \cdots dx_{N-1} \, e^{+i \frac{(x_n - x_{n-1})^2}{2m\Delta t}}$$

$$\xi_1 = x_1 - x_0 \qquad \xi_2 = x_2 - x_1 \quad \cdots \quad \xi_N = x_N - x_{N-1}$$

$$\int d\xi_1 \cdots d\xi_{N-1} \, d\xi_N \, e^{+i \frac{1}{2m\Delta t}\left(\xi_1^2 + \cdots + \xi_{N-1}^2 + \xi_N^2\right)} \, \delta(\xi_1 + \cdots + \xi_N - \Delta x)$$

$$\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int d\xi_1 \cdots d\xi_N \, e^{i\lambda\xi_1 + \cdots + i\lambda\xi_N - i\lambda\Delta x} \, e^{-i\frac{1}{2m\Delta t}\xi_1^2 \cdots - i\frac{1}{2m\Delta t}\xi_N^2}$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left(\sqrt{\frac{\pi 2m\Delta t}{-i}}\right)^N e^{-\frac{\lambda^2 2m\Delta t N}{4(-i)} - i\lambda\Delta x}$$

$$= (+2\pi i m\Delta t)^N \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, e^{-\frac{i}{2} m\lambda^2 (t_f - t_i) - i\lambda\Delta x}$$

$$= \frac{(+2\pi i m\Delta t)^N}{2\pi} \sqrt{\frac{\pi}{-\frac{im}{2}(t_f - t_i)}} \, e^{+\frac{\Delta x^2}{4\frac{i}{2}m(t_f - t_i)}}$$

$$= \frac{(+2\pi i m\Delta t)^N}{2\pi} \sqrt{\frac{2\pi i}{m(t_f - t_i)}} \, e^{\frac{i\Delta x^2}{2m(t_f - t_i)}}$$

same up to N factor.

H.O.

$$\int \mathcal{D}x \; e^{i\int dt \left(\frac{m\dot{x}^2}{2} + \frac{1}{2}m\omega^2 x^2\right)}$$

$$x = x_{cl} + \delta x \qquad \delta x(0) = \delta x(T) = 0 \qquad -m\ddot{x}\,\delta x + m\omega^2 x\,\delta x$$

$$\int \mathcal{D}x \; e^{i\int dt\left(\frac{m\dot{x}_{cl}^2}{2} + 2m(\dot{x}\,\delta\dot{x}) + \frac{1}{2}m\delta\dot{x}^2 + \frac{1}{2}m x_{cl}^2 + m\omega^2 x\,\delta x + \frac{1}{2}m\omega^2 \delta x^2\right.}$$

$$\int \mathcal{D}x \; e^{iS_{cl}} \; e^{i\int dt\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega^2 y^2\right)}$$

$$y(0) = y(T) = 0.$$

$\underline{S_{cl}}$:

$$x = A\cos\omega t + B\sin\omega t$$

$$x = x_i\cos\omega t + B\sin\omega t$$

$$x_f = x_i\cos\omega T + B\sin\omega T$$

$$B = \frac{1}{\sin\omega T}\left(x_f - x_i\cos\omega T\right)$$

$$x = x_i\cos\omega t + \frac{\sin\omega t}{\sin\omega T}\left(x_f - x_i\cos\omega T\right)$$

$$\dot{x} = -x_i\sin\omega t\,\omega + \frac{\omega\cos\omega t}{\sin\omega T}\left(x_f - x_i\cos\omega T\right)$$

$$S_{cl} = \int dt \left(\frac{m\dot{x}^2}{2} - \frac{1}{2} m\omega^2 x^2 \right) =$$

$$= \int dt \left(-\frac{m\ddot{x}x}{2} - \frac{1}{2} m\omega^2 x^2 + \partial_t \left(\frac{m x\dot{x}}{2} \right) \right)$$

$\frac{m\dot{x}\dot{x}}{2}$ $\frac{m\omega^2 x}{2}$

$$\ddot{x} = -\omega^2 x$$

$$= m\frac{x\dot{x}}{2}\Big|_0^T = \frac{m x_f}{2}\dot{x}(T) - \frac{m x_i}{2}\dot{x}(0)$$

$$= \frac{m}{2}\Big(x_f(-\omega x_i s\omega T) + x_f\omega\frac{c\omega T}{s\omega T}(x_f - x_i c\omega T) -$$

$$- x_i\frac{\omega}{s\omega T}(x_f - x_i c\omega T)\Big)$$

$$= \frac{m}{2}\Big(-\omega s\omega T\, x_i x_f + \omega x_f^2\frac{c\omega T}{s\omega T} - x_i x_f\,\omega\frac{c^2\omega T}{s\omega T} -$$

$$- \frac{x_i x_f\omega}{s\omega T} + x_i^2\omega\frac{c\omega T}{s\omega T}\Big)$$

$$= \frac{m}{2}\left[\frac{\omega x_i x_f}{s\omega T}\left(-s^2\omega T - c^2\omega T - 1\right) + \frac{\omega c\omega T}{s\omega T}(x_i^2 + x_f^2)\right]$$

$$S_{cl} = \frac{m\omega}{2}\cot\omega T\,(x_i^2+x_f^2) - \frac{m\omega\, x_i x_f}{\sin\omega T}$$

$$K = e^{\frac{im\omega}{2}\cot\omega T\,(x_i^2+x_f^2) - \frac{im\omega}{\sin\omega T}x_i x_f} \cdot \int_{y(0)=y(T)=0} \mathcal{D}y(t)\, e^{iS[y(t)]}$$

$$\int \mathcal{D}y(t)\, e^{i\int \frac{m}{2}\dot{y}^2 - \frac{1}{2}m\omega^2 y^2} = \int \mathcal{D}y(t)\, e^{-\frac{im}{2}\int y(\ddot{y}+\omega^2 y)}$$

$$= \int \mathcal{D}y(t)\, e^{-\frac{im}{2}\int y(\partial_t^2+\omega^2)y} = \det{}^{-1/2}\left(+i\frac{m}{2}(\partial_t^2+\omega^2)\right)$$

eigenvalues $\quad y = A\sin\left(\frac{n t \pi}{T}\right)$

$$= \prod_{n=1}^{\infty} \frac{1}{\sqrt{\frac{im}{2}\left(\omega^2 - \frac{n^2\pi^2}{T^2}\right)}} = \frac{1}{\left(\frac{im\omega^2}{2}\right)^{N/2}} \prod_{n=1}^{\infty} \frac{1}{\left(1+\frac{n^2\pi^2}{\omega^2T^2}\right)^{1/2}}$$

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$$

~~$\ln = \sum_{n=1}^{\infty} \ln\left(\frac{n^2\pi^2}{\omega^2T^2} - 1\right)$~~

Use them:

$$\frac{\det(\partial_t^2+\omega^2)}{\det(\partial_t^2)} = \prod_{n=1}^{\infty} \frac{\omega^2-\frac{n^2\pi^2}{T^2}}{-\frac{n^2\pi^2}{T^2}} = \prod_{n=1}^{\infty} \left(1-\frac{\omega^2T^2}{n^2\pi^2}\right)$$

$$= \frac{\sin(\omega T)}{\omega T}$$

$$\det{}^{-1/2}(\partial_t^2+\omega^2) = \sqrt{\frac{\omega T}{\sin\omega T}}\ \underbrace{\det{}^{-1/2}(\partial_t^2)}_{\text{free particle } K_{f.p.}(0t_f,0t_i)}$$

$$= e^{-\frac{i\pi}{4}}\sqrt{\frac{m}{2\pi T}}$$

$$K_{h.o.} = e^{-\frac{i\pi}{4}}\sqrt{\frac{m\omega}{2\pi\sin\omega T}}\ e^{\frac{im\omega}{2}\left[\cot(\omega T)(x_i^2+x_f^2)-\frac{2x_ix_f}{\sin\omega T}\right]}$$

↑ path-integral result agrees.

Derivation

$$\langle x_f | e^{-iHT} | x_i \rangle = \int dx_1 \ldots dx_{N-1} \, \langle x_f | e^{-iH\Delta t} | x_{N-1} \rangle \langle x_{N-1} | \ldots e^{-iH t} | x_0 \rangle$$

$$= \int dx_1 \ldots dx_{N-1} \, K(x_f \Delta t; x_{N-1}, 0) \ldots K(x_1 \Delta t; x_0, 0)$$

$$K(x_2 \Delta t, x_1, 0) = \langle x_2 | e^{-i\frac{p^2}{2m}\Delta t + iV(x)\Delta t} | x_1 \rangle =$$

$$= \langle x_2 | e^{-i\frac{p^2}{2m}\Delta t} \, e^{-iV(x)\Delta t} | x_1 \rangle =$$

to order Δt

$$= \int dp \, e^{-i\frac{p^2}{2m}\Delta t} \, e^{ip(x_2 - x_1)} \cdot e^{-iV(x_2)\Delta t}$$

$$= \int = \sqrt{\frac{2m\pi}{i\Delta t}} \, e^{-\frac{(x_2 - x_1)^2}{4\frac{i\Delta t}{2m}} - iV(x_2)\Delta t}$$

$$= \sqrt{\frac{2m\pi}{i\Delta t}} \, e^{\frac{im(x_2 - x_1)^2}{2\Delta t} - iV(x_2)\Delta t} = N_{\Delta t} \, e^{iS(x_2, x_1, \Delta t)}$$

$$K(x_f, T; x_i, 0) = \lim_{N\to\infty} N \int dx_1 \ldots dx_{N-1} \, e^{i\sum_j S(x_{j+1}, \Delta t; x_j, 0)}$$

$$= \lim_{N\to\infty} N \int dx_1 \ldots dx_{N-1} \, e^{i\int L}$$

What if

$$H = \frac{p^2}{2m} f(x) + V(x)$$

$$\langle x_2 | e^{-i \frac{p^2 f(x) \Delta t}{2m}} | p \rangle$$

$$\int dp \, e^{-\frac{i}{2m} f(x) p^2 \Delta t} \, e^{ip(x_2 - x_1)} = \sqrt{\frac{\pi}{\frac{i}{2m} f(x_i)}} \, e^{\frac{im}{2} \frac{(\Delta x)^2}{f \Delta t}}$$

$$\prod_i \frac{1}{\sqrt{f(x_i)}} = e^{\frac{1}{2} \sum_i \ln f(x_i)} = e^{\frac{1}{2} \frac{1}{\Delta t} \sum_i \ln f(x_i) \Delta t}$$

$$= e^{\frac{1}{2} \delta(0) \int dt \ln f(x(t))}$$

? extra term.

$$S_{eff} = \int dt \, L(q, \dot{q}) - \frac{i}{2} \delta(0) \int \ln f(q) \, dt$$

$$\int \left[\sqrt{f(q)} \, dq \right] e^{iS}$$