(1)

Spin 1/2 chain

$$H = -\lambda \frac{\hbar^2}{4} \sum_{j=-\infty}^{\infty} \vec{S}_j \vec{S}_{j+1} \qquad | \downarrow \uparrow \cancel{\downarrow} \uparrow \downarrow \uparrow \rangle$$

bases.

$j \quad j+1 \qquad S_x = \frac{\hbar}{2}\begin{array}{cc} \hbar & \uparrow \\ \end{array}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_x |\uparrow\rangle = |\downarrow\rangle \qquad \sigma_y |\uparrow\rangle = i |\downarrow\rangle \qquad \sigma_z |\uparrow\rangle = |\uparrow\rangle$

$\sigma_x |\downarrow\rangle = |\uparrow\rangle \qquad \sigma_y |\downarrow\rangle = -i |\uparrow\rangle \qquad \sigma_z |\downarrow\rangle = -|\downarrow\rangle$

$$(\sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)})|\uparrow\uparrow\rangle = \cancel{|\downarrow\downarrow\rangle} - \cancel{|\downarrow\downarrow\rangle} + |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle$$

$$|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle = 2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle$$

$$|\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$|\downarrow\downarrow\rangle = \cancel{|\uparrow\uparrow\rangle} - \cancel{|\uparrow\uparrow\rangle} + |\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle$$

$$H = -\lambda \sum_i H_{i,i+1}$$

$$H_{i,i+1} = \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{P}_{i,i+1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow H_{i,i+1} = \mathbb{1} + 2(-\mathbb{1} + \mathbb{P}_{i,i+1})$$

permutation

$$H = -\lambda \sum_j \mathbb{1} + 2\lambda \sum_j (\mathbb{1}_{j,j+1} - \mathbb{P}_{j,j+1}) = \underline{\underline{-\lambda \mathbb{1}}} + 2\lambda \sum_j (\mathbb{1}_{j,j+1} - \mathbb{P}_{j,j+1})$$

eliminate

Take $H = 2\lambda \sum_j (\mathbb{1} - \mathbb{P}_{j,j+1})$

Many groundstates $\Rightarrow$ all spin in the same state.

e.g. $|\downarrow\downarrow - \_\_\_\_\_\_ \downarrow\rangle$

$\underline{E_{g.s.} = 0}$ H preserves the # of spin up.

next: $|j\rangle = |\downarrow \dots \uparrow_j \dots \downarrow\rangle$ spin up in position j

$$H|j\rangle = \big(2\lambda|j\rangle - 2\lambda|j+1\rangle\big) + \big(2\lambda|j\rangle - 2\lambda|j-1\rangle\big)$$

$$+ 0 - - + - - 0$$

$$= 2\lambda\big(2|j\rangle - |j+1\rangle - |j-1\rangle\big)$$

$$|\psi\rangle = \sum_j a_j |j\rangle$$

$$H|\psi\rangle = 2\lambda\sum_j a_j \big(2|j\rangle - |j+1\rangle - |j-1\rangle\big)$$

$$= 2\lambda \sum_j (2a_j - a_{j-1} - a_{j+1})|j\rangle = \varepsilon \sum_j a_j |j\rangle$$

$$2a_j - a_{j-1} - a_{j+1} = \frac{\varepsilon}{2\lambda} a_j$$

$$a_j = q^j \qquad 2\cancel{q^j} - q^{j-1} - q^{j+1} = \frac{\varepsilon}{2\lambda}\cancel{q^j}$$

$$\boxed{2 - \frac{1}{q} - q = \frac{\varepsilon}{2\lambda}}$$

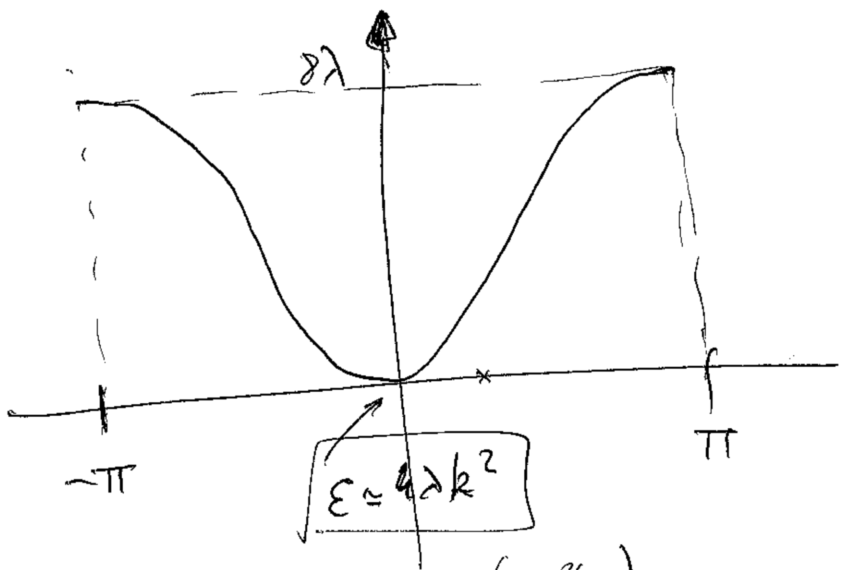
but q should be a phase, otherwise the probability diverges for $j \to \infty$, $j \to -\infty$

$1-c^2+s^2$

$q = e^{ik}$

$$\frac{\varepsilon}{2\lambda} = 2 - e^{-ik} - e^{ik} = 2 - 2\cos k = 2 \times 2\sin^2\frac{k}{2}$$

$$\boxed{\varepsilon = 8\lambda \sin^2\frac{k}{2}}$$



$$|k\rangle = \sum_{j=-\infty}^{\infty} e^{ikj} |j\rangle$$

$k \to k + 2\pi$ $\qquad |k+2\pi\rangle = \sum_{j=-\infty}^{\infty} e^{i(k+2\pi)j} |j\rangle = \sum_{j=-\infty}^{\infty} e^{ikj} |j\rangle = |k\rangle$

$$\boxed{-\pi < k < \pi}$$

<u>2 Spins</u>

$$|j_1\, j_2\rangle = |\cdots\downarrow\downarrow\uparrow_{j_1}\downarrow\cdots\uparrow_{j_2}\cdots\rangle$$

$$|\psi\rangle = \sum_{j_1=-\infty}^{\infty}\ \sum_{j_2=j_1+1}^{\infty} a_{j_1 j_2}\, |j_1, j_2\rangle$$

$$H|j_1\, j_2\rangle = 2\lambda \sum_{l=-\infty}^{\infty} \left(\mathbb{1}_{l,l+1} - \mathbb{P}_{l,l+1}\right) |\cdots\downarrow\uparrow_{j_1}\cdots\uparrow_{j_2}\downarrow\rangle$$

•) $j_2 > j_1+1$

$$H|j_1\, j_2\rangle = 2\lambda\Big(-|j_1-1, j_2\rangle - |j_1+1\; j_2\rangle - |j_1\; j_2-1\rangle - |j_1\; j_2+1\rangle + 4|j_1\, j_2\rangle\Big)$$

•) $j_2 = j_1+1$

$$H|j_1\, j_2\rangle = 2\lambda\Big(-|j_1-1\; j_2\rangle - |j_1\; j_2+1\rangle + 2|j_1\, j_2\rangle\Big)$$

$$H|\psi\rangle = \varepsilon|\psi\rangle$$

$$H|\psi\rangle = 2\lambda \sum_{j_1=-\infty}^{\infty}\ \sum_{j_2=j_1+2}^{\infty} a_{j_1 j_2}\Big(\overset{58}{\underset{\substack{58\\57}}{4|j_1 j_2\rangle}} - \underset{\substack{6\ 8\\ \cancel{6\ 7}}}{|j_1-1\, j_2\rangle} - \underset{\substack{4\ 8\\4\ 7}}{|j_1+1\, j_2\rangle} - \overset{j_1\ \ j_1+1}{\underset{\substack{5\ 9\\5\ 8}}{|j_1\, j_2-1\rangle}} - \underset{\substack{5\ 7\\ \cancel{5\ 6}}}{|j_1\, j_2+1\rangle}\Big)$$

$$+2\lambda \sum_{j_1=-\infty}^{\infty} \underset{\substack{6\ 7\\5\ 6}}{a_{j_1\, j_1+1}} \Big(2|j_1\, j_1+1\rangle - \underset{6\quad 6}{|j_1-1\; j_1+1\rangle} - \underset{5\quad 5}{|j_1\; j_1+2\rangle}\Big)$$

$$= \varepsilon \sum_{j_1=-\infty}^{\infty}\ \sum_{j_2=j_1+1}^{\infty} a_{j_1 j_2}\, |j_1\, j_2\rangle$$

Match coefficients:

$j_2 > j_1 + 1$

$$\varepsilon \underset{\substack{58\\57}}{a_{j_1 j_2}} = 2\lambda\left(4\underset{\substack{58\\57}}{a_{j_1 j_2}} - \underset{\substack{6\ 8\\6\ 7}}{a_{j_1+1\, j_2}} - \underset{\substack{4\ 8\\4\ 7}}{a_{j_1-1\, j_2}} - \underset{\substack{5\ 9\\5\ 8}}{a_{j_1\, j_2+1}} - \underset{\substack{5\ 8\\5\ 6}}{a_{j_1\, j_2-1}}\right).$$

$j_2 = j_1 + 1$

$$\varepsilon a_{j_1\, j_1+1} = 2\lambda\left(2a_{j_1 j_1+1} - a_{j_1-1\, j_1+1} - a_{j_1\, j_1+2}\right)$$

$$a_{j_1 j_2} = e^{i\theta/2}\, e^{ik_1 j_1 + ik_2 j_2} + e^{-i\theta/2}\, e^{ik_1 j_2 + ik_2 j_1}$$

$$\frac{\varepsilon}{2\lambda} e^{i\theta/2} e^{ik_1 j_1 + ik_2 j_2} + \frac{\varepsilon}{2\lambda} e^{-i\theta/2} e^{ik_1 j_2 + ik_2 j_1} = e^{i\theta/2} e^{ik_1 j_1 + ik_2 j_2} \times \Big($$

$$\left(4 - e^{ik_1} - e^{-ik_1} - e^{ik_2} - e^{-ik_2}\right) + e^{-i\theta/2} e^{ik_1 j_2 + ik_2 j_1}\left(4 - e^{ik_2} - e^{-ik_2} - e^{ik_1} - e^{-ik_1}\right)$$

$2(1-c_{k_1}) = 2(1-c^2+s^2) = 4s^2$

$$\frac{\varepsilon}{2\lambda} = 4 - e^{ik_1} - e^{-ik_1} - e^{ik_2} - e^{-ik_2} = 4 - 2\cos k_1 - 2\cos k_2 = 4\sin^2\frac{k_1}{2} + 4\sin^2\frac{k_2}{2}$$

$$\varepsilon = 8\lambda \sin^2\frac{k_1}{2} + 8\lambda\sin^2\frac{k_2}{2} = \varepsilon(k_1) + \varepsilon(k_2)$$

But also:

$$\frac{\varepsilon}{2\lambda}\left(e^{\frac{i\theta}{2}} \cancel{e^{ik_1 j_1 + ik_2 j_1}} e^{ik_2} + e^{-i\theta/2} \cancel{e^{ik_1 j_1 + ik_2 j_1}} e^{ik_1}\right) = \cancel{e^{ik_1 j_1 + ik_2 j_1}} \times$$

$$\times\left(2e^{i\theta/2} e^{ik_2} - e^{i\theta/2} e^{-ik_1} e^{ik_2} - e^{i\theta/2} e^{2ik_2}\right) + e^{-i\theta/2}\left(2e^{ik_1} - e^{-ik_2} e^{ik_1} - e^{2ik_1}\right)$$

$$(4 - e^{ik_1} - e^{-ik_1} - e^{ik_2} - e^{-ik_2})\left(e^{i\frac{\theta}{2}} e^{ik_2} + e^{-i\frac{\theta}{2}} e^{ik_1}\right) =$$

$$= 4e^{i\frac{\theta}{2}}e^{ik_2} + 4e^{-i\frac{\theta}{2}}e^{ik_1} - e^{i\frac{\theta}{2}}e^{ik_2+ik_1} - \cancel{e^{-i\frac{\theta}{2}}e^{2ik_1}} - \cancel{e^{i\frac{\theta}{2}}e^{ik_2-ik_1}} - e^{-i\theta/2}$$

$$- \cancel{e^{i\theta/2}e^{2ik_2}} - e^{-i\theta/2}e^{ik_1+ik_2} - e^{i\theta/2} - \cancel{e^{-i\theta/2}e^{ik_1-ik_2}}$$

$$= 2e^{i\theta/2}e^{ik_2} - \cancel{e^{i\theta/2}e^{ik_2-ik_1}} - \cancel{e^{i\theta/2}e^{2ik_2}} + 2e^{ik_1}e^{-i\theta/2} -$$

$$- \cancel{e^{-ik_2+ik_1-i\theta/2}} - \cancel{e^{2ik_1-i\theta/2}}$$

$$2e^{i\theta/2}e^{ik_2} + 2e^{-i\frac{\theta}{2}+ik_1} - \left(e^{i\theta/2}+e^{-i\theta/2}\right)e^{i(k_1+k_2)} - e^{-i\theta/2} - e^{i\theta/2} = 0$$

$$e^{i\theta/2}\left(2e^{ik_2} - e^{i(k_1+k_2)} - 1\right) = e^{-i\theta/2}\left(-2e^{ik_1} + e^{i(k_1+k_2)} + 1\right)$$

$$\boxed{e^{i\theta} = -\frac{e^{i(k_1+k_2)} - 2e^{ik_1} + 1}{e^{i(k_1+k_2)} - 2e^{ik_2} + 1}} \qquad \boxed{\theta(k_1,k_2) = -\theta(k_2,k_1)}$$

$K = k_1+k_2 \qquad q = \frac{k_2-k_1}{2} \qquad K+2q = 2k_2 \qquad K-2q = 2k_1$

$$e^{i\theta} = -\frac{e^{iK} - 2e^{i\frac{K}{2}-iq} + 1}{e^{iK} - 2e^{i\frac{K}{2}+iq} + 1} = -\frac{e^{iK/2} - 2e^{-iq} + e^{-iK/2}}{e^{iK/2} - 2e^{iq} + e^{-iK/2}}$$

$$e^{i\theta} = -\frac{\cancel{2}\cos K/2 - \cancel{2}e^{-iq}}{\cancel{2}\cos\frac{K}{2} - \cancel{2}e^{iq}} = -\frac{\cos K/2 - e^{-iq}}{\cos\frac{K}{2} - e^{iq}} \quad \} \text{ complex conjugate}$$

$$\frac{\theta}{2} + \frac{\pi}{2} = \arg\left(c\frac{K}{2} - e^{-iq}\right) = \arg\left(c\frac{K}{2} - \cos q + i s q\right)$$

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{2}\right) = \frac{s q}{c\frac{K}{2} - c q} \qquad \cancel{\tan\theta =}$$

$$-\cot\frac{\theta}{2} = \frac{s\left(\frac{k_2-k_1}{2}\right)}{c\left(\frac{k_1+k_2}{2}\right) - c\left(\frac{k_2-k_1}{2}\right)} = \frac{s_2c_1 - s_1c_2}{\cancel{c_1c_2 - s_1s_2 - c_2c_1 + s_2s_1}} = \frac{s_2c_1 - s_1c_2}{-2s_1s_2}$$

$$c_1c_2 - s_1s_2 \qquad c_2c_1 + s_1s_2$$

$$\cancel{c_1c_2} - s_1s_2 - \cancel{c_1c_2} - s_1s_2$$

$$\cot\frac{\theta}{2} = \frac{1}{2}\frac{c_1}{s_1} - \frac{1c_2}{2s_2} = \frac{1}{2}\cot k_1 - \frac{1}{2}\cot k_2$$

if k_1 & k_2 complex.

$$a_{j_1j_2} = e^{i\theta/2}\, e^{i\left(\frac{K}{2}-q\right)j_1 + i\left(\frac{K}{2}+q\right)j_2} + e^{-i\theta/2}\,(j_1 \leftrightarrow j_2)$$

$$= e^{i\theta/2}\, e^{i\frac{K}{2}(j_1+j_2) + iq(j_2-j_1)} + e^{-i\theta/2}\, e^{i\frac{K}{2}(j_1+j_2) - iq(j_2-j_1)}$$

$$a_{j_1j_2} = e^{i\frac{K}{2}(j_1+j_2)}\left(e^{i\theta/2 + iq(j_2-j_1)} + e^{-i\theta/2 - iq(j_2-j_1)}\right)$$

$$\boxed{a_{j_1j_2} = e^{i\frac{K}{2}(j_1+j_2)}\; 2\cos\left(\frac{\theta}{2} + q(j_2-j_1)\right)}$$

$K \to$ <u>real</u> otherwise $a_{j_1j_2} \to \infty$ $(j_1+j_2) \to \pm\infty$ together.

$\cos\left(\frac{\theta}{2} + q j\right)$ $\quad j = j_2 - j_1$

$\frac{e^{i\eta j} + e^{-i\eta j}}{2}$ $\qquad \frac{e^{i\eta j} - e^{-i\eta j}}{2i} = + \frac{i}{?}\,\mathrm{sh}(\eta j)$

$q = q_R + i\eta$ take $\eta > 0$

$$\cos\left(\frac{\theta}{2} + q_R j + i\eta j\right) = \cos\left(\frac{\theta}{2} + q_R j\right)\cos(i\eta j) - \sin\left(\frac{\theta}{2} + q_R j\right)\sin(i\eta j)$$

$$= \cos\left(\frac{\theta}{2} + q_R j\right)\mathrm{ch}(\eta j) - \sin\left(\frac{\theta}{2} + q_R j\right) i\,\mathrm{sh}(\eta j)$$

$j \to \infty$ $\qquad \cos\left(\frac{\theta}{2} + q_R j\right)\frac{e^{\eta j}}{2} - i \sin\left(\frac{\theta}{2} + q_R j\right) \frac{e^{\eta j}}{2}$

$q_R = 0$ $\qquad \cos\theta/2 - i\sin\theta/2 = 0$ ~~new~~ unless θ imaginary also.

$$e^{i\theta} = -\frac{cK/2 - e^{-iq_R} - e^{\eta}}{cK/2 - e^{iq_R} - e^{-\eta}}$$

$$e^{i\theta} = -\frac{cK/2 - e^{\eta}}{cK/2 - e^{-\eta}}$$ real $\Rightarrow \theta$ purely imag.

$\theta = i\alpha$

$$e^{-\alpha/2 - \eta(j_2 - j_1)} + e^{\alpha/2 + \underbrace{\eta(j_2 - j_1)}_{0}}$$ $\qquad e^{-\alpha} = -\frac{cK/2 - e^{\eta}}{cK/2 - e^{-\eta}}$

$j_2 - j_1 \to \infty$

$j_2 > j_1$ $\qquad \alpha = -\infty$ $\qquad e^{\alpha} = -\frac{cK/2 - e^{-\eta}}{cK/2 - e^{\eta}}$

$cK/2 = e^{-\eta}$

$$a_{l_1 l_2} = e^{i\frac{K}{2}(l_1+l_2)} \left(e^{-\eta(l_2-l_1)} + e^{\alpha+\eta(l_2-l_1)}\right)$$

$$\boxed{c\frac{K}{2} = e^{-\eta}} \qquad \boxed{\eta = -\ln c\frac{K}{2}}\ \checkmark$$

$$\frac{\varepsilon}{2\lambda} = 4 - e^{i\frac{K}{2}-iq} - e^{-i\frac{K}{2}+iq} - e^{i\frac{K}{2}+iq} - e^{-i\frac{K}{2}-iq})$$

$$= 4 - 2\cos\frac{K}{2}\, e^{iq} - 2\cos\frac{K}{2}\, e^{-iq} = 4 - 4\cos\frac{K}{2}\cos q$$

$$\frac{\varepsilon}{2\lambda} = 4 - \cancel{4}2\cos\frac{K}{2}\left(e^{-\eta} + e^{\eta}\right)$$

$$= 4 - 2\cos\frac{K}{2}\left(c\frac{K}{2} + \frac{1}{c\frac{K}{2}}\right) = 4 - 2\left(1 + c^2\frac{K}{2}\right)$$

$$= 2 - 2c^2\frac{K}{2} = 2s^2\frac{K}{2}$$

$$\boxed{\varepsilon = 4\lambda\, s^2\frac{K}{2}}\ \checkmark \qquad \varepsilon^{\text{unbound}} = 8\lambda s^2\frac{K}{4} + 8\lambda s^2\frac{K}{4} = 16\lambda s^2\frac{K}{4}$$

$$\cancel{4}\cancel{\lambda}s^2\frac{K}{2} \qquad \overset{4}{\cancel{16}}\cancel{\lambda}s^2\frac{K}{4}$$

$$\left(2s\frac{K}{4}c\frac{K}{4}\right)^2 \qquad 4s^2\frac{K}{4}$$

$$4\cancel{s^2\frac{K}{4}}c^2\frac{K}{4} \qquad 4\cancel{s^2\frac{K}{4}}\ 1$$

$$c^2\frac{K}{4} < 1$$

$$\varepsilon_b = 16\lambda s^2\frac{K}{4} - 4\lambda s^2\frac{K}{2}$$

$$= 16\lambda s^2\frac{K}{4} - 16\lambda s^2\frac{K}{4}c^2\frac{K}{4}$$

$$= 16\lambda s^2\frac{K}{4} s^2\frac{K}{4} = 16\lambda s^4\frac{K}{4}$$

$$\boxed{\varepsilon_b = 16\lambda s^4\frac{K}{4}}$$