Quantum dynamics (chapter 2)

Time evolution

$$|\psi, t_0\rangle \longrightarrow |\psi, t_0; t\rangle$$

$$|\psi, t_0; t\rangle = \mathcal{U}(t, t_0)\,|\psi, t_0\rangle$$

$\mathcal{U}$: linear, unitary operator. and $\boxed{\mathcal{U}(t_0, t_0) = \mathbb{1}}$

$$[\mathcal{U}(t, t_0)]^\dagger\, \mathcal{U}(t, t_0) = \mathbb{1}$$

Composition $\quad \mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\, \mathcal{U}(t_1, t_0)$

In particular:

$$\mathcal{U}(t_1 + \Delta t, t_0) = \mathcal{U}(t_1 + \Delta t, t_1)\, \mathcal{U}(t_1, t_0)$$

$\Delta t \to 0 \quad \mathcal{U}(t_1 + \Delta t, t_1) = \mathbb{1} + A \cdot \Delta t + \mathcal{O}(\Delta t^2)$

$$\mathcal{U}^\dagger \mathcal{U} = \mathbb{1} + (A^\dagger + A)\,\Delta t + \mathcal{O}(\Delta t^2) = \mathbb{1}$$

$A^\dagger = -A \quad \Rightarrow \quad A = -\frac{i}{\hbar} H(t_1)$ definition $\quad \boxed{H^\dagger = H} \quad$ H: hamiltonian

$$\mathcal{U}(t_1 + \Delta t, t_1) = \mathbb{1} - \frac{i}{\hbar} H(t_1)\,\Delta t + \mathcal{O}(\Delta t^2)$$

$$\mathcal{U}(t_1 + \Delta t, t_0) = \mathcal{U}(t_1, t_0) - \frac{i}{\hbar} \underbrace{H(t_1)\,\mathcal{U}(t_1, t_0)}_{\text{watch ordering.}}\,\Delta t + \mathcal{O}(\Delta t^2)$$

$$\frac{1}{\Delta t}\left(\mathcal{U}(t_1+\Delta t, t_0) - \mathcal{U}(t_1, t_0)\right) = -\frac{i}{\hbar} H(t_1)\, \mathcal{U}(t_1, t_0) + \mathcal{O}(\Delta t)$$

$\Delta t \to 0$

$$\boxed{\partial_t\, \mathcal{U}(t, t_0) = -\frac{i}{\hbar} H(t)\, \mathcal{U}(t, t_0)}$$

$$|\psi, t_0; t\rangle = \mathcal{U}(t, t_0)\, |\psi, t_0\rangle$$

$$\partial_t |\psi(t)\rangle = -\frac{i}{\hbar} H\, \mathcal{U}(t, t_0)\, |\psi, t_0\rangle = -\frac{i}{\hbar} H |\psi(t)\rangle$$

$$\boxed{\partial_t |\psi(t)\rangle = -\frac{i}{\hbar} H(t) |\psi(t)\rangle}$$

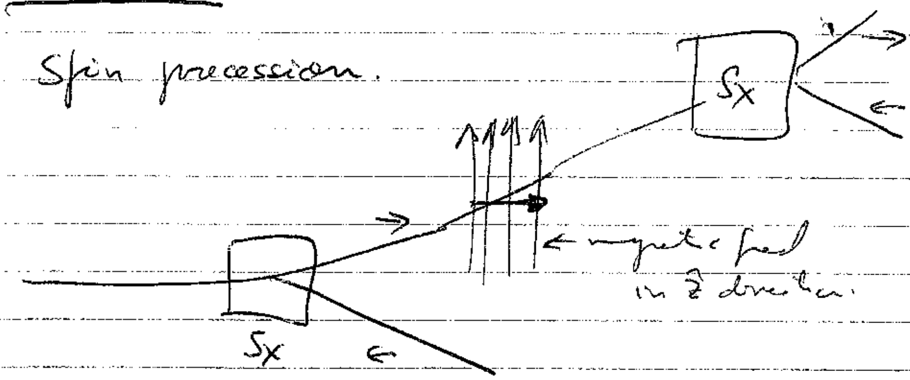
Quite frequently $\partial_t H = 0$ [conserved energy; No time dep. external fields]

$$\partial_t\, |\psi, t_0; t\rangle = -\frac{i}{\hbar} H\, |\psi, t_0; t\rangle$$

$$\boxed{|\psi, t_0; t\rangle = e^{-\frac{i}{\hbar} H (t - t_0)}\, |\psi, t_0\rangle} \leftarrow \partial_t H = 0$$

Example

Spin precession.



$$H = -\vec{\mu}\vec{B} \qquad \mu = -\frac{e}{m}\vec{S} \qquad \left(S_z = \pm\frac{\hbar}{2} \right)$$

$$H = \frac{e}{m} B\, S_z \ ; \quad \vec{B} = B\,\hat{z}$$

$$H|\uparrow\rangle = \frac{eB}{m}\frac{\hbar}{2}|\uparrow\rangle \qquad \boxed{E = \pm\frac{eB}{2m}\hbar}$$

$$H|\downarrow\rangle = -\frac{eB}{m}\frac{\hbar}{2}|\downarrow\rangle$$

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

$$\boxed{|\psi(t)\rangle = \alpha\, e^{-i\frac{eB}{2m}\Delta t}|\uparrow\rangle + \beta\, e^{+i\frac{eB}{2m}\Delta t}|\downarrow\rangle} \qquad \frac{iHt}{\hbar}$$

e.g. $|\rightarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$

$$|\rightarrow, t\rangle = \frac{1}{\sqrt{2}}\, e^{-i\frac{eB}{2m}\Delta t}|\uparrow\rangle + \frac{1}{\sqrt{2}}\, e^{+i\frac{eB}{2m}\Delta t}|\downarrow\rangle$$

$$\omega = \frac{eB}{m} \qquad |\rightarrow, t\rangle = \frac{1}{\sqrt{2}}\, e^{\frac{i}{2}\omega\Delta t}|\uparrow\rangle + \frac{1}{\sqrt{2}}\, e^{-\frac{i\omega}{2}\Delta t}|\downarrow\rangle$$

notice that $\Delta t = \frac{\pi}{\omega}$ same state but with (−) sign 0 → interference.

$P_\uparrow = \frac{1}{2} \quad P_\downarrow = \frac{1}{2} \qquad S_z$ centered

$P_\rightarrow = ?$

$$A_\rightarrow = \langle \rightarrow | \psi \rangle = \frac{1}{\sqrt{2}}\left(\langle\uparrow| + \langle\downarrow|\right)\left(\frac{1}{\sqrt{2}} e^{\frac{i\omega_0 t}{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}} e^{-\frac{i\omega_0 t}{2}}|\downarrow\rangle\right)$$

$$= \frac{1}{2}\left(e^{\frac{i\omega_0 t}{2}} + e^{-\frac{i\omega_0 t}{2}}\right) = \cos\left(\frac{\omega_0 t}{2}\right) \qquad (=1 \ \text{@} \ t=0)$$

$$\boxed{P_\rightarrow = |A_\rightarrow|^2 = \cos^2\left(\frac{\omega_0 t}{2}\right)}$$

$$\boxed{P_\leftarrow = \sin^2\left(\frac{\omega_0 t}{2}\right)}$$

$$\frac{\omega_0 t}{2} = \frac{1}{2}\frac{eB}{m}\Delta t$$

$$\frac{eB}{m} = \frac{1\,\text{eV}\cdot\text{s}\,(3\times10^8\,\text{m/s})^2}{0.5\,\text{MeV/m}^2} = 2\cdot10^{-6}\times9\times10^{16}\,\frac{1}{s} = 18\times10^{10}\,\frac{1}{s} = 1.8\times10^{11}\,\frac{1}{s}$$

$[B] \rightarrow$ tesla $\quad 1T = \frac{1\,V\cdot s}{m^2}$

$$\omega = 1.8\times10^{11}\,\frac{1}{s}\cdot B[T]$$

$$\frac{1}{2}\frac{eB}{m}\Delta t = 2\pi \qquad \Delta t = \frac{4\pi}{\omega} = \frac{12}{18\times10^{10}\,B[T]}\,s = \boxed{0.6\times10^{-10}\,s}$$

1 gauss $= 10^{-4}$ T $\quad \boxed{0.6\times10^{-6}\,s}$

J

$\frac{1}{\cancel{2}}MV^2 = \frac{3}{\cancel{2}}k_BT$

1300°K ~ ~~0.025eV~~

300°K → 0.025 eV

1300°K — $\frac{1300\times 0.025\,eV}{300} \approx \boxed{0.1\,eV}$

$MV^2 = 0.3\,eV$

$M = 107$

$M \simeq 100 \times m_p = 100 \times 10^3\,MeV = 10^5\,MeV$

$V^2 = \frac{0.3\,eV}{Mc^2}c^2 = \frac{0.3\,\cancel{eV}}{10^5\times 10^6\,\cancel{eV}}c^2 = 0.3\times 10^{-11}c^2 = 3\times 10^{-12}c^2$

$V \simeq 1.7\times 10^{-6}c = 1.7\times 10^{-6}\,3\times 10^8\,m/s = 5\times 10^2\,m/s$

ΔX $\qquad \Delta t = \frac{\Delta X}{V}$

$\Delta X \sim 1m \Rightarrow \quad \Delta t = \frac{1m}{5\times 10^2\,m/s} = 0.2\times 10^{-2}\,s = 2\times 10^{-3}\,s$

1 gauss $\quad$ 1 cm → $\Delta t = 2\times 10^{-5}\,s$

$\frac{1}{2}\frac{eB}{m}\Delta t = 2\pi n \qquad \cancel{n = \frac{0.6\times 10^{-6}s}{2\times 10^{-5}s}} =$

$n = \frac{2\times 10^{-5}}{0.6\times 10^{-6}} = \frac{10}{0.3}$ $3\times 10 = 30$ oscillations 1 cm and 1 gauss

neutrons are better. smaller μ.

$$\boxed{\psi(x,t=0) = \frac{1}{\pi^{1/4}\sqrt{\sigma}}\, e^{ikx - \frac{x^2}{2\sigma^2}}}$$

$$\int_{-\infty}^{\infty} |\psi|^2 dx = \frac{1}{\sqrt{\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-x^2/\sigma^2} dx = \frac{1}{\sqrt{\pi}\,\sigma}\sqrt{\frac{\pi}{1/\sigma^2}} = 1$$

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx\, \langle p|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\, \frac{e^{-\frac{ipx}{\hbar}}}{\sqrt{2\pi\hbar}}\, \frac{1}{\pi^{1/4}\sqrt{\sigma}}\, e^{ikx - \frac{x^2}{2\sigma^2}} \cancel{d} =$$

$$= \frac{1}{\sqrt{\cancel{2}\pi\hbar}}\, \frac{1}{\pi^{1/4}\cancel{\sqrt{\sigma}}} \sqrt{\frac{\pi}{1/2\sigma^2}}\, e^{(ik - ip/\hbar)^2 \frac{1}{4\frac{1}{2\sigma^2}}} = \frac{1}{\pi^{1/4}}\sqrt{\frac{\sigma}{\hbar}}\, e^{-\frac{\sigma^2}{2\hbar^2}(p-\hbar k)^2}$$

$$\boxed{\tilde{\psi}(p) = \frac{1}{\pi^{1/4}}\sqrt{\frac{\sigma}{\hbar}}\, e^{-\frac{1}{2}\frac{\sigma^2}{\hbar^2}(p-\hbar k)^2}} \quad \leftarrow \quad \langle p\rangle = \hbar k$$

$$\int_{-\infty}^{\infty} |\tilde{\psi}(p)|^2 dp = \frac{1}{\cancel{\sqrt{\pi}}}\, \frac{\cancel{\sigma}}{\cancel{\hbar}} \sqrt{\frac{\cancel{\pi}}{\cancel{\sigma^2/\hbar^2}}} = 1 \checkmark$$

$$\boxed{\tilde{\psi}(p,t) = \frac{1}{\pi^{1/4}}\sqrt{\frac{\sigma}{\hbar}}\, e^{-\frac{ip^2}{2m\hbar}t}\, e^{-\frac{1}{2}\frac{\sigma^2}{\hbar^2}(p-\hbar k)^2}}$$

(2)

$$\psi(x,t) = \int_{-\infty}^{\infty} dp \, \frac{e^{\frac{ipx}{\hbar}}}{\sqrt{2\pi\hbar}} \frac{1}{\pi^{\frac{1}{4}}} \sqrt{\frac{\sigma}{\hbar}} \, e^{-\frac{ip^2}{2m\hbar}t - \frac{1}{2}\frac{\sigma^2}{\hbar^2}(p-\hbar k)^2}$$

$$= \frac{\sqrt{\sigma}}{\sqrt{2\pi}\,\pi^{1/4}} \frac{1}{\hbar} \int_{-\infty}^{\infty} dp \, e^{\frac{ipx}{\hbar} - \frac{ip^2 t}{2m\hbar} - \frac{1}{2}\frac{\sigma^2 p^2}{\hbar^2} + \frac{\sigma^2 pk}{\hbar} - \frac{\sigma^2 k^2}{2}}$$

$$-\frac{1}{2}p^2\left(\frac{\sigma^2}{\hbar^2} + \frac{it}{m\hbar}\right) + p\left(\frac{ix}{\hbar} + \frac{\sigma^2 k}{\hbar}\right) - \frac{\sigma^2 k^2}{2}$$

$$= \frac{\sqrt{\sigma}}{\sqrt{\cancel{2\pi}}\,\pi^{1/4}} \frac{1}{\hbar} e^{-\frac{\sigma^2 k^2}{2}} \sqrt{\frac{\cancel{\pi}}{\frac{1}{\cancel{2}}\left(\frac{\sigma^2}{\hbar^2} + \frac{it}{m\hbar}\right)}} \, e^{\frac{\left(\frac{ix}{\hbar} + \frac{\sigma^2 k}{\hbar}\right)^2}{\frac{4}{2}\left(\frac{\sigma^2}{\hbar^2} + \frac{it}{m\hbar}\right)}}$$

$$\boxed{= \sqrt{\frac{\sigma}{\sigma^2 + \frac{it\hbar}{m}}} \, \frac{1}{\pi^{1/4}} \, e^{-\frac{\sigma^2 k^2}{2}} \, e^{\frac{1}{2}\frac{(ix + \sigma^2 k)^2}{\left(\sigma^2 + \frac{it\hbar}{m}\right)}}}$$

$$|\psi(x,t)|^2 = \frac{\sigma}{\sqrt{\sigma^4 + \frac{t^2\hbar^2}{m^2}}} \frac{1}{\sqrt{\pi}} e^{-\sigma^2 k^2} e^{\frac{1}{2}\frac{(ix+\sigma^2 k)^2}{(\sigma^2 + it\frac{\hbar}{m})} + \frac{1}{2}\frac{(-ix+\sigma^2 k)^2}{(\sigma^2 - it\frac{\hbar}{m})}}$$

$$= \frac{1}{\sqrt{\sigma^2 + \frac{t^2\hbar^2}{m^2\sigma^2}}} \frac{1}{\sqrt{\pi}} e^{-\sigma^2 k^2 + \frac{(ix+\sigma^2 k)^2(\sigma^2 - it\frac{\hbar}{m}) + (-ix+\sigma^2 k)^2(\sigma^2 + it\frac{\hbar}{m})}{2\left(\sigma^4 + \frac{t^2\hbar^2}{m^2}\right)}}$$

$$= 2\,\mathrm{Re}\left(\left(-x^2 + \sigma^4 k^2 + 2ix\sigma^2 k\right)\left(\sigma^2 - i\frac{t\hbar}{m}\right)\right) = 2\left(-x^2\sigma^2 + \sigma^6 k^2 + 2x\sigma^2 k\frac{t\hbar}{m}\right)$$

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\sigma^2 + t^2\hbar^2/m^2\sigma^2}} \; e^{\frac{-\cancel{2\sigma^6 k^2} - 2\sigma^2 k^2 t^2 \hbar^2/m^2 - 2x^2\sigma^2 + \cancel{2\sigma^6 k^2} + 4x\sigma^2 k t \hbar/m}{2(\sigma^4 + t^2\hbar^2/m^2)}} \quad 3$$

$$e^{-\frac{x^2\sigma^2 + \sigma^2 k^2 t^2 \hbar^2/m^2 - 2x\sigma^2 k t \hbar/m}{\sigma^4 + t^2\hbar^2/m^2}}$$

$$e^{-\frac{x^2 + k^2 t^2 \hbar^2/m^2 - 2x(kt\hbar/m)}{\sigma^2 + \frac{t^2\hbar^2}{m^2\sigma^2}}}$$

$$e^{-\frac{(x - kt\hbar/m)^2}{\sigma^2 + \frac{t^2\hbar^2}{m^2\sigma^2}}}$$

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\sigma(t)^2}} \; e^{-\frac{(x - x_0(t))^2}{\sigma^2(t)}}$$

$$\boxed{x_0(t) = \frac{\hbar k}{m} t \qquad \sigma(t) = \sqrt{\sigma^2 + \frac{t^2\hbar^2}{m^2\sigma^2}}}$$

$$\boxed{|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}\,\sigma(t)} \; e^{-\frac{1}{\sigma^2(t)}(x - x_0(t))^2}}$$

$$\int_{-\infty}^{\infty} |\psi|^2 dx = \frac{1}{\sqrt{\pi}\,\sigma} \sqrt{\frac{\pi}{1/\sigma^2}} = 1 \quad ✓$$

①

$$\langle px+xp\rangle = \int_{-\infty}^{\infty} \psi^*(x)\left[-i\hbar\partial_x(x\psi) + x(-i\hbar\partial_x\psi)\right]dx$$

$$= \int_{-\infty}^{\infty} \psi^*(x)\left[-i\hbar\psi - 2i\hbar x\partial_x\psi\right]dx$$

$$= -i\hbar\int_{-\infty}^{\infty}|\psi|^2 - 2i\hbar\int_{-\infty}^{\infty}\psi^* x\partial_x\psi\,dx$$

$$x\partial_x\psi = x\frac{1}{\cancel{2}}\frac{\cancel{2}(ix+\sigma^2 k)}{\sigma^2+\frac{it\hbar}{m}}\,i$$

$$\langle px+xp\rangle = -i\hbar - 2i\hbar\int_{-\infty}^{\infty}|\psi|^2\frac{(-x^2+i\sigma^2 kx)}{\sigma^2+\frac{it\hbar}{m}}dx$$

$$|\psi|^2 = \frac{1}{\sqrt{\pi}\,\sigma(t)}e^{-\frac{1}{\sigma^2(t)}(x-x_0(t))^2}$$

$$\int x|\psi|^2 = \int(x-x_0+x_0)|\psi|^2 = x_0(t)$$

$$\int x^2|\psi|^2 = \int e^{-\frac{\cancel{x}^2}{\sigma^2}}\frac{\cancel{x}^2}{\sqrt{\pi}\,\sigma}dx = \frac{1}{\cancel{\sqrt{\pi}}\,\sigma}\frac{\cancel{\sqrt{\pi}}}{2}\sigma^3 = \frac{\sigma^2}{2}$$

$$\int e^{-\alpha x^2}x^2dx = -\partial_\alpha\sqrt{\frac{\pi}{\alpha}} = \frac{\sqrt{\pi}}{2\alpha^{3/2}}$$

②

$$\langle px+xp\rangle = -i\hbar - 2i\hbar \frac{1}{\sigma^2 + i\frac{t\hbar}{m}} \left(-\langle x^2\rangle + i\sigma^2 k\langle x\rangle\right)$$

$$\langle (x-x_0)^2\rangle = \frac{\sigma^2(t)}{2}$$

$$\langle x^2 - 2xx_0 + x_0^2\rangle = \frac{\sigma^2(t)}{2} \quad \Rightarrow \quad \langle x^2\rangle = \frac{\sigma^2(t)}{2} + x_0^2$$

$$\langle px+xp\rangle = -i\hbar - 2i\hbar \frac{1}{\sigma^2 + i\frac{t\hbar}{m}} \left(-\frac{\sigma^2(t)}{2} - x_0^2 + i\sigma^2 k x_0\right)$$

$$= -i\hbar - 2i\hbar \frac{1}{\sigma^2 + i t\frac{\hbar}{m}} \left(-\frac{\sigma^2}{2} - \frac{t^2\hbar^2}{2m^2\sigma^2} - \frac{\hbar^2 k^2}{m^2}t^2 + i\frac{\sigma^2 t\hbar k^2 t}{m}\right)$$

$$= \frac{-i\hbar}{\sigma^2 + i t\frac{\hbar}{m}} \left(\sigma^2 + i t\frac{\hbar}{m} + 2\left(-\frac{\sigma^2}{2} - \frac{\hbar^2 t^2}{2m^2\sigma^2} - \frac{\hbar^2 k^2}{m^2}t^2 + i\frac{\sigma^2 t\hbar k^2 t}{m}\right)\right)$$

$$= -\frac{i\hbar}{\sigma^2 + i t\frac{\hbar}{m}} \left(\cancel{\sigma^2} + i t\frac{\hbar}{m} - \cancel{\sigma^2} - \frac{\hbar^2 t^2}{m^2\sigma^2} - \frac{2\hbar^2 k^2 t^2}{m^2} + 2i\frac{\sigma^2 t\hbar k^2 t}{m}\right)$$

$$= -\frac{i\hbar}{\sigma^2 + i t\frac{\hbar}{m}}\, i t\frac{\hbar}{m}\left(1 + \frac{i t\hbar}{m\sigma^2} + 2i\frac{\hbar k^2 t}{m} + 2\sigma^2 k^2\right)$$

$$= \frac{t\hbar^2/m}{\sigma^2 + it\hbar/m}\left(1 + \frac{it\hbar}{m\sigma^2} + 2\sigma^2 k^2\left(1 + \frac{i\hbar t}{m\sigma^2}\right)\right)$$

$$= \frac{t\hbar^2/m}{\cancel{\sigma^2 + it\hbar/m}}(1 + 2\sigma^2 k^2)\frac{1}{\sigma^2}\cancel{\left(\sigma^2 + \frac{it\hbar}{m}\right)}$$

$$= \frac{t\hbar^2}{m\sigma^2}(1 + 2\sigma^2 K^2)$$

$$\frac{\partial \sigma^2(t)}{2\,\partial t} = \frac{1}{m}\left\{\frac{t\hbar^2}{m\sigma^2}(1 + 2\sigma^2 k^2) - 2\, x_0(t)\hbar k\right\}$$

$$= \frac{1}{m^2}\frac{t\hbar^2}{\sigma^2}(1 + 2\sigma^2 k^2) - 2\,\frac{(\hbar k)^2}{m^2}t$$

$$= \frac{\hbar^2 t}{m^2\sigma^2} + \cancel{2\frac{t\hbar^2 k^2}{m^2}} - \cancel{2\frac{(\hbar k)^2 t}{m^2}}$$

$$\boxed{\frac{\partial \sigma^2(t)}{\partial t} = \frac{2\hbar^2 t}{m^2\sigma^2}}$$

$$\frac{\partial \sigma^2}{\partial t} = \frac{2t\hbar^2}{m^2\sigma^2}$$

evolution of mean values:

$$\frac{\partial}{\partial t}\left(\langle\psi|A|\psi\rangle\right) = \left(\frac{\partial\langle\psi|}{\partial t}\right) A\,|\psi\rangle + \langle\psi|A\,\frac{\partial|\psi\rangle}{\partial t}$$

$$\frac{\partial|\psi\rangle}{\partial t} = -i\frac{H}{\hbar}|\psi\rangle \qquad = i\,\langle\psi|HA|\psi\rangle - i\,\langle\psi|AH|\psi\rangle$$

$$\frac{\partial\langle\psi|}{\partial t} = \frac{i}{\hbar}\langle\psi|H \qquad = \frac{i}{\hbar}\langle\psi|[H,A]|\psi\rangle$$

if $[H,A]=0$

$\Rightarrow \frac{\partial\langle A\rangle}{\partial t} = $ costante

conserved!

$$H = \frac{p^2}{2m} + V(x) \qquad [p,x] = -i\hbar$$

$$\frac{\partial\langle x\rangle}{\partial t} = \frac{i}{2m\hbar}\langle[x,p^2]\rangle = -\frac{i}{m}\frac{i\hbar}{\hbar}\langle p\rangle = +\frac{\langle p\rangle}{m}$$

$$\frac{\partial\langle p\rangle}{\partial m} = \frac{i}{\hbar}\langle\psi|[V(x),p]|x\rangle = \frac{i}{\hbar}\langle\partial_x V\rangle i\hbar = -\langle\partial_x V\rangle$$

$$\begin{cases} \dfrac{\partial\langle x\rangle}{\partial t} = \dfrac{\langle p\rangle}{m} \\[2ex] \dfrac{\partial\langle p\rangle}{\partial t} = -\langle\partial_x V\rangle \Rightarrow \boxed{m\dfrac{\partial^2\langle x\rangle}{\partial t^2} = -\langle\partial_x V\rangle} \end{cases} \qquad \text{Ehrenfest}$$

For harmonic osc. $\langle x\rangle$ oscillates classically,

$$\frac{\partial\langle x^2\rangle}{\partial t} = \frac{i}{2m\hbar}\langle [p^2, x^2]\rangle = \frac{i}{2m\hbar}\langle p[p, x^2] + [p, x^2]p\rangle$$

$$= \frac{2\cancel{i}}{2m\cancel{\hbar}}\langle p(-\cancel{i\hbar})x + (-\cancel{i\hbar})xp\rangle = \frac{2}{2m}\langle px + xp\rangle$$

$$\frac{\partial\langle x\rangle}{\partial t} = \frac{\langle p\rangle}{m}$$

$$\frac{\partial(\langle x^2\rangle - \langle x\rangle^2)}{\partial t} = \frac{1}{m}\langle px + xp\rangle - 2\langle x\rangle\frac{\langle p\rangle}{m}$$

$$= \frac{1}{m}\Big(\langle px\rangle - \langle p\rangle\langle x\rangle + \langle xp\rangle - \langle x\rangle\langle p\rangle\Big)$$

$$\frac{\partial(\sigma^2)}{\partial t} = \frac{1}{m}\Big\{\underbrace{\langle px + xp\rangle}_{D} - 2\langle x\rangle\langle p\rangle\Big\}$$

Heisenberg picture.

$$\langle \psi | A | \psi \rangle (t) = \langle \psi, 0 | U^{-1}(t)\, A\, U(t)\, |\psi, 0\rangle$$

Define $A(t) = U^{-1}(t)\, A\, U(t) = U^{+}(t)\, A\, U$

$$A(t) = e^{\frac{iHt}{\hbar}} A\, e^{-\frac{iHt}{\hbar}}$$

~~$\frac{\partial A}{\partial t} = e^{\frac{iHt}{\hbar}}$~~

$$\frac{\partial U}{\partial t} = -\frac{iH(t)}{\hbar} U$$

$$\frac{\partial U^{+}}{\partial t} = +\frac{i}{\hbar} U^{+} H$$

if $[A, B] = C$

$$[U^{+}AU, U^{+}BU] = U^{+}CU$$

$$\Rightarrow \boxed{[A(t), B(t)] = C(t)}$$

preserve comm. relation

$\boxed{[x(t), p(t)] = i\hbar \quad \text{etc.}}$

$$\frac{\partial A(t)_H}{\partial t} = +\frac{i}{\hbar} U^{+} H A U - \frac{i}{\hbar} U^{+} A H U = \frac{i}{\hbar} U^{+} [H, A]\, U$$

$$= \frac{i}{\hbar} U^{+} H U U^{+} A U - \frac{i}{\hbar} U^{+} A U U^{+} H U = \frac{i}{\hbar} [U^{+} H U, A(t)]$$

$$= \frac{i}{\hbar} [\tilde{H}, A(t)_H] \qquad H_H = U^{+} H U \qquad \boxed{H = U H_0 U^{+}}$$

constant

$$\frac{\partial H_H}{\partial t} = \frac{i}{\hbar} U^{+} H H U - \frac{i}{\hbar} U^{+} H H U = 0$$

again $[H, A] = 0 \Rightarrow \partial A / \partial t = 0$ conserved!

example a, a^+ in h.o

$H = \hbar\omega(a_H^+ a_H + 1/2)$

$$\boxed{U^+ H U = H}$$

$[a, H] = \hbar\omega a$

$[a^+, H] = -\hbar\omega a^+$

$$\frac{\partial a_H}{\partial t} = \frac{i}{\hbar}[H, a_H] = -\frac{i}{\cancel{\hbar}}\cancel{\hbar}\omega a = -i\omega a$$

$$\boxed{\begin{aligned} a_H(t) &= e^{-i\omega t} a_s \\ a_H^+(t) &= e^{i\omega t} a_s \end{aligned}}$$

$\rightarrow$ ~~$x(t) =$~~

$[p, x] = -i\hbar$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

~~$[H,p]$~~ $[H, p] = m\omega^2 x(+i\hbar) = i\hbar m\omega^2 x$

$[H, x] = \frac{1}{m} p(-i\hbar)$

$$\frac{\partial x_H}{\partial t} = \frac{\cancel{i}}{\cancel{\hbar}}(\cancel{-i\hbar})\frac{p}{m} = p/m$$

$$\frac{\partial p_H}{\partial t} = \frac{i}{\cancel{\hbar}} i\cancel{\hbar} m\omega^2 x = -m\omega^2 x_H$$

$$\frac{\partial^2 x_H}{\partial t^2} = -\omega^2 x_H$$

$$\boxed{\begin{aligned} x_H &= \cos(\omega t) x_s + \sin(\omega t) p_s / m\omega \\ p_H &= \cos(\omega t) p_s - \sin\omega t \; m\omega^2 x_s \end{aligned}}$$

Time dep sch. eq.

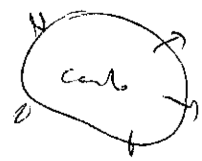
$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}\langle x|H|\psi\rangle = -\frac{i}{\hbar}\left(-\nabla^2\psi + V(x)\psi(x)\right)$$

$$\frac{\partial|\psi|^2}{\partial t} = \psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi = \frac{i}{\hbar}\psi^*\nabla^2\psi \cancel{-\frac{i}{\hbar}V\psi^*\psi} +$$

$$+\frac{i}{\hbar}\left(-\psi\nabla^2\psi^* \cancel{+V\psi^*\psi}\right)$$

$$= \frac{i}{\hbar}\left(\psi^*\nabla^2\psi - \nabla^2\psi^*\psi\right) = \frac{i}{\hbar}\nabla\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$\boxed{\frac{\partial\rho}{\partial t} = \nabla j} \Rightarrow \boxed{\frac{\partial}{\partial t}\int_V \rho = \oiint_{\partial V} j\,d\vec{s}}$$



local conservation law.

$\int|\psi|^2$: probability   conservation of probability

$$\boxed{j = \frac{i}{\hbar}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)}$$ probability flux.

charged particle $j = \frac{ie}{\hbar}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$

electric current.

In an eigenstate

$\psi = e^{iEt/\hbar}\psi_0$ $\boxed{|\psi| = \text{constant}}$ $\rightarrow$ $\boxed{\nabla j = 0}$