Homework V hyperfine transition.

$$V = -\vec{\mu}\cdot\vec{B} \quad \text{perturbation}$$

$$\vec{\mu} = -\frac{e}{mc}\,\vec{S}_e \quad \Rightarrow \quad V = \frac{e}{mc}\,\vec{B}\cdot\vec{S}$$

$$\vec{B} = \nabla\times\vec{A} = \sqrt{\frac{4\pi}{V}}\sum_{\vec{k},\alpha} c\sqrt{\frac{\hbar}{2\omega}}\left[a_{\vec{k},\alpha}\,(i\vec{k}\times\vec{\epsilon}^{(\alpha)})\,e^{i\vec{k}\vec{x}} + a^{\dagger}_{\vec{k},\alpha}\,(-i\vec{k}\times\vec{\epsilon}^{(\alpha)})\,e^{-i\vec{k}\vec{x}}\right]$$

$$\langle\vec{k},\alpha|\vec{B}|0\rangle = \sqrt{\frac{4\pi}{V}}\,c\sqrt{\frac{\hbar}{2\omega}}\,(-i\vec{k}\times\epsilon^{(\alpha)})\,e^{-i\vec{k}\vec{x}}$$

1 photon

vacuum (of photons)

$$\langle 1s|\,e^{-i\vec{k}\vec{x}}\,|1s\rangle \simeq 1$$

Spin take initial state $\frac{1}{\sqrt{2}}\left(|\overset{e^-}{\uparrow}\overset{p^+}{\downarrow}\rangle + |\overset{e^-}{\downarrow}\overset{p^+}{\uparrow}\rangle\right)$

final state $\frac{1}{\sqrt{2}}\left(|\overset{e^-}{\uparrow}\overset{p^+}{\downarrow}\rangle - |\overset{e^-}{\downarrow}\overset{p^+}{\uparrow}\rangle\right)$

$$\frac{1}{\sqrt{2}}\left(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|\right)\,\vec{S}_e\left(\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)\right) = 0 \quad \text{for } S_x, S_y$$

and $\vec{S}_z$ gives $\rightarrow \frac{\hbar}{2}$

So

$$\langle f|V|i\rangle = \sqrt{\frac{4\pi}{V}}\, c\sqrt{\frac{\hbar}{2\omega}}\,(-i\vec{k}\times\epsilon^{(\alpha)})_z\,\frac{\hbar}{2}\,\frac{e}{mc}$$

$$w_{i\to f} = \frac{2\pi}{\cancel{\hbar}}\,\frac{4\pi}{V}\,\cancel{c^2}\,\frac{\cancel{\hbar}}{2\omega}\left|(-i\vec{k}\times\epsilon^{(\alpha)})_z\right|^2\,\frac{\hbar^2}{\cancel{4}}\,\frac{e^2}{m^2\cancel{c^2}}\,\delta(E_f-E_i)$$

$$= \frac{\pi^2}{V}\,\frac{\hbar^2 e^2}{m^2\omega}\left(k_x\epsilon_y^{(\alpha)} - k_y\epsilon_x^{(\alpha)}\right)^2\delta(E_f-E_i)$$

$$\int_f w_{i\to f} = \cancel{V}\int\frac{d^3k}{(2\pi)^3}\sum_{\alpha=1,2}\frac{\pi^2}{\cancel{V}}\,\frac{\hbar^2 e^2}{m^2\omega}\left(k_x\epsilon_y^{(\alpha)} - k_y\epsilon_x^{(\alpha)}\right)^2\delta(E_f-E_i)$$

$$= \frac{1}{8\pi}\,\frac{\hbar^2 e^2}{m^2\omega}\sum_\alpha\int_0^\infty k^2dk\int d\Omega\; k^2\left(\hat{k}_x\epsilon_y^{(\alpha)} - \hat{k}_y\epsilon_x^{(\alpha)}\right)^2\delta(\hbar\omega-\Delta E)$$

$$\delta(\hbar kc - \Delta E)$$

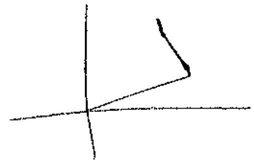
$$\frac{1}{\hbar c}\,\delta\!\left(k - \frac{\Delta E}{\hbar c}\right)$$

$$= \frac{\hbar e^2}{8\pi m^2\omega c}\,k^4\sum_\alpha\int d\Omega\left(\hat{k}_x\epsilon_y^{(\alpha)} - \hat{k}_y\epsilon_x^{(\alpha)}\right)^2$$

$$k = \frac{\omega}{c}$$

$$= \frac{\hbar e^2\omega^3}{8\pi m^2 c^5}\sum_\alpha\int d\Omega\left(\hat{k}_x\epsilon_y^{(\alpha)} - \hat{k}_y\epsilon_x^{(\alpha)}\right)^2$$

Take



$$\hat{k} = (s\theta c\varphi,\ s\theta s\varphi,\ c\theta)$$

$$\epsilon^{(1)} = (-s\varphi,\ c\varphi,\ 0)$$

$$\epsilon^{(2)} = (c\theta c\varphi,\ c\theta s\varphi,\ -s\theta)$$

$$\hat{k}_x \epsilon^{(1)}_y - \hat{k}_y \epsilon^{(1)}_x = s\theta c^2\varphi + s\theta s^2\varphi = s\theta$$

$$\hat{k}_x \epsilon^{(2)}_y - \hat{k}_y \epsilon^{(2)}_x = s\theta c\varphi c\theta s\varphi - s\theta s\varphi c\theta c\varphi = 0$$

$$\sum_\alpha \int d\Omega \left(\hat{k}_x \epsilon^{(\alpha)}_y - \hat{k}_y \epsilon^{(\alpha)}_x\right)^2 = \int s\theta d\theta d\varphi\ s^2\theta = 2\pi \int_{-1}^{1} (1-\mu^2) d\mu$$

$$= 2\pi \left(2 - \frac{2}{3}\right) = \frac{8\pi}{3}$$

$$\int_f w_{i\to f} = \frac{\hbar e^2 \omega^3}{3 m^2 c^5} = \frac{e^2}{\hbar c} \frac{\hbar^2 \omega^3}{3 m^2 c^4} = \frac{\alpha}{3} \frac{(\hbar\omega)^3 c}{(mc^2)^2 \hbar c}$$

$$\frac{\mathrm{MeV}^3 \ \ \mathrm{m/s}}{\mathrm{MeV}^2\ \mathrm{MeV\,fm}} = 1/s \quad ✓$$

$$\tau = \frac{3}{\alpha} \frac{(mc^2)^2 \hbar c}{(\hbar\omega)^3\ c} = 3\times 137 \frac{(0.511\ \mathrm{MeV})^2\ 197\ \mathrm{MeV\,fm}}{(5.859\times10^{-6})^3 \mathrm{eV}^3\ 3\times10^8\ \mathrm{m/s}} =$$

$$= 3.5\times10^{14}\ \mathrm{s} \simeq 11\times10^6\ \text{years}.$$

$$\nu = 1420\ \mathrm{MHz} \qquad \omega = \frac{2\pi}{T} = 2\pi\nu \quad \to \hbar\omega = 2\pi \frac{\hbar c}{c} \nu = 2\pi \frac{197\ \mathrm{MeV\,fm}}{3\times10^8\ \mathrm{m/s}} 1420\times10^6 \frac{1}{s} =$$

$$= 5.859\times10^{-6}\ \mathrm{eV}$$