Fermigas

$$p = \frac{g\, k_B T}{h^3}\, 4\pi \int_0^\infty p^2 dp \ln\left(1 + e^{-\frac{\beta p^2}{2m} + \beta\mu}\right)$$

$$\frac{1}{v} = \frac{4\pi}{h^3} g \int_0^\infty p^2 dp \frac{1}{e^{\beta p^2/2m - \beta\mu} + 1}$$

g = degeneracy. (e.g. $2s+1$)

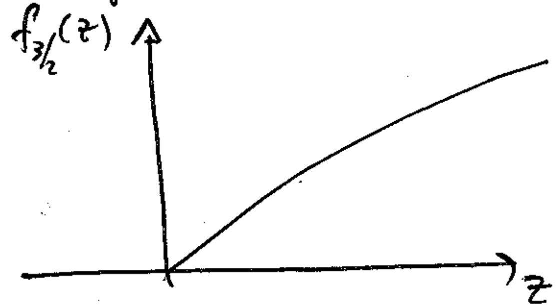
$$\lambda = \sqrt{\frac{h^2}{2m\pi k_B T}}$$ thermal wave length.

$z = e^{\beta\mu}$ fugacity. ; $v = \frac{V}{N}$

$$\frac{P}{k_B T} = \frac{g}{\lambda_T^3} f_{5/2}(z) \qquad z > 0$$

$$\frac{1}{v} = \frac{g}{\lambda_T^3} f_{3/2}(z)$$

if N is given one should invert



$$f_a(z) = \sum_{l=1}^{\infty} \frac{(-)^{l+1} z^l}{l^a}$$

Also

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^{\infty} dx \, \underbrace{\frac{x^2}{z^{-1} e^{x^2} + 1}}_{\geq 0}$$

$z \uparrow \quad (z^{-1} e^{x^2} + 1) \downarrow \quad \frac{1}{(z^{-1} e^{x^2} + 1)} \uparrow \quad f_{3/2}(z) \uparrow$

$f_{3/2}(z)$ monotonically increasing with z.

$z \to 0$

$$f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots$$

$z \to \infty$?

$z = e^{\beta\mu} = e^{\nu} \quad ; \nu = \beta\mu$

$\nu \to \infty$ $\qquad y = x^2 \qquad 2x\,dx = dy$

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^{\infty} dx \frac{x^2}{e^{x^2 - \nu} + 1} = \frac{4}{\sqrt{\pi}} \int_0^{\infty} \frac{dy}{2\sqrt{y}} \frac{y}{e^{y-\nu} + 1}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} dy \frac{\sqrt{y}}{e^{y-\nu} + 1} = \frac{2}{\sqrt{\pi}} \frac{2}{3} \int_0^{\infty} dy \frac{\partial_y y^{3/2}}{e^{y-\nu} + 1}$$

$$= \frac{4}{3\sqrt{\pi}} \left[\underbrace{\int_0^\infty dy\, \partial_y \left(\frac{y^{3/2}}{e^{y-\nu}+1} \right)}_{0} + \int_0^\infty dy \frac{e^{y-\nu}\, y^{3/2}}{(e^{y-\nu}+1)^2} \right]$$ (3)

$$= \frac{4}{3\sqrt{\pi}} \int_0^\infty dy \frac{y^{3/2} e^{y-\nu}}{(e^{y-\nu}+1)^2}$$

$$\left[f_a(z) = \frac{1}{\Gamma(a)} \int_0^\infty dy \frac{y^{a-1}}{z^{-1}e^{y}+1} \right]$$

$\nu \to \infty$ $\quad y < \nu$ $\quad$ integrand exponentially small $e^{-\nu}$

so. $\quad y \gg \nu$ $\quad e^{-y}$ also small

integral comes from $\boxed{y \sim \nu}$ Sommerfeld.

$x = y - \nu$

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} \int_{-\nu}^\infty dx \frac{(x+\nu)^{3/2} e^{x}}{(e^{x}+1)^2}$$

$x < -\nu \quad \nearrow \sim e^{-\nu}$

(4)

$$f_{3/2}(z) \underset{z\to\infty}{\simeq} \frac{4}{3\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \frac{(x+\nu)^{3/2} e^{x}}{(e^{x}+1)^2} + \mathcal{O}(e^{-\nu})$$

$\downarrow 1/z$

$$= \frac{4}{3\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \frac{(x+\nu)^{3/2}}{\left(\frac{e^{x/2}+e^{-x/2}}{2}\right)^2}$$

$$= \frac{1}{3\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \frac{(x+\nu)^{3/2}}{ch^2(x/2)} = \frac{\nu^{3/2}}{3\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \frac{(1+x/\nu)^{3/2}}{ch^2(x/2)}$$

$$\simeq \frac{\nu^{3/2}}{3\sqrt{\pi}} \left[\int_{-\infty}^{\infty} \frac{dx}{ch^2(x/2)} + \underbrace{\frac{3}{2} \int_{-\infty}^{\infty} dx \, \frac{x/\nu}{ch^2(x/2)}}_{0} + \frac{3}{8} \int_{-\infty}^{\infty} dx \, \frac{x^2/\nu^2}{ch^2(x/2)} \right.$$

$x^{3/2}$ $\quad \frac{3}{2} x^{1/2}$ $\quad \frac{3}{4} x^{-1/2}$

0 $\quad$ 1 $\quad$ 2

$$\simeq \frac{\nu^{3/2}}{3\sqrt{\pi}} \left[\underbrace{\int_{-\infty}^{\infty} \frac{dx}{ch^2(\frac{x}{2})}}_{4} + \frac{3}{8\nu^2 \cdot 2} \underbrace{\int_{-\infty}^{\infty} \frac{x^2}{ch^2 \frac{x}{2}} dx}_{4\pi^2/3} + \cdots \right]$$

$$\simeq \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} + \frac{1}{6} \pi^{3/2} (\ln z)^{-1/2} + \cdots + \mathcal{O}(1/z)$$

(5)

$$f_{3/2}(z) \simeq \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} + \frac{1}{6} \pi^{3/2} (\ln z)^{-1/2} + \ldots$$

$z \to \infty$

<u>lowest order</u>

$$\frac{1}{v} = \frac{g}{\lambda_T^3} \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2}$$

$$\frac{3\sqrt{\pi}}{4g} \frac{\lambda_T^3}{v} = (\beta\mu)^{3/2}$$

$$\mu = k_B T \left(\frac{3\sqrt{\pi}}{4g} \frac{\lambda_T^3}{v} \right)^{2/3}$$

$$= k_B T \lambda_T^2 \left(\frac{3\sqrt{\pi}}{4g} \frac{1}{v} \right)^{2/3}$$

$$= \cancel{k_B T} \frac{h^2}{2m\pi \cancel{k_B T}} \left(\frac{3\sqrt{\pi}}{4g} \frac{1}{v} \right)^{2/3}$$

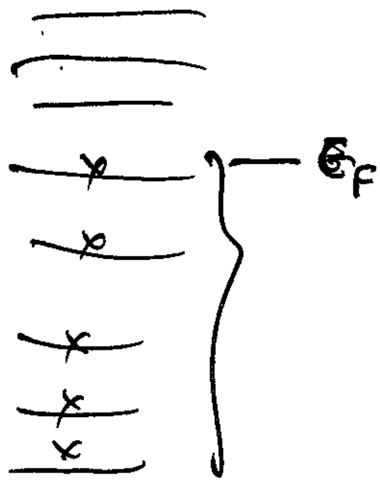
μ = indep of T as $z \to \infty$

$$e^{\beta\mu} = e^{\mu/k_B T}$$

$T \to 0$ $\mu \to \epsilon_F$

$$\epsilon_F = \mu(T=0) = \frac{1}{2m} \left(\frac{3\sqrt{\pi}}{4g} \frac{h^3}{v\, \pi^{3/2}} \right)^{2/3}$$

$$\boxed{\epsilon_F = \frac{1}{2m} \left(\frac{3h^3}{4g\, v\, \pi} \right)^{2/3}}$$



$$N = g\int_0^{p_F} \frac{V\,d^3p}{h^3}$$

$$= \frac{4\pi g V}{h^3}\int_0^{p_f} p^2\,dp = \frac{4\pi g V p_f^3}{3h^3}$$

$$p_f = \left(\frac{3h^3 N}{4\pi g V}\right)^{1/3}$$

$$\epsilon_F = \frac{p_f^2}{2m} = \frac{1}{2m}\left(\frac{3h^3 N}{4\pi g V}\right)^{2/3} \quad ✓$$

$$\frac{1}{v} = \frac{g}{\lambda_T^3}\,\frac{4}{3\sqrt{\pi}}(\ln z)^{3/2}\left(1 + \frac{3\sqrt{\pi}}{4(\ln z)^{3/2}}\,\frac{\pi^{3/2}}{6}(\ln z)^{-1/2} \ldots\right)$$

$$\frac{N}{V} = \frac{g}{\lambda_T^3}\,\frac{4}{3\sqrt{\pi}}\,(\ln z)^{3/2}\left(1 + \frac{\pi^2}{8(\ln z)^2}\right)$$

$$\frac{3\sqrt{\pi}}{4g}\,\frac{N}{V}\,\lambda_T^3 = (\beta\mu)^{3/2}\left(1 + \frac{\pi^2}{8(\beta\mu)^2}\right)$$

$\beta \to \infty$.

$$\left(\frac{3\sqrt{\pi}}{4g}\frac{N}{V}\right)^{2/3}\lambda_T^2 = \beta\mu\left(1+\frac{2\pi^2}{24(\beta\mu)^2}\right)$$

$$\beta\mu \cong \left(\frac{3\sqrt{\pi}}{4g}\frac{N}{V}\right)^{2/3}\frac{h^2}{2m\pi k_B T}\left(1-\frac{2\pi^2}{24(\beta\mu)^2}\right)$$

$$\beta\mu \simeq \underbrace{\frac{1}{2m}\left(\frac{3h^3N}{4\pi gV}\right)^{2/3}}_{\epsilon_F}\frac{1}{k_B T}\left(1-\frac{\pi^2}{12(\beta\mu)^2}\right)$$

$$\beta\mu \simeq \frac{\epsilon_F}{k_B T}$$

$$\beta\mu \simeq \frac{\epsilon_F}{k_B T}\left(1-\frac{\pi^2}{12}\frac{(k_B T)^2}{\epsilon_F^2}\right)$$

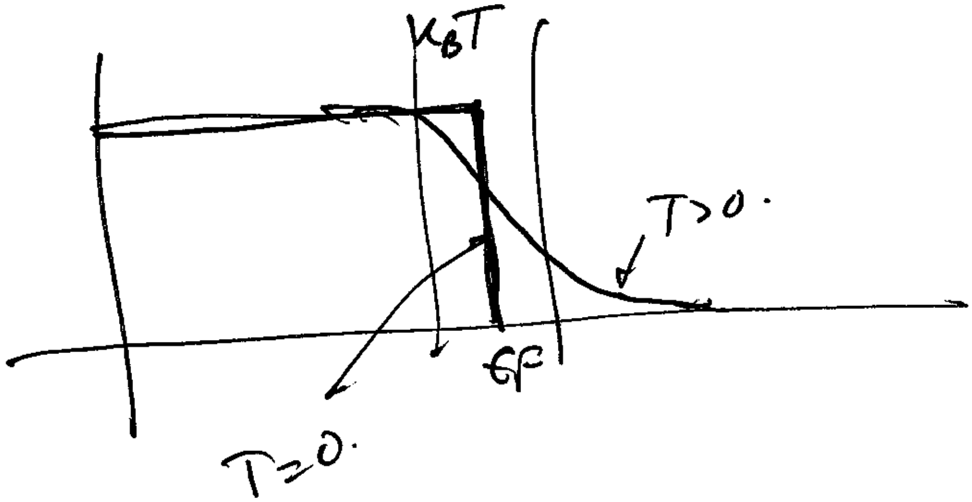
$$\mu = \epsilon_F\left(1-\frac{\pi^2}{12}\frac{(k_B T)^2}{\epsilon_F^2}\right)$$

$$\boxed{\mu = \epsilon_F\left(1-\frac{\pi^2}{12}\frac{T^2}{T_f^2}\right)}$$

$$\boxed{k_B T_f = \epsilon_F}$$

$T \to 0$

$$\langle n_p \rangle = \frac{1}{e^{\beta \epsilon_p - \beta \mu} + 1}$$



$$E = g \frac{4\pi V}{h^3} \int_0^\infty p^2 \frac{p^2}{2m} dp \, \frac{1}{e^{\beta \epsilon_p - \beta \mu} + 1}.$$

$$= \frac{4\pi g V}{2m h^3} \int_0^\infty \frac{p^4 dp}{e^{\beta \epsilon_p - \beta \mu} + 1}.$$

$$= \frac{4\pi g V}{2m h^3} \frac{1}{5} \int_0^\infty \frac{(\partial_p p^5) \, dp}{e^{\beta p^2/2m - \beta \mu} + 1}$$

$$= + \frac{4\pi g V}{2m h^3 5} \int_0^\infty \frac{p^5 dp \, e^{-\beta \mu} \frac{2\beta p}{2m} e^{\frac{\beta p^2}{2m}}}{\left(e^{\beta p^2/2m - \beta \mu} + 1\right)^2}$$

$$= \frac{4\pi g V}{10 m h^3} \frac{\beta}{m} \int_0^\infty \frac{p^6 e^{\frac{\beta p^2}{2m} - \beta\mu}}{\left(e^{\beta p^2/2m - \beta\mu} + 1\right)^2} dp$$

$$y = \frac{\beta p^2}{2m} - \beta\mu \qquad p = \sqrt{\frac{2m}{\beta}(y + \beta\mu)}$$

$$= \frac{4\pi g V}{5 m h^3} \frac{\beta}{m} \int_{-\beta\mu}^{\infty} \left(\frac{2m}{\beta}\right)^3 \frac{(y+\beta\mu)^3 e^y \sqrt{\frac{2m}{\beta}}\, dy}{(e^y+1)^2 \; 2\sqrt{y+\beta\mu}}$$

$$E \simeq \frac{\pi g V}{5 m h^3} \frac{\beta}{m} \left(\frac{2m}{\beta}\right)^{7/2} \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^y}{\left(\frac{e^y+1}{2}\right)^2} (y+\beta\mu)^{5/2} dy$$

$$1 + \frac{5}{2}\frac{y}{\beta\mu} + \frac{15}{8}\frac{y^2}{(\beta\mu)^2}$$

$$\simeq \frac{\pi g V}{20 m h^3} 2^{7/2} \left(\frac{m}{\beta}\right)^{5/2} (\beta\mu)^{5/2} \int_{-\infty}^{\infty} \frac{(1 + 5/2\, y/\beta\mu)}{\mathrm{ch}^2 y/2} dy$$

$$4 + \frac{15}{8}\frac{4\pi^2}{3}\frac{1}{(\beta\mu)^2}$$

$$\int_{-\infty}^{\infty} \frac{dy}{\mathrm{ch}^2 y/2} = 4 \qquad \int_{-\infty}^{\infty} \frac{y^2 dy}{\mathrm{ch}^2 y/2} = \frac{4\pi^2}{3}$$

$$\simeq \frac{\pi g V}{5 m h^3} \left(2 m \mu\right)^{5/2} \cdot 2 \left(1 + \frac{5}{8} \frac{\pi^2}{(\beta\mu)^2}\right)$$

$$= \frac{\pi g V}{5 m h^3}\, 2 \left(2 m \epsilon_F\right)^{5/2} \left(1 - \frac{\pi^2}{12} \frac{5}{2} \frac{T^2}{T_f^2} + \frac{5}{8} \frac{\pi^2 T^2}{T_f^2}\right)$$

$$\frac{5}{8} - \frac{5}{24} = \frac{5}{8}\left(1 - \frac{1}{3}\right) = \frac{2 \times 5}{3 \times 8}$$

$$= 5/12$$

$$\frac{E}{N} = \frac{\pi g V}{5 m h^3 N}\, 2 \left(2 m \epsilon_F\right)^{5/2} \left(1 + \frac{5\pi^2}{12} \frac{T^2}{T_f^2}\right)$$

$$\frac{E}{N} = \frac{2\pi g}{5 m h^3} \frac{(2 m \epsilon_F)^{5/2}}{(2 m \epsilon_F)^{3/2}} \frac{3 h^3}{4 \pi g} \left(1 + \frac{5\pi^2}{12} \frac{T^2}{T_f^2}\right)$$

$$= \frac{2}{5 m}\, 2 m \epsilon_F\, \frac{3}{4} \left(1 + \frac{5\pi^2}{12} \frac{T^2}{T_f^2}\right)$$

$$= \frac{3}{5} \epsilon_F \left(1 + \frac{5\pi^2}{12} \frac{T^2}{T_f^2}\right)$$

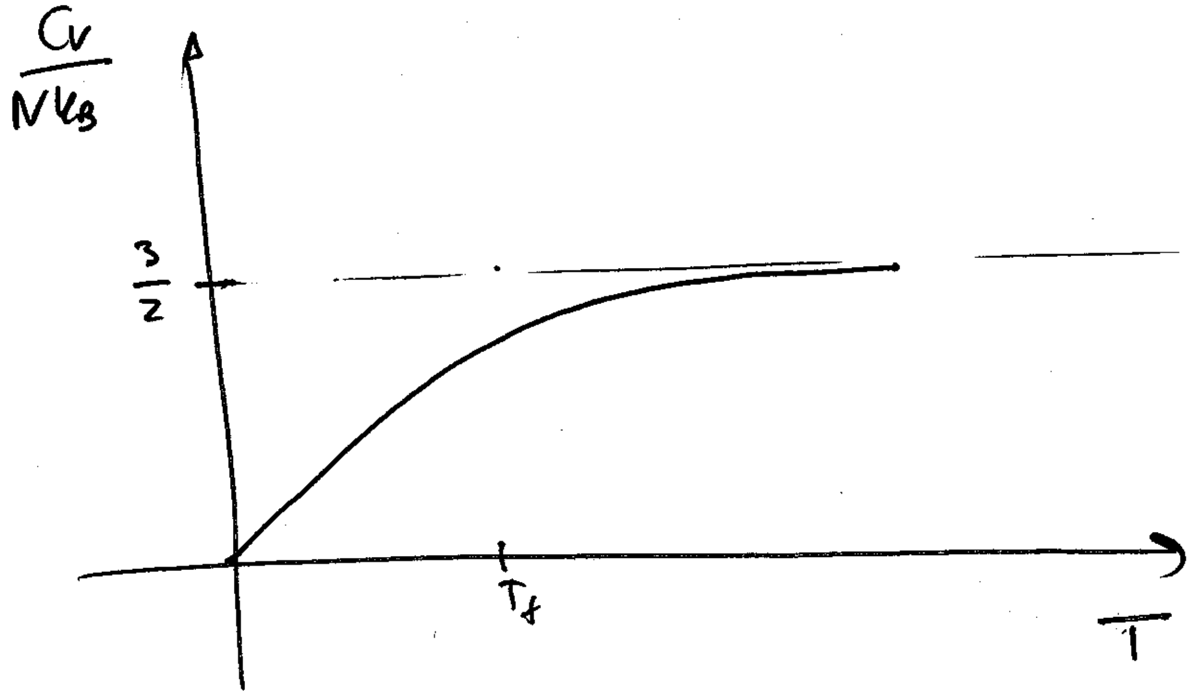
$$\boxed{\frac{E}{N} = \frac{3}{5} \epsilon_F \left(1 + \frac{5\pi^2}{12} \frac{T^2}{T_f^2}\right)}$$

Specific heat at low temp.

$$c_v = \frac{1}{N}\frac{\partial E}{\partial T}\Big|_V = \frac{3}{5}\epsilon_F \frac{5\pi^2}{12}\cdot\frac{2T}{T_f^2}$$

$$= \frac{\pi^2}{2}\epsilon_F\frac{T}{T_f^2} = \frac{\pi^2}{2} k_B \frac{T}{T_f}$$

$$\boxed{\frac{C_v}{k_B} = \frac{\pi^2}{2} N \frac{T}{T_f}}$$



3N quadratic d.o.f $\quad T \to \infty \quad E = \frac{3}{2} N k_B T$

$$\boxed{C_v = \frac{3N k_B}{2}}$$

(a)

$$f_a(z) = \frac{1}{\Gamma(a)} \int_0^\infty dy \, \frac{y^{a-1}}{z^{-1}e^{y}+1}$$

$z \to \infty \qquad z = e^{\nu} \; ; \; \nu = \beta\mu$

$$f_a(z) = \frac{1}{a\Gamma(a)} \int_0^\infty dy \, \frac{\partial_y \, y^{a}}{z^{-1}e^{y}+1}$$

$$= \frac{1}{\Gamma(a+1)} \left. \frac{y^a}{z^{-1}e^{y}+1} \right|_0^\infty + \frac{1}{\Gamma(a+1)} \int_0^\infty dy \, y^a \, \frac{z^{-1}e^{y}}{(z^{-1}e^{y}+1)^2}$$

$(a>0)$ $\quad = 0$

$$= \frac{1}{\Gamma(a+1)} \int_0^\infty dy \, \frac{y^a e^{y-\nu}}{(e^{y-\nu}+1)^2}$$

$x = y-\nu$

$$= \frac{1}{\Gamma(a+1)} \int_{-\nu}^{\infty} dx \, \frac{(x+\nu)^a \, e^{x}}{4\left(\frac{e^{x}+1}{2}\right)^2}$$

$$= \frac{1}{\Gamma(a+1)} \int_{-\nu}^{\infty} dx \, \frac{(x+\nu)^a}{4\,\mathrm{ch}^2(x/2)}$$

$x^a, \quad a x^{a-1}, \quad a(a-1)x^{a-2} + \ldots$

$(1+\epsilon)^a = 1 + a\varepsilon + \frac{1}{2}a(a-1)\epsilon^2 + \ldots$

$$\simeq \frac{1}{\Gamma(a+1)} \int_{-\infty}^{\infty} dx \, \frac{\nu^a \left(1 + a x/\nu + \frac{1}{2}a(a-1)x^2/\nu^2 + \ldots\right)}{2\cosh^2 x/2}$$

$$\simeq \frac{\nu^a}{\Gamma(a+1)} \left\{ 1 + \frac{a}{\nu}\cdot 0 + \frac{1}{2}\frac{a(a-1)}{\nu^2}\frac{\pi^2}{3} + \ldots \right\}$$

$$f_a(z) = \frac{\nu^a}{\Gamma(a+1)} \left(1 + \frac{\pi^2}{6}\frac{a(a-1)}{\nu^2} + \ldots\right)$$

$$f_a(z) \underset{z\to\infty}{\simeq} \frac{(\ln z)^a}{\Gamma(a+1)} \left(1 + \frac{\pi^2}{6}\frac{a(a-1)}{(\ln z)^2} + \ldots\right)$$

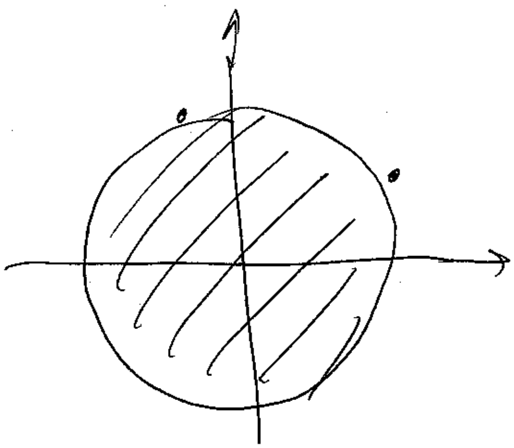
$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left(1 + \frac{\pi^2}{6}\frac{3}{2}\frac{1}{2}\frac{1}{(\ln z)^2}\right)$$

$\Gamma(5/2) = 3/2 \, \frac{1}{2} \Gamma(1/2) = \frac{3\sqrt{\pi}}{4}$

$$= \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} + \frac{\pi^{3/2}}{6} \frac{1}{(\ln z)^{1/2}} + \ldots$$

✓

Cooper pairs



attractive force

⇒ bound state.

(as opposed to free case)

$$\psi_0(\vec{r}_1, \vec{r}_2) = \sum_k g_k e^{ik r_1 - ik r_2}$$

(total $k = 0$.)

Consider singlet →

$$\psi_0(\vec{r}_1 - \vec{r}_2) = \sum_{k > k_F} g_k \cos(\vec{k} \cdot (\vec{r}_1 - \vec{r}_2)) \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$H = \sum_i \frac{\vec{p}^2}{2m} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j)$$

$$\left(-\frac{\hbar^2}{2m}(\partial_{r_1}^2 + \partial_{r_2}^2) + V(\vec{r}_1 - \vec{r}_2)\right)\psi_0(\vec{r}_1 - \vec{r}_2) = E\,\psi_0(\vec{r}_1 - \vec{r}_2)$$

~~$\left(\sum_k \frac{\hbar^2 k^2}{m} + V(\vec{r}_1 - \vec{r}_2)\right)\psi_0(\vec{r}_1 - \vec{r}_2) = E\,\psi_0(\vec{r}_1 - \vec{r}_2)$~~

$$\sum_{k>k_f} \frac{\hbar^2 k^2}{m} g_k \cos(\vec{k}(\vec{r}_1-\vec{r}_2)) + \sum_k V(\vec{r}_1-\vec{r}_2)\, g_k \cos(\vec{k}(\vec{r}_1-\vec{r}_2)) =$$

$$= E \sum_{k>k_f} g_k \cos(\vec{k}(\vec{r}_1-\vec{r}_2))$$

$$\frac{\hbar^2 k^2}{m} g_k + \sum_{k'} V_{kk'} g_{k'} = E g_k$$

$$(E-2\epsilon_k)\, g_k = \sum_{k'>k_f} V_{kk'} g_{k'}$$

$$V_{kk'} = \Omega^{-1}\int V(r)\, e^{i(\vec{k}'-k)\vec{r}}\, d\vec{r}$$

$V_{kk'} = -V$ in a region around $\underline{\epsilon_F}$

$$(E-2\epsilon_k)\, g_k = -V\sum_{k'} g_{k'}$$

$$g_k = -V\,\frac{\sum_{k'} g_{k'}}{E-2\epsilon_k}$$

$$\sum_k g_k = -V\sum_k g_k \sum_k \frac{1}{E-2\epsilon_k}$$

$$\sum_{k>k_F} \frac{1}{-E+2\epsilon_k} = \frac{1}{V}$$

$$\frac{1}{V} = N(0)\int_{\epsilon_F}^{\epsilon_F+\hbar\omega_c} \frac{d\epsilon}{2\epsilon - E} = \frac{1}{2} N(0) \ln(2\epsilon-E)\Big|_{\epsilon_F}^{\epsilon_F+\hbar\omega_c}$$

density of states

$$\therefore \frac{1}{V} = \frac{1}{2} N(0) \ln\left(\frac{2E_F + 2\hbar\omega - E}{2E_F - E}\right)$$

Assume $N(0)V \ll 1$ small V.

$$2 = N(0)V \ln\left(\frac{2E_F + 2\hbar\omega - E}{2E_F - E}\right)$$

large

$$E = 2E_F + \delta$$

$$\frac{2}{N(0)V} = \ln\left(\frac{2\hbar\omega - \delta}{-\delta}\right) = \ln\left(\frac{2\hbar\omega}{-\delta}\right)$$

We have $\underline{\delta < 0}$

$$\frac{2\hbar\omega}{-\delta} = e^{2/N(0)V} \Rightarrow \delta = -2\hbar\omega e^{-\frac{2}{N(0)V}}$$

$$E = 2E_F - 2\hbar\omega e^{-\frac{2}{N(0)V}}$$

↑ bound state. for $V \to 0$ but $V \neq 0$