Cardinality Bundling with Spence-Mirrlees Reservation Prices

Jianqing "Fisher" Wu Mohit Tawarmalani Karthik N. Kannan

Abstract

We study the pricing of cardinality bundles, where firms set prices that depend only on the size of the purchased bundle, a practice that is increasingly being adopted by industry. The model we study, where consumer choices are discrete, was originally proposed by Hitt and Chen (2005) and it requires that consumers' preferences obey Spence-Mirrlees Single Crossing Property (SCP). We correct prior approaches and develop various structural and managerial insights. We develop a fast combinatorial technique to obtain the optimal prices. We extend our analysis to address a quantity discount problem originally proposed in Spence (1980). We provide examples that demonstrate that the proposed approach of Spence (1980) only identifies local optima without providing guidance on selecting the globally optimal pricing function. Our insights from the discrete model are extended to this context to develop a scheme that provides solutions within an arbitrary pre-specified tolerance. Consequently, we also solve the continuous version of the cardinality bundling problem.

1 Introduction

Bundling and its benefits have been studied extensively in the literature. For example, Bakos and Brynjolfsson (1999) show that when products are synergistic, offering bundles of products can yield higher profits than selling them separately. The earliest work on bundling (e.g., Stigler, 1963, Adams and Yellen, 1976, McAfee et al., 1989) focused on *mixed bundling*, wherein every combination of goods is sold at a possibly different price. However, because the number of combinations quickly increases with the number of goods, the pricing problem becomes intractable except for a small number of goods (Hanson and Martin, 1990). So, alternate bundling schemes – such as component pricing, where only the components are sold; or pure bundling, where only the bundle is sold – have also been studied and deployed. The focus of this work is to study another bundling scheme called *cardinality bundling* or, in short, CB.

In CB, bundles of equal cardinality or size are sold at the same price. That is, for a firm that sells J goods, consumer may purchase any one good for a listed price, a bundle of any two goods for a different price, and so on and so forth. In contrast to mixed bundling (which requires pricing $2^J - 1$ bundles), CB only requires prices for J bundles. Perhaps because of the simplicity of the pricing scheme, CB has been adopted in practice. Pricing for theme parks within entertainment complexes such as Disney World are based on CB. Consumers can purchase multi-day (2, 3, 4 or 5 day) passes and can choose to visit any of the four theme parks each day. Similarly, Eastlink cable TV allows its consumers to choose their channel combinations within the cardinality bundles (12 or 20 channels) purchased.

The current literature on CB is relatively sparse and we review it briefly here. Most relevant to the current paper is Hitt and Chen (2005). They explore conditions under which CB can attain the same profit as mixed bundling. Further assuming that consumers' reservation price satisfy Spence-Mirrlees Single Crossing Property (SCP), they propose and analyze a readily computable pricing strategy. Wu et al. (2008) seek to solve the CB pricing problem as a nonlinear mixed-integer program. They use Lagrangian relaxation, subgradient ascent, and heuristic methods to derive bounds for the problem. Hitt and Chen (2005) and Wu et al. (2008) restrict the consumers to purchase at most one bundle. Chu et al. (2011) consider a CB model where unit prices for bundles decrease with increasing size. They use computations and real data to argue that profit from their CB model is almost the same as that from mixed bundling.

We first consider the model of Hitt and Chen (2005) for cardinality bundling, which assumes that reservation prices obey SCP and consumers are restricted to choose bundles of a discrete size. The proposed pricing strategy of Hitt and Chen (2005) is not correct because it does not take into account the reservation prices of non-adjacent consumer types while allocating a bundle to a consumer. We show that such a myopic pricing strategy is inherently deficient. Nonetheless, we expose many structural properties for the problem and utilize them to (i) reformulate the cardinality bundling problem as a linear program (LP), and (ii) develop a combinatorial fast solution approach to the problem. Our analysis provides many novel managerial insights into cardinality bundling, including the impact of changing reservation prices and costs on the vendor's profit. Later, we consider the model of Spence (1980) addressing the quantity discount problem and extend it to model a variant of the cardinality bundling problem where the bundle sizes can take fractional values. For this problem and its quantity-discount variant, we show that although the optimality conditions proposed by Spence (1980) only identify local optimal solutions (that are necessary only under certain technical restrictions). The problem of selecting the globally optimal pricing function remains open due to the multiplicity of local optima. We adapt and extend the ideas that led to the solution of the discrete cardinality bundling problem to the continuous context. This enables us to identify the globally optimal pricing function within any arbitrary pre-specified tolerance.

2 CB Discrete Case: Model & Analysis

A customized cardinality bundling strategy models a situation where a vendor offers a menu of products that may be purchased in a bundle, whose price is determined by its size. The consumer is free to choose any products as long as the number of goods she chooses matches the bundle size for which she has paid. This model was originally proposed by Hitt and Chen (2005), where they assume that the consumers can be ordered such that a consumer of higher type not only assigns a higher value to bundles of a given size but also derives higher marginal value from increasing the bundle size. When the consumers can be ordered this way, their reservation prices are said to satisfy the Spence-Mirrlees Single Crossing Property (SCP).

In this section, we consider the cardinality bundling problem, which is modeled to optimally choose the sizes and prices of the bundles a vendor should offer in the market. Our basic model is the same as that in Hitt and Chen (2005) and we review it here for the sake of completeness. Consider a vendor who sells J products and assume that there are I consumers in the market. In the following, we denote the bundle of size j as Bundle j. We assume WLOG that all bundles, $1, \ldots, J$ are offered in the market and the vendor decides their prices. We denote the price of Bundle j as p_j. Obviously, the consumer does not pay anything for Bundle 0, whose price is therefore fixed at 0. We assume that the cost of the Bundle j for vendor is c_j and that the total cost to the vendor is the sum of the costs for all the bundles sold. Clearly, c_0 is 0. The model makes a reasonable assumption that a consumer's willingness-to-pay (WTP) is non-decreasing with the bundle size,[1] which would be trivially true if extra units can be freely disposed. The model further

[1] Hitt and Chen (2005) imposes WTP for each consumer to be concave in j, which we relax in our model.

assumes that each consumer can purchase at most one bundle.

Let $w_{ij} \geq 0$ denote the WTP of Consumer i for Bundle j. For every i, we set w_{i0} to zero to denote that consumers, who do not purchase anything, do not derive any value out of the vendor's products. Since WTPs are non-decreasing with bundle size, $w_{ij} \geq w_{ij'}$ for $j \geq j'$. Since the choice of the bundle rests with the consumer, if Consumer i purchases Bundle j_i, this bundle must maximize her consumer surplus, *i.e.*, $j_i \in \arg\max_j \{w_{ij} - p_j\}$. Let J_i be the set of bundles Consumer i prefers with price vector p. If $|J_i| > 1$, we assume that Consumer i purchases a Bundle j_i that belongs to $\arg\max_j \{p_j - c_j \mid j \in J_i\}$, *i.e.*, the surplus-maximizing bundle that yields the most profit to the vendor. This assumption is typical in the literature and is without loss of generality.[2]

Let x_{ij} be 1 if Consumer $i \in \{1, 2, \ldots, I\}$ buys Bundle $j \in \{0, 1, 2, \ldots, J\}$ and 0 otherwise. Then, CBP can be formulated as follows (see Hitt and Chen, 2005):

$$\text{CBP1}: \quad \underset{x_{ij}, p_j}{\text{Max}} \quad \sum_{i=1}^{I} \sum_{j=0}^{J} x_{ij}(p_j - c_j)$$

$$\text{s.t.} \quad \sum_{j'=0}^{J} (w_{ij'} - p_{j'}) x_{ij'} \geq w_{ij} - p_j \ \forall i, \ \forall j \tag{1}$$

$$\sum_{j=0}^{J} x_{ij} = 1 \ \forall i \tag{2}$$

$$p_0 = 0 \tag{3}$$

$$x_{ij} \in \{0, 1\} \ \forall i, \ \forall j. \tag{4}$$

Let (x^*, p^*) be a solution that generates the maximum profit for the vendor. Assuming (2), Constraints (1) enforce incentive compatibility (IC) and individual rationality (IR) for Consumer i. The left hand side

[2] To see this, let $J'(j) = \{j' \mid p_{j'} - c_{j'} < p_j - c_j\}$ be the set of bundles that provides less profit to vendor than j. Observe that since the number of consumers and bundles is finite, there exists an $\epsilon > 0$ such that even if the price of a bundle that a consumer does not prefer is reduced by $J\epsilon$, the consumer continues to prefer the bundles in J_i after the change. Now, consider a new pricing scheme p', where the price of Bundle j is set to $p'_j = p_j - |J'(j)|\epsilon$. Then, it is easy to verify that, when the prices are p', Consumer i prefers the Bundle $j_i \in \arg\max\{p_j - c_j \mid j \in J_i\}$ over other bundles in J_i and, since $|J'(j)| < J$, this preference is also over bundles not in J_i. Further, the vendor does not lose more than $JI\epsilon$ in the profit when he prices the bundles using p' instead of p. Since ϵ can be chosen to be arbitrarily small, this yields a sequence of solutions for which vendor's profit converges to the one obtained under our assumption.

models the consumer surplus from the purchase decision and the right hand side models the consumer surplus from the purchase of alternate bundles. The case with $j = 0$ ensures that consumer only purchases bundles with non-negative surplus. Constraints (2) enforce that each consumer purchases only one bundle. Observe that $\mathrm{CBP1}$ is a mixed integer nonlinear program (MINLP) since the price vector p_j and consumer decisions x_{ij} are variables and their products appear in the objective and in Constraint (1).

Like in other nonlinear pricing problems, Hitt and Chen (2005) assume that consumer valuations satisfy the Spence-Mirrlees Single Crossing Property (SCP) (see Spence, 1980). We also make the same assumption, which imposes the following ordering on the consumers' WTP for the bundles:

$$w_{ij} \geq w_{i'j} \; \forall i > i', \tag{5}$$

$$w_{ij} - w_{ij'} \geq w_{i'j} - w_{i'j'} \; \forall i > i', \; \forall j > j'. \tag{6}$$

The interpretation of these conditions is straightforward. A consumer with a higher index has a (weakly) higher WTP for any bundle. Also, the WTP exhibits increasing differences, *i.e.*, as bundle size increases, the WTP for a higher-indexed consumer increases more rapidly than the WTP for a lower-indexed consumer. Essentially, this assumption states that consumers can be ordered by types, with higher type consumers valuing the products and marginal changes in bundle sizes more than the lower type ones. Before we develop an efficient solution for this problem, we review the currently available approaches using examples.

Table 1: Willingness-to-pay for Example 1

Bundle	Consumers' WTP			
size	I_1	I_2	I_3	I_4
0	0	0	0	0
1	26	36	58	120
2	47	62	91	180
3	58	77	113	221
4	62	83	123	240

Example 1 *Consider a scenario with $I = 4$ consumers, $J = 4$ bundle sizes, and costs $c_j = 0$ for all j.*

Suppose the WTP for the consumers are as given in Table 1. It can be verified easily that they satisfy SCP. We use BARON (Tawarmalani and Sahinidis, 2002) to solve the MINLP formulation of CBP1*. (Note that BARON guarantees that it finds the global optimal solution at termination.) The optimal solution thus found is to set* $p_1^* = p_2^* = 47$*,* $p_3^* = 62$*, and* $p_4^* = 72$*. It is easy to check that, with these prices, Consumer* 1*,* 2*,* 3*, and* 4 *buy Bundles* 2*,* 3*,* 4*, and* 4 *respectively. The optimal profit for the vendor is* 253*.*[3]

We now make a small change to the setting of Example 1 and illustrate that the optimal assignment for a consumer depends on the WTP of all other consumers.

Example 2 *In the setting of Example 1, change* w_{41} *from* 120 *to* 100*, so that WTPs still satisfy SCP. If* CBP1 *is now solved using BARON, the optimal solution assigns Consumer* 1 *to Bundle* 0 *yielding a profit of* 256*.*[4] *There is no optimal allocation that assigns Bundle* 2 *to Consumer* 1*.*[5] *Any allocation that ignores the WTP of Consumer* 4 *while allocating bundle to Consumer* 1 *will thus not yield optimal profit.* [6,7]

The only available approaches to solve CBP1 either use an MINLP solver directly or solve the

[3]Result 3 in Hitt and Chen (2005) claims that the following approach optimally solves CBP1, which we show later isn't always the case. Consumer i is assigned to the largest bundle size j that satisfies the following condition:

$$(I - i + 1)(w_{ij} - w_{i,j-1}) - (I - i)(w_{i+1,j} - w_{i+1,j-1}) \geq c_j - c_{j-1}. \tag{7}$$

We remark that, when Consumer i is assigned a bundle, the WTP of consumers other than i and $i+1$ are ignored. Here, the right hand side is 0 since we assume $c_{j'} = 0$ for all $1 \leq j' \leq J$. The left hand side values are shown in Table 2.

Bundle size	LHS values I_1	I_2	I_3	I_4
0				
1	-4	-8	16	120
2	6	12	-14	60
3	-1	1	3	41
4	-2	-2	1	19

Table 2: Left hand side values of Equation (7)

For Example 1, the above approach yields the same solution as the optimal solution found earlier using BARON.

[4]The optimal assignment of Consumer 1, 2, 3, and 4 is to Bundles 0, 0, 1, and 4 respectively. The corresponding prices are $p_1^* = 58$ and $p_2^* = p_3^* = p_4^* = 198$.

[5]In fact, if Consumer 1 is restricted to purchase Bundle 2, the vendor cannot obtain a profit more than 253.

[6]Hitt and Chen (2005) claims that it is optimal to assign Consumer 1 to Bundle 2 even in this case. This is so, because for $i = 1$, Equation (7) is independent of w_{41}. However, as shown above, this is not an optimal assignment.

[7]In the proof of Result 3, Hitt and Chen (2005) modify the procedure when higher type consumers do not buy larger sized bundles. This modification does not apply here.

linearized formulation using an MIP solver. The MINLP/MIP-based approach is, however, not amenable to comparative statics because global optimality certificates are typically neither small nor easy to obtain. In this section, we develop an alternate solution approach that is efficient, guarantees optimality, and is amenable to comparative statics.

2.1 Properties of the Optimal Solution

First, we identify some properties of the optimal solution.[8]

Proposition 3 *There exists an optimal pricing scheme that is nondecreasing with bundle size.*

Proposition 4 *There exists an optimal solution to* CBP1 *that satisfies:*

$$\sum_{j'=j}^{J} x_{i+1j'} \geq \sum_{j'=j}^{J} x_{i,j'} \; i = 1, \ldots, I-1, \; \forall j. \tag{8}$$

That is, there exists an optimal solution where the mapping from consumer types to bundle sizes is non-decreasing, i.e., for any $i < I$, if Consumer i buys Bundle j, then Consumer $i+1$ buys a Bundle j' such that $j' \geq j$. Further, for any given price vector, there exists a feasible allocation of bundle sizes to consumer types that is non-decreasing.

Proposition 5 *There exists an optimal pricing scheme such that if two bundle sizes j and j' are bought by some consumers and $j' > j$ then $p_{j'} - c_{j'} > p_j - c_j$.*

Proposition 6 *Among the consumers purchasing a non-zero bundle size, the lowest indexed one is charged at her WTP in every optimal solution.*

Proposition 4 is particularly interesting, since it provides redundant, yet rather important, constraints that facilitate the solution of CBP1. Further, Proposition 4 applies to other bundling problems where WTPs follow SCP, including those where consumers may purchase more than one bundle (Wu et al., 2014a).

[8] All the proofs are provided in the appendix.

Propositions 3, 4, and 5 imply that prices do not decrease with bundle size; the higher type consumers purchase weakly larger-sized bundles; and larger bundles yield higher profits to the vendor.

2.2 A Solution Approach

In this section, we reformulate CBP1 into a linear programming problem. Then, the dual formulation is used to develop a fast fully combinatorial algorithm that solves CBP1 in strongly polynomial time.

2.2.1 Reformulating the MINLP to a 0-1 IP

We first provide some intuition into what makes it possible to solve CBP1 quickly. First, assume that the vendor fixes a certain bundle size that the first consumer will purchase. Since the first consumer must purchase one of Bundles $0, \ldots, J$, this yields $J + 1$ problems for the vendor to solve. The key property that enables the vendor to solve the problem is that once the first consumer is allocated Bundle j, the remaining problem can be solved by solving a smaller cardinality bundling problem, *i.e.*, one which has Consumers $2, \ldots, I$ and Bundles $j, \ldots, J$. This subproblem can then be solved recursively using the same technique. Before we provide a formal proof of our algorithm, we build some intuition into the problem structure.

Consider the cardinality bundling problem where the vendor only considers Consumers $i', \ldots, I$ and prices the bundles so that each of these consumers buys one of the Bundles $j', \ldots, J$. To accomplish this, by Proposition 4, it suffices to restrict i' to purchase a bundle of size at least j' and to remove Consumers $1, \ldots, i' - 1$. More generally, assume that the vendor wishes to ensure that i' buys one of the options from a set of bundle sizes, say J'. Then, the corresponding problem can be formulated by adding the constraint, $\sum_{j \in J'} x_{i'j} = 1$, to CBP1. We denote this problem as CBP$(i', j' \mid J')$ and the corresponding optimal profit as $\Pi^{\text{CBP}(i',j'|J')}$. Obviously, $\Pi^{\text{CBP}(i',j'|\{j',\ldots,J\})} = \max_{j \geq j'} \Pi^{\text{CBP}(i',j'|\{j\})}$.[9] Therefore, it suffices to find a way to solve CBP$(i', j' \mid \{j'\})$, whose solution can in turn be obtained by solving CBP$(i'+1, j' \mid \{j', \ldots, J\})$. As it turns out, this is because the purchasing decision of Consumers

[9] Further, by Proposition 4, $\Pi^{\text{CBP}(i',j'|\{j\})} = \Pi^{\text{CBP}(i',j|\{j\})}$ because if i purchases j, then every higher type consumer purchases a bundle j or, higher and $j \geq j'$.

$i' + 1, \ldots, I$ are the same in the two problems. If we denote the set $\{j', \ldots, J\}$ as $j'_{\geq}$:

$$\Pi^{\text{CBP}(i',j'|\{j'\})} = \Pi^{\text{CBP}(i'+1,j'|j'_{\geq})} + \underbrace{(w_{i'j'} - c_{j'})}_{\text{sale of } j' \text{ to } i'} + \underbrace{(I - i')(w_{i'+1j'} - w_{i'j'})}_{\text{restrictions on prices}}. \tag{9}$$

The first adjustment is because of the revenue and cost from selling j' to i' and the second is because the price of Bundle j' is constrained to the WTP of Consumer i' in CBP$(i', j' \mid \{j'\})$ whereas it is constrained to the WTP of Consumer $i' + 1$ in CBP$(i' + 1, j' \mid j'_{\geq})$. In order to make the result also apply to the case when $i' = I$, we define $w_{I+1j} = w_{Ij}$. To capture this difference succinctly, we let $v_{i'j'}$ denote $w_{i'j'} - (I - i')(w_{i'+1j'} - w_{i'j'})$ and rewrite (9) as: $\Pi^{\text{CBP}(i',j'|\{j'\})} = \Pi^{\text{CBP}(i'+1,j'|j'_{\geq})} + (v_{i'j'} - c_{j'})$.

Now, we formally show that the cardinality bundling problem can be linearized into a 0-1 integer program using the above notation.

Proposition 7 CBP1 *can be reformulated as the following 0-1 integer linear problem:*

$$\text{CBP2} : \text{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \;\middle|\; (2), (4), (8) \right\}.$$

Let x^* be an optimal solution to CBP2. *Let $\{i_0, \ldots, i_k\}$ be the lowest type consumers that purchase a certain bundle size and, for any j, let $r'(j) = \arg\min_r \left\{ i_r \mid \sum_{j'=j}^{J} x^*_{ij'} = 1 \right\}$. If there is no feasible solution, set $p_j = w_{IJ} + \epsilon$ for an arbitrary $\epsilon > 0$. Otherwise, let $j(i) = \sum_{j=0}^{J} j x_{ij}$ and*

$$p_j = w_{i_0 j(i_0)} + \sum_{r=1}^{r'(j)} \left(w_{i_r j(i_r)} - w_{i_r j(i_{r-1})} \right). \tag{10}$$

Converting CBP1 into CBP2 is possible because $\sum_{i=1}^{I} \sum_{j=0}^{J} v_{ij} x_{ij}$ captures the total revenue for any feasible x_{ij}. So, v_{ij} can be interpreted as the incremental revenue from selling Bundle j to Consumer i.

We return to the setting of Example 2 to illustrate the application of Proposition 7 and compute the maximum profit for the vendor in this case. Table 3 shows v_{ij} values for Example 2. So, to compute the profit, the appropriate v_{ij} values are summed up. For example, if a vendor tries to serve Consumers 1, 2, 3, 4 with Bundles 1, 2, 3, 4 respectively, then the total vendor profit is $v_{11}+v_{22}+v_{33}+v_{44} = 245$. The maximum profit is the summation of v_{ij} that yields the maximum value and is such that x_{ij} satisfy Constraints (2), (4), and (8). In particular, this implies that the only admissible strategies are such that higher type consumers are served larger-sized bundles. In this case, the maximum profit evaluates to $v_{1,0} + v_{2,0} + v_{3,1} + v_{4,4} = 256$.

Table 3: Computing v_{ij} for Example 2

Bundle	v_{ij}			
size	I_1	I_2	I_3	I_4
0	0	0	0	0
1	-4	-8	16	100
2	2	4	2	180
3	1	5	5	221
4	-1	3	6	240

We now show that CBP2 can be solved without the binary restrictions (4) because its constraint matrix is totally unimodular.

Proposition 8 *The constraint matrix of* CBP2 *is totally unimodular.*

Since the constraint matrix of CBP2 is totally unimodular, we can relax its binary restrictions.

2.2.2 Linear Program

The linear programming formulation of the cardinality bundling problem is first presented by introducing a_{ij}, which is 1 if and only if Consumer i purchases a bundle no larger than j and $i+1$ purchases a bundle strictly larger than j.

Proposition 9 *Let* $a_{ij} = \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'}$, *where* $x_{I+1,j}, \forall j \neq J$ *is understood to be* 0 *and*

$x_{I+1,J}$ *is understood to be* 1. *Then,* $\mathrm{CBP2}$ *is equivalent to the following* $\mathrm{CBP2a}$*:*

$$\mathrm{CBP2a}: \quad \underset{x_{ij},a_{ij}}{\mathit{Max}} \quad \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij}-c_j)x_{ij}$$

$$\text{s.t.} \quad a_{ij} - a_{i,j+1} + x_{ij} - x_{i+1,j} = 0 \;\forall (i,j) \neq (I,J) \tag{11}$$

$$a_{IJ} + x_{IJ} = 1 \tag{12}$$

$$a_{iJ+1} = 0 \;\forall i \tag{13}$$

$$a_{ij} \geq 0;\; x_{ij} \geq 0 \;\forall i,\; \forall j. \tag{14}$$

Recall that initially we formulated the cardinality bundling problem as an MINLP, but have now transformed it into an LP, $\mathrm{CBP2a}$. This enables us to draw upon the comparative statics from the LP literature to derive managerial insights into the CBP. We defer this analysis to the next subsection, but observe here that one does not need a full-blown LP solver to solve CBP. Instead, a simple fast algorithm is revealed by the dual of $\mathrm{CBP2a}$.

Theorem 10 $\mathrm{CBP2d}$ *is the dual of* $\mathrm{CBP2a}$ *and can be solved within* $O(IJ)$ *time.*

$$\mathrm{CBP2d}: \underset{l_{ij}}{\mathit{Min}} \quad l_{I,J}$$

$$\text{s.t.} \quad l_{ij} \geq l_{ij-1} \qquad i = 0,\ldots,I;\; j = 1,\ldots,J \tag{15}$$

$$l_{ij} \geq l_{i-1j} + v_{ij} - c_j \qquad i = 1,\ldots,I;\; j = 0,\ldots,J \tag{16}$$

$$l_{00} = 0.$$

2.3 Comparative Statics

Invoking the sensitivity results from LP, we can infer that the vendor profit is convex in c_j and v_{ij} (Theorem 5.3 in Bertsimas et al., 1997). In the following paragraphs, we consider some additional comparative statics.

Consider the cost parameters first. We say that for any two cost vectors c' and c'', the marginal cost of c' is less than that of c'' if for all $j \geq 1$, $c'_j - c'_{j-1} \leq c''_j - c''_{j-1}$. We say the marginal cost is strictly less if the inequality is strict. Although the solution approach of Hitt and Chen (2005) is inadequate, their insight regarding the weak reduction in the size of cardinality bundles with increasing marginal cost still holds.

Corollary 11 *Assume that marginal cost of c' is less than that of c''. Then, for every optimal allocation x' with c' there exists an optimal allocation s with c'' such that each consumer is allocated a bundle of weakly smaller size in s than in x'. Similarly, for every optimal allocation x'' with c'' there exists an optimal allocation t with c' such that each consumer is allocated a bundle of weakly larger size in t than in x''. If the marginal cost of c' is strictly less than that of c'' then every optimal allocation x' with c' allocates a bundle of size no smaller than any optimal allocation x'' with c''.*

Next, we study how changes to consumers' WTP affect the solution (e.g., when the vendor pursues advertising efforts). Since the optimal value of $\mathrm{CBP2d}$ is convex in v_{ij} and v_{ij} is a linear transformation of w_{ij}, the value function of cardinality bundling problem is convex in WTP. When WTPs do not satisfy SCP, this value function may not be convex in WTPs.[10]

Since v and w are linearly related, we now investigate whether the assumed SCP property of WTPs manifests itself in a natural form over the v variables. It turns out that most v values can be chosen arbitrarily and still the corresponding WTPs satisfy SCP. Incidentally, this is useful for quickly generating example values of WTPs that satisfy SCP. It also gives an insight into why the prior approaches to solve the cardinality

[10]In the following example, we illustrate that when WTPs do not satisfy SCP, $\mathrm{CBP2}$ may not be convex in the WTPs. Consider a scenario with $I = 2$ consumers, $J = 3$ bundle sizes, and costs $c_j = 0$ for all j. Suppose $\mathcal{W}_1$ is one WTP matrix as given in the second and third columns of Table 4 and $\mathcal{W}_2$ is another WTP matrix as given in the fourth and fifth columns of Table 4. Notice, the WTP of Consumer 1 in $\mathcal{W}_2$ is the same as that of Consumer 2 in $\mathcal{W}_1$ and the WTP of Consumer 2 in $\mathcal{W}_2$ is the same as that of Consumer 1 in $\mathcal{W}_1$. The optimal solutions for both problems are $p_1^* = p_2^* = 12, p_3^* = 34$ and the optimal profits are $\Pi_1^* = \Pi_2^* = 46$. Let $\mathcal{W}_3 = \frac{1}{2}\mathcal{W}_1 + \frac{1}{2}\mathcal{W}_2$, as shown in the last two columns in Table 4. The optimal solution is $p_1^* = p_2^* = p_3^* = 26$ and the optimal profit is $\Pi_3^* = 52 > \frac{1}{2}\Pi_1^* + \frac{1}{2}\Pi_2^*$.

Table 4: Willingness-to-pay

Bundle	$\mathcal{W}_1$		$\mathcal{W}_2$		$\mathcal{W}_3$	
size	I_1	I_2	I'_1	I'_2	I''_1	I''_2
1	10	16	16	10	13	13
2	12	18	18	12	15	15
3	12	40	40	12	26	26

bundling problem were unsuccessful since they did not look at the v beyond those in the adjacent column. As mentioned before, given WTPs, $v_{ij} = w_{ij} - (I-i)(w_{i+1j} - w_{ij})$. So,

$$\sum_{i'=i}^{I} v_{i'j} = \sum_{i'=i}^{I}(I-i'-1)w_{ij} - \sum_{i'=i}^{I-1}(I-i')w_{i+1j} = \sum_{i'=i}^{I}(I-i'-1)w_{ij} - \sum_{i'=i+1}^{I}(I-i'-1)w_{ij} = (I-i-1)w_{ij}.$$

Therefore,

$$w_{ij} = \frac{1}{I-i-1}\sum_{i'=i}^{I} v_{i'j}. \tag{17}$$

Given the relationship, we show next that we may choose v arbitrarily for the first $I-1$ consumers and still find WTPs that satisfy SCP and are increasing in j.

Proposition 12 *Given v_{ij} for $i \in \{1, \ldots, I-1\}$ and $j \in \{1, \ldots, J\}$, there exist w_{ij} for $i \in \{1, \ldots, I\}$ amd $j \in \{0, \ldots, J\}$ that satisfy SCP and are increasing in j.*

Now, we investigate the impact of changing WTP on vendor profit. As seen in Examples 1 and 2, increasing WTP (even if it is subject to SCP) does not guarantee an increase in vendor profit. It follows from (10) that increasing consumers' WTP (of course, subject to SCP) for the purchased bundles will weakly increase vendor profits. These insights reveal WTPs that should be targeted by marketing strategies.

Next, consider the scenario when the vendor cannot increase the WTPs but can only shift the WTP from one consumer type to the other (for example, vendor pursues homogenization efforts). We first study the profit implications in the context of information goods, where $c_j = 0\ \forall j$. Let w denote a given $I \times J$ WTP matrix. Define $w' = \mathcal{W}(i_1, i_2, w)$ as a function which maps w to another $I \times J$ matrix w', such that, for any j,

$$w'_{ij} = \begin{cases} w_{ij} & \text{if } i < i_1 \text{ or } i > i_2 \\ \frac{1}{i_2-i_1+1}\sum_{i'=i_1}^{i_2} w_{i'j} & \text{if } i_1 \le i \le i_2. \end{cases}$$

That is, consumers indexed between i_1 and i_2 are homogenized so that their individual WTPs in the transformed setting is the average of their original WTPs; whereas the other consumers remain unaffected. Let $\Pi^*_{CBP}(w)$ denote the optimal profit of the CBP problem for a given w WTP matrix.

Proposition 13 *When $c_j = 0$, for each i', $\Pi^*_{CBP}(\mathcal{W}(i', I, w)) \geq \Pi^*_{CBP}(w)$.*

Proposition 13 shows that homogenizing improves the vendor profit only if it involves the highest consumer type.[11] A corollary is that homogenizing across all consumer types (*i.e.*, using $\mathcal{W}(1, I, w)$ as WTP) will weakly increase the profit. Notice that, if costs are non-zero, even when the highest consumer type is included for homogenization, the vendor profit can decrease.[12]

3 Continuous Case: Model and Analysis

We now investigate a continuous version of the problem treated in Section 2. One application of the continuous problem is in quantity discount pricing, which was explored by Spence (1980).[13] The continuous version can also be applied in cardinality bundling, when the goods are not discrete. For example, many restaurants charge based on weight (for e.g., kilos in Brazil) regardless of the kind of food chosen by the consumer on their plate. The main difference is that bundle sizes are not restricted to integer values $1, \ldots, J$ but can take any real value. The problem for the vendor is then to identify the optimal pricing function for all real-valued sizes, which turns out to be significantly more difficult. Nevertheless, we show that the new

[11] We illustrate that merging may decrease the vendor profit when the highest consumer types are not involved by using the following example.

Table 5: Willingness-to-pay

Bundle size	WTP I_1	WTP I_2	WTP I_3
1	2	10	13
2	4	12	20

Consider a scenario with $I = 3$ consumers, $J = 2$ bundle sizes, and costs $c_j = 0$ for all j. Suppose the WTP for the consumers are as given in Table 5. Obviously, the optimal solution is $p_1^* = 10, p_2^* = 17$ and the optimal profit is 27. If we merge Consumer 1 and 2, then $w'_{11} = w'_{21} = 6, w'_{12} = w'_{22} = 8$, which leads to a new optimal solution of $p_1'^* = 6, p_2'^* = 13$ and a lower optimal profit of 25.

[12] Consider a scenario with $I = 2$, $J = 1$, and costs $c_1 = 10$. Suppose $w_{11} = 4$ and $w_{21} = 20$. the optimal solution is $p_1^* = 20$ and the optimal profit is 10. If we merge Consumer 1 and 2, then $w'_{11} = w'_{21} = 12$, which leads to a new optimal solution of $p_1'^* = 12$ and a lower optimal profit of 4.

[13] The discrete case analyzed in Hitt and Chen (2005) was heavily inspired by Spence (1980).

insights developed in Section 2 can be used to approach this problem.

3.1 Prior Related Work

The model we investigate here is motivated by Spence (1980) and is similar to that in the previous section except that we use a continuous variable $y \in \mathbb{R}_+$ to represent the bundle sizes, instead of using an index j to denote discrete sizes. Every variable that had an index j before now becomes a function of y instead. In particular: $p(y)$ represents the price of bundle size y; $c(y)$ the cost of Bundle y; $w_i(y)$ the Consumer i's WTP for bundle size y. We also define y_i to denote the bundle size Consumer i purchases and corresponds to $j(i) = \sum_{j=0}^{J} j x_{ij}$ in the discrete case. Spence (1980) also assumes WTPs satisfy SCP and models it as $w_i'(y) < w_{i+1}'(y)$ for all y. We relax these conditions slightly to the weak inequality and generalize them to the non-differentiable case as follows:[14]

$$0 = w_i(0) \leq w_i(y) \quad \leq \quad w_{i+1}(y)\ \forall y \tag{18}$$

$$w_i(y+d) - w_i(y) \quad \leq \quad w_{i+1}(y+d) - w_{i+1}(y)\ \forall y\ \forall d \geq 0. \tag{19}$$

Assuming $p(0) = 0$, the vendor's decision problem is then as follows:

$$\begin{aligned} \text{CBPc1}: \quad \underset{y_i, p(y)}{\text{Max}} \quad & \sum_{i=1}^{I} (p(y_i) - c(y_i)) \\ \text{s.t.} \quad & w_i(y_i) - p(y_i) \geq w_i(y) - p(y)\ \forall i\ \forall y. \end{aligned} \tag{20}$$

We first review the approach suggested in Spence (1980). Assuming that WTPs satisfy SCP conditions with a strict inequality, he shows that every optimal solution must satisfy $y_{i+1} \geq y_i$ for all $i < I - 1$. Then, given y_i, $i = 1, \ldots, I$, he substitutes the optimal prices, obtaining the optimization problem in the space of y variables. Then, the paper ignores the constraints $y_{i+1} \geq y_i$ to obtain an unconstrained optimiza-

[14] Since $w_i'(y) \leq w_{i+1}'(y)$ for all y, it follows that $\int_y^d w_i'(y')dy' \leq \int_y^d w_{i+1}'(y')dy'$ for all y and d, which in turn implies that $w_i(y+d) - w_i(y) \leq w_{i+1}(y+d) - w_{i+1}(y)$ for all y and d. On the other hand, $w_{i+1}(y+d) - w_{i+1}(y) \geq w_i(y+d) - w_i(y)$ implies that $\lim_{d\to 0} \frac{w_{i+1}(y+d) - w_{i+1}(y)}{d} \geq \lim_{d\to 0} \frac{w_i(y+d) - w_i(y)}{d}$ or that $w_{i+1}'(y) \geq w_i'(y)$.

tion problem and sets its derivative to zero, yielding the following local optimality condition:

$$(I-i+1)w_i'(y_i) - (I-i)w_{i+1}'(y_i) = c'(y_i). \tag{21}$$

We now interpret the approach of Spence (1980) using our results in Section 2. Assume that the optimal bundle sizes the consumers buy are given by y_i^*, $i = 1, \ldots, I$. Then, CBPc1 restricted to these bundle sizes reduces to a discrete problem. Since y_i^*, $i = 1, \ldots, I$ must be optimal to this restricted problem, the results of our previous section still apply. Therefore, with the slightly relaxed SCP conditions (18) and (19), the results of Spence (1980) still hold. In particular, Proposition 4 shows that there exists an optimal solution with $y_{i+1}^* \geq y_i^*$ for all $i < I$ and Proposition 7 shows that CBPc1 can be rewritten as:

$$\begin{aligned} \text{CBPcy}: \qquad & \underset{y_i}{\text{Max}} \ \sum_{i=1}^{I} (v_i(y_i) - c(y_i)) \\ & \text{s.t.} \ \ y_{i+1} \geq y_i \qquad 1 \leq i \leq I-1, \end{aligned} \tag{22}$$

where $v_i(y_i) = w_i(y_i) - (I-i)(w_{i+1}(y) - w_i(y))$ and $w_{I+1}(y)$ is assumed to be $w_I(y)$. Then, Equation (21) is the same as setting the derivative of the objective of CBPcy to zero, *i.e.*, $v_i'(y_i) = c'(y_i)$.[15]

Solving (21) may not seem hard since each consumer's decision is independent of others. However, this approach only works if Constraints (22) are automatically satisfied by the solution. Otherwise, the optimality conditions do not decompose. Once the optimal Lagrangian multipliers are known, the remaining optimality conditions (those of the inner problem of the Lagrangian dual) can still be decomposed. However, for a given i, the Lagrangian multiplier of $y_{i+1} \geq y_i$ gets multiplied with the decision of both Consumers i and $i+1$. Therefore, the problem of determining the optimal multipliers links the consumers together.

Besides Constraints (22) being ignored in the optimality conditions, there is another subtle issue with Spence's approach. The optimality condition in (21 is a local optimality condition, which would be

[15] More generally, when v and c are not necessarily differentiable, then the above optimality condition generalizes to zero belonging to the subdifferential of $v_i(y_i) - c(y_i)$.

reasonable, if the objective had a unique local maximum (for example if it was strictly concave). However, as shown in the next example, $v_i(\cdot)$ is often nonconvex, and there may be many points where the derivative of the objective of CBPcy is zero.

Example 14 *In* $\mathrm{CBPc1}$*, assume that consumers can choose any bundle size* y*, as long as* $0 \leq y \leq J$*, where* J *is an even number, and let* $c(y)$ *be identically zero. Let* $w_i(y) = 1 + \frac{I}{I-i+1}(\pi y + \log(1+y)) - \cos(\pi y)\ \forall i$*. Each consumer's WTP is increasing in* y *and the WTPs satisfy SCP. It follows that* $v_i(y) = 1 - \cos(\pi y)$*. Therefore, if* y_i *is even, it satisfies (21). Since every consumer can be assigned Bundles* $\{0, 2, \ldots, J\}$*,* $\left(\frac{J}{2}+1\right)^I$ *solutions satisfy Condition (21). Moreover, let* $J = 4I - 2$*, and observe that there are exponentially many solutions that satisfy Condition (21) and satisfy Constraint (22). To see this, consider* 2^I *solutions obtained by allocating bundle sizes in* $\{4(i-1), 4i-2\}$ *to Consumer* i*.*

Spence (1980) does not mention the fact that there may be many solutions that satisfy Condition (21). There is, thus, no guidance available on selecting the best solution among them. If one ignores Constraint (22), this situation can be remedied by selecting, for Consumer i, the bundle size y_i that maximizes $v_i(y) - c_i(y)$ by solving a one-dimensional global optimization problem. However, in the presence of Constraint (22), the situation is significantly more complex. Thus, the approach based on Condition (21) is deficient in that it ignores Constraint (22) and does not provide any way of selecting the global optimal solution from many possible local optima.

3.2 Reformulation and Approximation

We assume that the vendor only provides bundles of size Y or smaller. This assumption is reasonable since the vendor is typically limited by a production capacity. In other words, we include the constraint $0 \leq y_i \leq Y$ for all i in CBPcy. As illustrated in Section 3.1, solving $\mathrm{CBPc1}$ is challenging since it requires the determination of the optimal price function $p(y)$ instead of pricing a discrete set of bundles and has infinitely many incentive compatibility constraints of the type (20), one for each y. These issues can be somewhat sidestepped by reformulating CBPc1 as CBPcy which has finitely many continuous variables. However, since the resulting functions $v_i(\cdot)$ are in general non-convex, the problem remains challenging to solve, especially in the presence of Constraints (22).

First, we remark that it is possible to extend the approach used in formulating CBP2d to solve the continuous case. Assume $v_i(\cdot)$ and $c(\cdot)$ are bounded. Then, an equivalent problem aims to find functions $l_i(\cdot)$, $i = 1, \dots, I$, such that:

$$\begin{aligned}
\text{CBPcyd}: \quad & \min_{l_i(y)} \quad l_I(Y) \\
& \text{s.t.} \quad l_i(y) \geq l_{i-1}(y) + v_i(y) - c(y) && i = 1, \dots, I,\ 0 \leq y \leq Y && (23) \\
& \qquad l_0(y) = 0 && 0 \leq y \leq Y && (24) \\
& \qquad l_i(y) \text{ is non-negative and non-decreasing} && i = 1, \dots, I,\ 0 \leq y \leq Y && (25)
\end{aligned}$$

Appendix A.12 shows that CBPcyd is a valid reformulation. The above approach solves the continuous cardinality bundling problem by computing $l_i(y) = \sup\{l_{i-1}(y') + v_i(y') - c(y') \mid y' \leq y\}$ for each i.

We remark that the convex reformulation CBP2 (without the integrality constraints) for the discrete case does not extend easily to the continuous case. Note that, for CBP2, the bundle size that Consumer i buys is $y_i = \sum_{j=0}^{J} jx_{ij}$. At the binary values of x_{ij}, these reduce to $\sum_{j=0}^{J} jx_{i+1j} \geq \sum_{j=0}^{J} jx_{ij}$. However, when x_{ij} take continuous values, Constraints (8) are tighter: $\sum_{j=0}^{J} jx_{i+1j} = \sum_{j=1}^{J} \sum_{j'=1}^{j} x_{i+1j} = \sum_{j'=1}^{J} \sum_{j=j'}^{J} x_{i+1j} \geq \sum_{j'=1}^{J} \sum_{j=j'}^{J} x_{ij} = \sum_{j=1}^{J} \sum_{j'=1}^{j} x_{ij} = \sum_{j=0}^{J} jx_{ij}$, where the inequality follows from Constraints (8). The converse does not hold for continuous values of x_{ij}.[16] This explains why CBPcy is not convex although CBP2 is a convex program when the superfluous binary restrictions are removed.

For a set, S, let $\operatorname{conv}(S)$ and $\operatorname{proj}_x S$ denote respectively the convex hull of S and the projection of S to the space of x variables. Let $\{k_j\}_{j=0}^{J} \in [0, Y]^{J+1}$, where $0 = k_0 < \cdots < k_J = Y$. Consider $y' \in \mathbb{R}^I$,

[16] To see this, let $J = 2$ and define $x_{i0} = x_{i2} = 0.5$, $x_{i1} = x_{i+1,0} = x_{i+1,2} = 0$, and $x_{i+1,1} = 1$. Then, although $\sum_{j=0}^{J} jx_{ij} \leq \sum_{j=0}^{J} jx_{i+1j}$, $x_{iJ} \not\leq x_{i+1J}$.

with $0 \le y_i' \le Y$ for all i that satisfies Constraints (22) and extend y' to $(y', x') \in \mathbb{R}^I \times \mathbb{R}^{I \times J}$ so that

$$x'_{ij} = \begin{cases} 0 & \text{if } y_i' \le k_{j-1} \text{ or } y_i' \ge k_{j+1} \\ \frac{y_i' - k_{j-1}}{k_{j-1} - k_j} & \text{if } k_{j-1} < y_i' < k_j \\ \frac{k_{j+1} - y_i'}{k_{j+1} - k_j} & \text{if } k_j \le y_i' < k_{j+1}, \end{cases} \tag{26}$$

where k_{-1} and k_{J+1} are understood to be 0 and $Y+1$ respectively. Define

$$S = \left\{ (y,x) \;\middle|\; \begin{array}{l} y_i = \sum_{j=0}^{J} k_j x_{ij}, \forall i; \quad \sum_{j=0}^{J} k_j x_{ij} \ge \sum_{j=0}^{J} k_j x_{i+1j}, i = 1, \ldots, I-1 \\ \sum_{j=0}^{J} x_{ij} = 1, \forall i; \quad x_{ij} x_{ij'} = 0, \forall i, j, j' \ge j+2; \quad x_{ij} \ge 0, \forall i, j \end{array} \right\}, \tag{27}$$

and observe that (y', x') is the only solution in S that projects to y'. Next, we compute $\operatorname{conv}(S)$.

Lemma 15 *The convex hull of S is given by:*

$$S' = \left\{ (y,x) \;\middle|\; \begin{array}{l} y_i = \sum_{j=0}^{J} k_j x_{ij}, \forall i; \quad \sum_{j'=j}^{J} x_{ij'} \le \sum_{j'=j}^{J} x_{i+1j'}, \forall j, i = 1, \ldots, I-1; \\ \sum_{j=0}^{J} x_{ij} = 1, \forall i; \quad x_{ij} \ge 0, \forall i, j \end{array} \right\}. \tag{28}$$

Let $A = \{y \mid (22), 0 \le y_i \le Y, \forall i\}$. *Then,* $\operatorname{proj}_y S' = \operatorname{proj}_y S = A$. *Further,* $\operatorname{conv}(\operatorname{proj}_x S) = \operatorname{proj}_x S'$.

By Lemma 15, the continuous cardinality bundling problem can be written as:

$$\text{CBPcx} : \text{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \left(v_i \left(\sum_{j=0}^{J} k_j x_{ij} \right) - c \left(\sum_{j=0}^{J} k_j x_{ij} \right) \right) \;\middle|\; x \in \operatorname{proj}_x S. \right\}.$$ [17]

We now show that when $w_i(\cdot)$ and $c(\cdot)$ are piecewise linear functions whose breakpoints form a

[17] $\text{Max}_{y_i} \left\{ \sum_{i=1}^{I} \big(v(y_i) - c(y_i) \big) \mid y \in A \right\}$ reformulates to $\text{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \Big(v\big(\sum_{j=0}^{J} k_j x_{ij}\big) - c\big(\sum_{j=0}^{J} k_j x_{ij}\big) \Big) \;\Big|\; (y,x) \in S \right\}$, which reduces to CBPcx since the objective only depends on x.

subset of $\{k_1, \ldots, k_J\}$, then CBPcx can be solved quickly. First, observe that $k_j \le y \le k_{j+1}$,

$$w_i(y) = \frac{k_{j+1} - y}{k_{j+1} - k_j} w_i(k_j) + \frac{y - k_j}{k_{j+1} - k_j} w_i(k_{j+1}) = x_{ij} w_i(k_j) + x_{ij+1} w_i(k_{j+1}) = \sum_{j'=0}^{J} x_{ij'} w_i(k_{j'}),$$

where the second equality is from (26), and the third equality is because it follows from (26) that $x_{ij'} = 0$ for all $j' \notin \{j, j+1\}$. Similarly $c(y) = \sum_{j=0}^{J} x_{ij} c(k_j)$. We define $w_{ij} = w_i(k_j)$, $c_j = c(k_j)$, and $v_{ij} = w_{ij} - (I - i)(w_{i+1j} - w_{ij})$, where w_{I+1j} is understood to be w_{Ij}. Then, CBPcx can be rewritten as:

$$\text{CBPcxL} : \text{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \;\middle|\; x \in \text{proj}_x S. \right\}$$

Now, since the objective is linear, by Lemma 15, we replace $\text{proj}_x S$ with $\text{proj}_x S'$ and rewrite CBPcxL as:

$$\text{CBPcxL2} : \text{Max}_{x_{ij}} \left\{ \sum_{i=1}^{I} \sum_{j=0}^{J} (v_{ij} - c_j) x_{ij} \;\middle|\; x \in \text{proj}_x S' \right\}.$$

Thus, we have shown the following result.

Theorem 16 *When $w_i(\cdot)$ and $c(\cdot)$ are piecewise linear functions, whose breakpoints form a subset of $\{k_1, \ldots, k_J\}$, the continuous cardinality bundling problem can be solved as* CBPcxL2.

Observe that CBPcxL2 is identical to the discrete cardinality bundling problem CBP2 for which we developed an $O(IJ)$ algorithm in Section 2.2.2. Therefore, it follows from Theorem 16 that the continuous cardinality bundling problem with piecewise-linear functions can be solved in $O(IJ)$ time.

Corollary 17 *When $w_i(\cdot)$ and $c(\cdot)$ are piecewise linear functions, whose breakpoints form a subset of $\{k_1, \ldots, k_J\}$, there exists an optimal solution where every consumer purchases a bundle in $\{k_1, \ldots, k_J\}$, i.e., $y_i \in \{k_1, \ldots, k_J\}$ for all i.*

Now, we relax the assumption that w_i and c are piecewise linear functions and consider the more general case of Lipschitz continuous functions. Recall that a function $f(x)$ is said to be Lipschitz continuous with Lipschitz constant L_f on an interval $[a, b]$, if there is a non-negative constant L_f such that $|f(x_1) - f(x_2)| \leq L_f|x_1 - x_2|$ for all x_1, x_2 that belong to $[a, b]$. We assume that $w_i(y)$ and $c(y)$ are Lipschitz continuous with Lipschitz constant β. We will construct piecewise linear approximation for $w_i(y)$ (resp. $c(y)$). Say, we wish to approximate the solution within ϵ. Then, we choose $k = \frac{\epsilon}{I(2I+1)\beta}$ and $J = \lceil \frac{Y}{k} \rceil$. We let $k_j = jk$ for $j \in 0, \ldots, J-1$ and $k_J = Y$. Then, for $k_j \leq y \leq k_{j+1}$, we define

$$w_i^k(y) = \frac{k_{j+1} - y}{k_{j+1} - k_j} w_i(k_j) + \frac{y - k_j}{k_{j+1} - k_j} w_i(k_{j+1}) \text{ and } c^k(y) = \frac{k_{j+1} - y}{k_{j+1} - k_j} c(k_j) + \frac{y - k_j}{k_{j+1} - k_j} c(k_{j+1}).$$

Observe that $w_i^k(\cdot)$ and $c^k(\cdot)$ are piecewise linear functions. Let Π^c be the optimal value of CBPcx and Π^k denote the optimal profit when $w_i^k(y)$ and $c^k(y)$ are the WTP for Consumer i and the cost for producing y.

Theorem 18 *For a given ϵ, define $k = \frac{\epsilon}{I(2I+1)\beta}$. Then, $\Pi^k \leq \Pi^c \leq \Pi^k + \epsilon$. Further, Π^k can be computed in* O $\left(\frac{I^2(I+2)\beta Y}{\epsilon} + I\right)$ *time.*

4 Conclusion

Pricing of cardinality bundles has not been widely studied in literature although this bundling scheme is increasingly being adopted in industry. Our paper provides a comprehensive analysis of the problem when the consumer's willingness to pay satisfies Spence-Mirrlees condition and consumers are restricted to buy only one bundle. We first study the cardinality bundling problem in the context of discrete bundle sizes. This problem was first considered in Hitt and Chen (2005) and we demonstrate that their proposed approach is not correct. Instead, we observe that the consumer allocations depend on reservation prices of other consumers in a much more complex manner than previously believed. Even then, we find that the problem can be reformulated as a linear program and admits an efficient solution procedure. The structural properties we discover are then used to develop managerial insights regarding the impact of reservation prices and costs on vendor profit. We also use the solution procedure to revisit the quantity discount problem, originally proposed in Spence (1980), and derive insights and a mechanism to obtain solutions that are within an arbitrarily specified tolerance. As a consequence, we obtain results for the continuous cardinality bundling problem as well.

Appendix

A Proofs

A.1 Proof of Proposition 3

Proof. Assume p' is an optimal price vector that is not non-decreasing and k, the smallest index for which $p'_k > p'_{k+1}$, is the largest among all optimal price vectors. We claim that for every feasible solution to CBP1 and for all i, $x_{ik} = 0$. Otherwise, Constraint (2) implies that $x_{ik'} = 0$ for all $k' \neq k$. Since $w_{ik} \leq w_{ik+1}$, $w_{ik} - p'_k < w_{ik+1} - p'_{k+1}$ which violates Constraint (1). Therefore, $x_{ik} = 0$. Consider now a price vector p such that $p_j = p'_j$ for all $j \neq k$ and $p_k = p_{k+1}$. Let (x, p') be feasible to CBP1. Since $x_{ik} = 0$, the objective value for x is the same for both p' and p. We claim that (x, p) is also feasible to CBP1 and therefore the optimal value with price p does not decrease. This is because $\sum_{j'=0}^{J}(w_{ij'} - p_{j'})x_{ij'} \geq \sum_{j'=0}^{J}(w_{ij'} - p'_{j'})x_{ij'} \geq w_{ik+1} - p'_{k+1} \geq w_{ik} - p_k$, where the first inequality follows since $p' \geq p$, the second because (x, p') is feasible, and the last because $w_{ik+1} \geq w_{ik}$ and $p_k = p'_{k+1}$. Further, existence of $k' > k$ such that $p_{k'} > p_{k'+1}$ contradicts the choice of p'. Therefore, p must be non-decreasing. ■

A.2 Proof of Proposition 4

Proof. We show the result for a fixed price vector. Then, the first part follows by applying the argument to an optimal price vector. Observe that there are finitely many solutions to CBP1 in the x-space for a given p. We consider the allocations that yield the most profit and order them arbitrarily. Let $j_k(i')$ denote the bundle Consumer i' buys in the k^{th} such solution to CBP1. Then, let $k' = \arg\max_k \min_{i''}\{i'' \mid j_k(i'') > j_k(i''+1)\}$. This means that k' is the optimal solution where the first consumer that buys a larger sized bundle than her immediate successor is of the highest type. Let $i \in \arg\min_{i''}\{i'' \mid j_{k'}(i'') > j_{k'}(i''+1)\}$. Now, construct the solution $j(\cdot)$ where $j(i') = j_{k'}(i')$ when $i' \neq i+1$ and $j(i+1) = j_{k'}(i)$. We show that

$j(\cdot)$ is a feasible assignment of bundles to consumers which achieves at least the same objective function value, thus deriving a contradiction to the choice of k'. Since we do not change the assignment for any $i' \neq i+1$, we only need to verify that $j(\cdot)$ satisfies $w_{i+1j(i+1)} - p_{j(i+1)} \geq w_{i+1j} - p_j$ for all j. Now, consider the following chain of inequalities:

$$\begin{aligned} 0 &\geq w_{i+1j_{k'}(i)} - p_{j_{k'}(i)} - w_{i+1j_{k'}(i+1)} + p_{j_{k'}(i+1)} \\ &\geq w_{ij_{k'}(i)} - p_{j_{k'}(i)} - w_{ij_{k'}(i+1)} + p_{j_{k'}(i+1)} \\ &\geq 0, \end{aligned}$$

where the first inequality follows because $i+1$ chooses $j_{k'}(i+1)$, the second inequality because $j_{k'}(i) > j_{k'}(i+1)$ implies by SCP that $w_{i+1j_{k'}(i)} - w_{i+1j_{k'}(i+1)} \geq w_{ij_{k'}(i)} - w_{ij_{k'}(i+1)}$ and the last inequality because i chooses $j_{k'}(i)$. Therefore, equality holds throughout. Then, for any j, it follows that

$$w_{i+1j(i+1)} - p_{j(i+1)} = w_{i+1j_{k'}(i)} - p_{j_{k'}(i)} = w_{i+1j_{k'}(i+1)} - p_{j_{k'}(i+1)} \geq w_{i+1j} - p_j,$$

where the first equality follows because $j(i+1) = j_{k'}(i)$, the second equality follows from the argument above, and the first inequality because $i+1$ chooses $j_{k'}(i+1)$ under the feasible solution $j_{k'}(\cdot)$. Therefore, we have shown that $j(\cdot)$ is a feasible assignment of bundles to consumers. Now, we show that the corresponding objective value does not decrease. This follows since

$$\sum_{i'} \left(p_{j(i')} - c_{j(i')}\right) = \sum_{i' \neq i+1} \left(p_{j_{k'}(i')} - c_{j_{k'}(i')}\right) + p_{j_{k'}(i)} - c_{j_{k'}(i)} \geq \sum_{i'} \left(p_{j_{k'}(i')} - c_{j_{k'}(i')}\right),$$

where the first equality follows by the definition of $j(\cdot)$. The first inequality follows because $p_{j_{k'}(i)} - c_{j_{k'}(i)} \geq p_{j_{k'}(i+1)} - c_{j_{k'}(i+1)}$ is implied by $w_{ij_{k'}(i)} - p_{j_{k'}(i)} - w_{ij_{k'}(i+1)} + p_{j_{k'}(i+1)} = 0$ and optimality of $j_{k'}(\cdot)$ for p. Otherwise, $j_{k'}(i+1)$ yields the same surplus to i' as $j_{k'}(i)$, which means $j'(i') = j_{k'}(i')$ for $i' \neq i$ and

$j'(i) = j_{k'}(i+1)$ is feasible, yielding a strictly higher objective value than $j_{k'}(\cdot)$. ■

A.3 Proof of Proposition 5

Proof. Consider an optimal solution such that no other optimal solution allocates a subset of the bundle sizes to the consumers. Assume that the bundle sizes sold are $\{j_k, \ldots, j_1\}$ where $j_k < \cdots < j_1$ and the corresponding price vector is p'. If $J \notin \{j_k, \ldots, j_1\}$, we assume without loss of generality that $p'_J = w_{IJ} + \epsilon$ for some $\epsilon > 0$. Similarly, we assume that for $j \notin \{j_k, \ldots, j_1\} \cup \{J\}$, the price is $\min\{p_{j'} \mid \exists j' \geq j, j' \in \{j_k, \ldots, j_1\} \cup \{J\}\}$. So, by optimality of j', no consumer purchases any bundles not in $\{j_k, \ldots, j_1\}$.

We assume that $k \geq 2$ since there is nothing to show otherwise. We show by induction on r that $p'_{j_{r+1}} - c_{j_{r+1}} < p'_{j_r} - c_{j_r}$ for all $r < k$. Consider $r = 1$. By Proposition 4, Consumers $i, \ldots, I$ purchase Bundle j_1 for some $i \leq I$. Construct a price vector p'' where $p''_j = p'_j$ for $j < j_1$ and $p''_{j_1} = w_{Ij_1} + \epsilon$. Any consumer that does not purchase j_1 does not alter her decision since the surplus of non-preferred bundles only decreased with p''. Since Consumer $i-1$ continues to buy Bundle j_2, by Proposition 4, Consumers $i, \ldots, I$ only consider bundles j_2 or higher. Since j_1 does not offer any surplus, all these consumers will purchase Bundle j_2. Observe that $p'_{j_2} - c_{j_2} \leq p'_{j_1} - c_{j_1}$. Otherwise, the optimal solution with p'' attains a strictly higher profit. If $p'_{j_2} - c_{j_2} = p'_{j_1} - c_{j_1}$, the optimal profit attained with p'' is the same as that with p'. However, this contradicts the selection of the optimal solution with minimal number of bundles allocated to consumers. Therefore, $p'_{j_2} - c_{j_2} < p'_{j_1} - c_{j_1}$. Now, for the induction step, we assume that $p'_{j_r} - c_{j_r} < p'_{j_{r-1}} - c_{j_{r-1}}$ and show that $p'_{j_{r+1}} - c_{j_{r+1}} < p'_{j_r} - c_{j_r}$. Let $\{i_1, \ldots, i_t\}$ be the consumers that purchase Bundle j_r. Then, consider the price vector p'' such that $p''_j = p'_j$ for $j \neq j_r$ and $p''_{j_r} = p'_{j_{r-1}}$. Observe that any consumer who does not purchase j_r does not change their decision since, by Proposition 3, the surplus of non-preferred items only reduced with the price change. It follows from Proposition 4 that any consumer in $\{i_1, \ldots, i_t\}$ now purchases one of the bundles $\{j_{r+1}, j_r, j_{r-1}\}$. We first show that with p'', no consumer strictly prefers j_r. Let $i \in \{i_1, \ldots, i_t\}$. Then, $w_{ij_r} - p''_{j_r} = w_{ij_r} - p'_{j_{r-1}} \leq w_{ij_{r-1}} - p'_{j_{r-1}} = w_{ij_{r-1}} - p''_{j_{r-1}}$. Therefore, Consumer i weakly prefers Bundle j_{r-1} over j_r under price p''. Since we assumed that consumers purchase bundle sizes that offer most profit to the vendor (among the sizes that offer maximum surplus), it follows from the induction hypothesis that each consumer prefers Bundle j_{r-1} over j_r. Now, assume that $p'_{j_{r+1}} - c_{j_{r+1}} \geq p'_{j_r} - c_{j_r}$, *i.e.*, Bundle j_{r+1} offers more profit to the vendor as compared to j_r. Since all the consumers in $\{i_1, \ldots, i_t\}$ now purchase either Bundle j_{r-1} or j_{r+1}, both of which offer either same or more profit to the vendor compared to $p'_{j_r} - c_{j_r}$, the profit under p'' must be optimal, and thus contradicts the minimality of the bundles allocated to consumers. Therefore, $p'_{j_{r+1}} - c_{j_{r+1}} < p'_{j_r} - c_{j_r}$. ■

A.4 Proof of Proposition 6

Proof. Let i_1 be the lowest indexed consumer who purchases a bundle of non-zero size, say $j_1 > 0$. Let p^* be the optimal price vector. Clearly, $p^*_{j_1} \leq w_{i_1 j_1}$. Now, assume that $p^*_{j_1} < w_{i_1 j_1}$. Consider $p' = p^* + \Delta$, where $\Delta = w_{i_1 j_1} - p^*_{j_1} > 0$ and a consumer i' that purchased a bundle, $j' > 0$. Then, $w_{i'j'} - p^*_{j'} \geq w_{i'j_1} - p^*_{j_1} \geq w_{ij_1} - p^*_{j_1} = \Delta$, where the first inequality is because i' prefers j' over j_1 and the second inequality follows from SCP and $i' > i_1$. Therefore, $w_{i'j'} - p'_{j'} = w_{i'j'} - p^*_{j'} - \Delta \geq 0$. This shows that any consumer that purchases j' with p^* still prefers j' to not purchasing anything. For any consumer, the relative preference between bundles of non-zero size does not change. Therefore, all consumers that purchased any product still purchase the same product. The consumers that did not purchase a product with p^* do not have incentive to purchase a product with p' because the surpluses have reduced. Therefore, the consumer purchasing decisions do not change. If I' is the set of consumers that purchase a bundle of non-zero size, the vendor makes an additional $|I'|\Delta$ profit due to the increase in price. Since $i_1 \in I'$, it follows that $|I'| \geq 1$. However, this yields a contradiction to the optimality of p^* since p' yields a strictly higher profit. ■

A.5 Proof of Proposition 7

Proof. For a given x that satisfies (2), (4), and (8), we obtain the optimal prices. Let J' be the set of bundles of non-zero size that some consumers buy. We will derive the prices for the bundles in J' by solving an optimization model. Given the prices of the bundles in J', we show how to price the remaining bundles. If

$J \notin J'$, the price for Bundle J is assigned to be $w_{IJ} + \epsilon$. The price of Bundle 0 is fixed at 0. Now consider a remaining bundle, $j \in \{1, \ldots, J\} \backslash (J' \cup \{J\})$. The vendor does not want any consumer to purchase this bundle. Therefore, he may price j at the price of Bundle $j' = \min\{j'' \mid j'' \geq j, j'' \in J' \cup \{J\}\}$. Since $j \leq J$, it follows that the minimum in the definition of j' is attained.

Now, we compute prices for the bundles in J' by solving CBP1 with x variables fixed to the values given. To emphasize that optimization is in the space of the p variables, we refer to this formulation as CBP$_p$. We will show that CBP$_p$ can be reformulated into a model that is much simpler. We replace the consumers that do not purchase any bundle with the highest type consumer that does not purchase any bundle. (Clearly, if this consumer does not have an incentive to purchase a bundle, the lower-type consumers will not either.) If every consumer purchases some bundle, we create a consumer whose WTP for all bundles is 0 and therefore does not buy any bundle. Then, we reindex the consumers to $1, \ldots, I'$. We denote the reindexed WTP as w' and $w'_{I'+1j} = w'_{I'j}$. We denote by $j(i)$ the bundle that is assigned to Consumer i. We reformulate CBP$_p$ as:

$$\text{CBP1a}: \quad \underset{p_{j(i)}}{\text{Max}} \quad \sum_{i=1}^{I'} (p_{j(i)} - c_{j(i)}) \tag{29}$$

$$\text{s.t.} \quad w'_{ij(i)} - p_{j(i)} \geq w'_{ij(i')} - p_{j(i')} \; 1 \leq i, i' \leq I' \tag{30}$$

$$p_0 = 0 \tag{31}$$

It can be verified easily that CBP1a and CBP$_p$ are equivalent. We assume without loss of generality, by re-indexing the bundles, that the bundles sizes are $\{0, 1, \ldots, |J'|\}$.

Let $\{i_0, i_1, \ldots, i_{J'}\}$ be the lowest-type consumers who buy Bundle j, where by definition, $i_0 = 1$. Now, we rewrite Constraint (30) as $w'_{ij(i)} - p_{j(i)} \geq w'_{ij} - p_j$ for all i and $j \in \{0, \ldots, |J'|\}$. Since the constraint for $j = j(i)$ holds trivially, we decompose this constraint for a Consumer i as follows:

$$w'_{ij(i)} - p_{j(i)} \geq w'_{ij} - p_j \quad \forall j < j(i) \tag{32}$$

$$w'_{ij(i)} - p_{j(i)} \geq w'_{ij} - p_j \quad \forall j > j(i). \tag{33}$$

We show that all constraints in (32) are redundant except those corresponding to some $i \in \{i_1, \ldots, i_{|J'|}\}$ and $j = j(i) - 1$. Note that there is no constraint of the type (32) for $i = 1$. Observe that:

$$w'_{ij(i)} - w'_{ij} = \sum_{j'=j}^{j(i)-1} (w'_{ij'+1} - w'_{ij}) \geq \sum_{j'=j}^{j(i)-1} (w'_{i_{j'+1}j'+1} - w'_{i_{j'+1}j'}) \geq \sum_{j'=j}^{j(i)-1} (p_{j'+1} - p_{j'}) = p_{j(i)} - p_j,$$

where the first inequality is because of SCP and $i_{j'} \leq i$ for all $j' \leq j(i)$, and the second inequality uses (32) for some $i \in \{i_1, \ldots, i_{|J'|}\}$ and where $j = j(i) - 1$. Therefore, we replace (32) by the following:

$$w'_{i_j j} - p_j \geq w'_{i_j j-1} - p_{j-1} \quad 1 \leq j \leq |J'|. \tag{34}$$

We will now show that in every optimal solution the inequalities in (34) are binding. We consider a feasible p to $\mathrm{CBP1a}$ where at least one of the (34) is not binding. Then, we show that p is not optimal by constructing p' which is feasible, has at least one more (34) binding, and has a higher objective function value than p. Let $j' = \arg\min\{j \mid w'_{i_j j} - p_j > w'_{i_j j-1} - p_{j-1}\}$, the index of the first inequality that is not binding, and $\Delta = w'_{i_{j'} j'} - p_{j'} > w'_{i_{j'} j'-1} - p_{j'-1}$. Then, consider the price vector p', where $p'_j = p_j$ for $j < j'$ and $p'_j = p_j + \Delta$ otherwise. Then, it is easy to see that for $j \neq j'$, the left hand side of (34) changes by the same amount as the right hand side. Therefore, if the inequality was binding for p then it remains binding for p'. Further, the adjustment of $p'_{j'}$ guarantees that (34) is binding for $j = j'$. Now, we show that p' is also feasible to (33). If $j(i) < j'$ the inequality follows since the surplus of bundles that the consumer does not buy only increases. If $j(i) \geq j'$, then both sides of the inequality decrease by the same amount. Therefore, the constraint holds. Now,

$$\sum_{i=1}^{I'} p_{j(i)} - c_{j(i)} < \sum_{i=1}^{i_{j'}-1} (p'_{j(i)} - c'_{j(i)}) + \sum_{i=i_{j'}}^{I'} (p'_{j(i)} - c'_{j(i)}),$$

where the inequality follows because $i_{j'} \leq I'$ and $p'_i > p_i$ for $i \geq i_{j'}$.

Constraint (33) is redundant since:

$$\begin{aligned} w'_{ij} - w'_{ij(i)} &= \sum_{j'=j(i)+1}^{j} (w'_{ij'} - w'_{ij'-1}) \\ &\leq \sum_{j'=j(i)+1}^{j} (w'_{i_{j'}j'} - w'_{i_{j'}j'-1}) \\ &= \sum_{j'=j(i)+1}^{j} (p_{j'} - p_{j'-1}) \\ &= p_j - p_{j(i)}. \end{aligned}$$

Here, the first inequality follows from SCP and that $i_{j'} \geq i$ whenever $j' \geq j(i) + 1$, the second equality because (34) is tight at an optimal solution.

For any $j \in J'$, we give a closed-form formula for p_j. Since $p_{j(1)} = 0$ and Constraint (34) is binding,

$$p_j = \sum_{r=1}^{j} \left(w'_{i_r j(i_r)} - w'_{i_r j(i_{r-1})} \right).$$

It is easy to verify that the above formula is equivalent to (10). We define $i_{|J'|+1} = I' + 1$ and let $w'_{I'+1j} =$

$w'_{I'j}$. We now evaluate the objective function value for CBP1a.

$$\begin{aligned}
\sum_{i=1}^{I'}\left(p_{j(i)}-c_{j(i)}\right) &= \sum_{j=1}^{|J'|}\sum_{i=i_j}^{i_{j+1}-1} p_j - \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}x_{ij} \\
&= \sum_{j=1}^{|J'|}\sum_{i=i_j}^{i_{j+1}-1}\sum_{r=1}^{j}\left(w'_{i_r j(i_r)} - w'_{i_r j(i_{r-1})}\right) - \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}x_{ij} \\
&= \sum_{r=1}^{|J'|}\sum_{j=r}^{|J'|}\sum_{i=i_j}^{i_{j+1}-1}\left(w'_{i_r j(i_r)} - w'_{i_r j(i_{r-1})}\right) - \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}x_{ij} \\
&= \sum_{r=1}^{|J'|}\left((I'-i_r+1)w'_{i_r j(i_r)} - (I'-i_{r+1}+1)w'_{i_{r+1} j(i_r)}\right) - \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}x_{ij} \\
&= \sum_{r=1}^{|J'|}\sum_{i=i_r}^{i_{r+1}-1}\left((I'-i+1)w'_{i_r j(i_r)} - (I'-i)w'_{i_r+1 j(i_r)}\right) - \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij}x_{ij} \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(v_{ij}-c_{ij}\right)x_{ij}.
\end{aligned}$$

Here, the forth equality uses $w'_{10} = 0$ and $w'_{I'+1j} = w'_{I'j}$. By Proposition 4, every consumer i', who purchases a bundle of non-zero size, is reindexed to some consumer in $\{1,\ldots,I'\}$. Let this index be i and observe that $I - i' = I' - i$. Therefore, the last equality follows. ■

A.6 Proof of Proposition 8

Proof. Consider any subset, T of the allocation variables, x_{ij}. By Theorem III.1.2.7 in Nemhauser and Wolsey (1988), the constraint matrix of CBP2 is totally unimodular if and only if T can be partitioned into two subsets T_1 and T_2 such that for every constraint, $\sum_{i=1}^{I}\sum_{j=1}^{J} d_{ij}x_{ij} \le d_0$, in CBP2:

$$\left|\sum_{(i,j)\in T_1} d_{ij} - \sum_{(i,j)\in T_2} d_{ij}\right| \le 1. \tag{35}$$

We construct such a partition. For Consumer i, let T contain $\{x_{ij_1}\ldots,x_{ij_{k_i}}\}$. If k_i is odd, we include $\{x_{ij_1},x_{ij_3},\ldots,x_{ij_{k_i}}\}$ in T_1. Otherwise, we include $\{x_{ij_2},x_{ij_4},\ldots,x_{ij_{k_i}}\}$. The remaining variables are in T_2. We do the same for every consumer. Now, consider the variables in T that have a non-zero coefficient in

Constraint (8). Among these, let the number of variables for Consumer i that belong to T_1 (resp. T_2) be a_i (resp. b_i). Clearly, $b_i \in \{a_i - 1, a_i\}$ and the same conclusion holds for Consumer $i+1$'s allocation. Then, for Constraint (8), the sum of the coefficients for variables in T_1 minus the sum of coefficients for variables in T_2 equals $a_i - b_i - a_{i+1} + b_{i+1}$. Then,

$$-1 \leq -a_{i+1} + b_{i+1} \leq a_i - b_i - a_{i+1} + b_{i+1} \leq a_i - b_i \leq 1.$$

We have thus verified (35) for Constraint (8). Verification for Constraint (2) is easy since $\lceil \frac{k_i}{2} \rceil - \lfloor \frac{k_i}{2} \rfloor \leq 1$. Further, (35) holds for bound constraints since they have only one variable with non-zero coefficient. ■

A.7 Proof of Proposition 9

Proof. We first show that CBP2 can be reformulated to the following problem:

$$\begin{aligned}
\text{CBP2b}: \quad \underset{x_{ij}}{\text{Max}} \quad & \sum_{i=1}^{I}\sum_{j=0}^{J}(v_{ij} - c_j)x_{ij} \\
\text{s.t.} \quad & (8) \\
& \sum_{j=0}^{J} x_{Ij} \leq 1 && (36) \\
& x_{ij} \geq 0 \ \forall i, \ \forall j. && (37)
\end{aligned}$$

Note that we have dropped the binary restriction, (4), using Proposition 8 and assuming that the solution technique provides an extreme point solution to CBP2. We only need to show that (2) may be relaxed to (36) and x_{ij} do not need to be upper-bounded by one explicitly. We start by observing that $x_{ij} \leq \sum_{j'=0}^{J} x_{ij} = 1$. Therefore, $x_{ij} \leq 1$ can be dropped from CBP2. We now prove that adding (2) in the presence of (36) and (8) does not alter the optimal objective function value. Clearly, for any i,

$$\sum_{j=0}^{J} x_{ij} \leq \sum_{j=0}^{J} x_{Ij} \leq 1.$$

The first inequality follows from (8) and the second from (36). Observe that the extreme points of the feasible region of CBP2 are binary-valued since $\sum_{j=0}^{J} x_{ij} \leq 1$ and $x_{ij} \geq 0$ imply that $x_{ij} \leq 1$ and the constraint matrix of CBP2b includes a subset of the constraints in CBP2. Now, assume, by contradiction, that x is optimal extreme point of CBP2b but does not satisfy $\sum_{j=0}^{J} x_{ij} = 1$. Then, define $x'_{ij} = x_{ij}$ if $j \neq 0$ and $x'_{i0} = 1 - \sum_{j=0}^{J} x_{ij}$. Since x' satisfies (8), (36), and (37), it is feasible to CBP2b. Further, since x'_{i0} does not participate in the objective function of CBP2b, x' is also optimal to CBP2b. By construction, x' satisfies (2) and is therefore feasible to CBP2. Given that the optimal value of CBP2 is already shown to be no more than that of CBP2b, it follows that x' is optimal to CBP2. Therefore, given an extreme point optimal solution to CBP2b, it is easy to construct a solution optimal to CBP2 with the same objective function value.

Next, we prove that CBP2a is equivalent to CBP2b. Let x be feasible to CBP2b. We construct (a, x) that is feasible to CBP2a and has the same objective value. Let $a_{ij} = \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'}$. Here, we assume that $x_{I+1,J}$ is 1 and for $j \neq J$, $x_{I+1,j}$ is 0. Observe that (11) is satisfied by definition:

$$
\begin{aligned}
& a_{ij} - a_{i,j+1} + x_{ij} - x_{i+1,j} \\
= \; & \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'} - \left(\sum_{j'=j+1}^{J} x_{i+1,j'} - \sum_{j'=j+1}^{J} x_{ij'} \right) + x_{ij} - x_{i+1,j} \\
= \; & x_{i+1,j} - x_{ij} + x_{ij} - x_{i+1,j} \\
= \; & 0.
\end{aligned}
$$

Further, (12) is satisfied because of $a_{IJ} + x_{IJ} = 1 - x_{IJ} + x_{IJ} = 1$. Observe that (13) follows since both summations in the definition are empty. Finally, (14) follows easily from (8) and (37).

Now, given (a, x) feasible to CBP2a, we show that x is feasible to CBP2b with the same objective value. Since the objective functions match in the two problems, we only need to show that x is feasible to CBP2b. Let $i \neq I$, and observe that $0 \leq a_{ij} = a_{ij} - a_{iJ+1} = \sum_{j'=j}^{J} x_{i+1,j'} - \sum_{j'=j}^{J} x_{ij'}$. The first inequality is by (14), first equality is by (13), second equality is by summing (11) for j' from j to J. Therefore, x satisfies (8). Now, consider

$$0 \leq a_{I0} = a_{I0} - a_{IJ} + a_{IJ} = 0 - \sum_{j'=0}^{J-1} x_{Ij'} + 1 - x_{IJ},$$

where the first inequality is by (14), second equality is by summing (11) with $i = I$ and j' ranging from 0 to $J-1$ with (12). Therefore, x satisfies (36). Clearly, $x_{ij} \geq 0$ by (14). ■

A.8 Proof of Theorem 10

Proof. In order to derive the dual formulation, let $l_{ij}, (i,j) \neq (I,J)$ be the multipliers for the constraints in (11) and l_{IJ} be the multiplier for (12). Then, it follows easily that CBP2d is the LP-dual of CBP2a. We have introduced l_{0j}, for $j = 0, \ldots, J$ to make the interpretation of the dual easier. However, it is easy to verify that there exists an optimal solution where all these variables are set to zero. This is because these variables only appear in the right-hand-side of the constraint (15).

We next show that there exists an optimal solution to CBP2d such that for each (i,j), $l_{ij} = \max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$, where, as mentioned before, l_{0j} is understood to be zero. For each (i,j) define $H(i,j) = i(J+1) + j$ and observe that the mapping from $(i,j) \in \mathbb{Z} \times \{0, 1, \ldots, J\}$ to $H(i,j)$ is one-to-one. For a given l' optimal to CBP2d but that does not satisfy $l'_{ij} = \max\{l'_{ij-1}, l'_{i-1j} + v_{ij} - c_j\}$ for all (i,j), let $h(l') = \min_{(i,j)}\{H(i,j) \mid l'_{ij} > \max\{l'_{ij-1}, l'_{i-1j} + v_{ij} - c_j\}\}$. Then, define $l''_{ij} = \max\{l'_{ij-1}, l'_{i-1j} + v_{ij} - c_j\}$ if $H(i,j) = h(l')$ and $l''_{ij} = l'_{ij}$ otherwise. It is easy to verify that l'' is feasible to CBP2d because, if $H(i,j) = h(l')$, by construction, $l''_{ij} \geq l''_{ij-1}$ and $l''_{ij} \geq l''_{i-1j} + v_{ij} - c_j$, $l''_{ij} < l'_{ij}$, and l''_{ij} only appears on the right-hand-side of the remaining constraints if at all. Further, $h(l'') > h(l')$ and $l''_{IJ} \leq l'_{IJ}$. Therefore, in finitely many iterations of this procedure, we find l that satisfies $l_{ij} = \max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$ and is optimal to CBP2d.

Implicit in the argument above, is an algorithm that can be used to optimize CBP2d. The algorithm computes l_{ij}s in increasing order of $H(i,j)$. Since $H(i,j-1) < H(i,j)$ and $H(i-1,j) < H(i,j)$, it follows that $\max\{l_{ij-1}, l_{i-1j} + v_{ij} - c_j\}$ can be computed in constant time. Altogether, the number of such computations required is IJ, if one additionally realizes that l_{0j} and l_{i0} can be assumed to be zero. Therefore, the problem is solved in $O(IJ)$ time. ■

A.9 Proof of Corollary 11

Proof. Let c' and c'' be two (non-decreasing) cost vectors such that, for any $j \geq 1$, $c'_j - c'_{j-1} \leq c''_j - c''_{j-1}$. Let x' be optimal with c'. For c'', we construct an optimal allocation, s, where consumers purchase bundles of weakly decreasing size compared to x'. Assume, x'' is optimal with cost c'' and some consumer purchases a bundle smaller than in x'. Since $c_0 = 0$,

$$\sum_{i=1}^{I}\sum_{j=1}^{J} c_j x'_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J}(c_j - c_0)x'_{ij} = \sum_{i=1}^{I}\sum_{j=1}^{J} x'_{ij}\sum_{j'=1}^{j}(c_{j'} - c_{j'-1}) = \sum_{i=1}^{I}\sum_{j'=1}^{J}(c_{j'} - c_{j'-1})\sum_{j=j'}^{J} x'_{ij}. \quad (38)$$

Now, consider

$$s_{ij} = \min\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\} - \min\left\{\sum_{j'=j+1}^{J} x'_{ij'}, \sum_{j'=j+1}^{J} x''_{ij'}\right\}$$

and let

$$t_{ij} = \max\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\} - \max\left\{\sum_{j'=j+1}^{J} x'_{ij'}, \sum_{j'=j+1}^{J} x''_{ij'}\right\}.$$

Both s_{ij} and t_{ij} are feasible and $s_{ij} + t_{ij} = x'_{ij} + x''_{ij}$. Then:

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{j=1}^{J} v_{ij}(s_{ij} + t_{ij}) - \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(c'_j - c'_{j-1})\sum_{j'=j}^{J} t_{ij'} + (c''_j - c''_{j-1})\sum_{j'=j}^{J} s_{ij'}\right] \\
&\quad \leq \sum_{i=1}^{I}\sum_{j=1}^{J} v_{ij}(x'_{ij} + x''_{ij}) - \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(c'_j - c'_{j-1})\sum_{j'=j}^{J} x'_{ij'} + (c''_j - c''_{j-1})\sum_{j'=j}^{J} x''_{ij'}\right] \\
&\quad \leq \sum_{i=1}^{I}\sum_{j=1}^{J} v_{ij}(s_{ij} + t_{ij}) - \sum_{i=1}^{I}\sum_{j=1}^{J}\left[(c'_j - c'_{j-1})\sum_{j'=j}^{J} t_{ij'} + (c''_j - c''_{j-1})\sum_{j'=j}^{J} s_{ij'}\right],
\end{aligned} \quad (39)$$

where the first inequality is by optimality of x' with cost c' and the optimality of x'' with cost c'' and the second inequality follows $s_{ij} + t_{ij} = x'_{ij} + x''_{ij}$ and rearrangement inequality because $c'_j - c'_{j-1} \le c''_j - c''_{j-1}$, $\sum_{j'=j}^{J} s_{ij'} = \min\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\}$, and $\sum_{j'=j}^{J} t_{ij'} = \max\left\{\sum_{j'=j}^{J} x'_{ij'}, \sum_{j'=j}^{J} x''_{ij'}\right\}$. Therefore, equality holds throughout. Since x' and x'' are optimal with c' and c'' respectively, s is a feasible allocation which yields optimal profit to the vendor when the cost is c''. Since

$$\sum_{j'=0}^{j} s_{ij'} = 1 - \min\left\{\sum_{j'=j+1}^{J} x'_{ij'}, \sum_{j'=j+1}^{J} x''_{ij'}\right\} \ge 1 - \sum_{j'=j+1}^{J} x'_{ij'} = \sum_{j'=0}^{j} x'_{ij'},$$

it follows that s allocates smaller bundle sizes to all consumers compared to x'. Similarly, for every optimal allocation x'' with c'', there exists an optimal allocation t with c' where each consumer buys a bundle of size at least as large as in x''.

Moreover, if $c'_j - c'_{j-1} < c''_j - c''_{j-1}$ and there is a consumer i such that $\sum_{j'=j}^{J} x'_{ij} = 0$ and $\sum_{j'=j}^{J} x''_{ij'} = 1$, then the second inequality in (39) is strict and yields a contradiction. Therefore, if the marginal cost of selling an additional unit (from $j - 1$ to j) with c' is strictly smaller than that with c'', then no consumer purchases a bundle size less than j with c' but at least j with c''. If, for all $j \ge 1$, $c'_j - c'_{j-1} < c''_j - c''_{j-1}$, then with c' no consumer purchases a bundle size smaller than with c''. Or, in every optimal solution with c' consumers purchase a bundle size smaller than in any optimal solution with c''. ■

A.10 Proof of Proposition 13

Proof. Let w be an arbitrary set of WTPs satisfying SCP and v be a set of the corresponding v_{ij} values. Consider $w' = \mathcal{W}(i', I, w)$, wherein WTPs of consumers indexed i' through I are homogenized. Then, the corresponding v'_{ij} values of w' can be written as:

$$v'_{ij} = \begin{cases} v_{ij} & \text{if } i \leq i'-2 \\ (I-i'+2)w_{i'-1j} - \sum_{i''=i'}^{I} w_{i''j} & \text{if } i = i'-1 \\ \frac{1}{I-i'+1}\sum_{i''=i'}^{I} w_{i''j} & \text{if } i \geq i'. \end{cases}$$

Call the CBP problem with WTPs w as $CBP(w)$ and that with w' as $CBP(w')$. Let $j(i)$ denote the bundle size that Consumer i purchases in an optimal solution of $CBP(w)$. Consider an allocation $j'(i)$ such that $j'(i) = j(i)$ when $i < i'$ and $j'(i) = J$ when $i \geq i'$. Obviously, $j'(i)$ is a feasible allocation for $CBP(w')$. We next show that $j'(i)$ in $CBP(w')$ leads to a profit $\sum_{i=1}^{I} v'_{ij'(i)}$ that is weakly higher than $\sum_{i=1}^{I} v_{ij(i)}$, the optimal profit of $CBP(w)$.

$$\begin{aligned}
\sum_{i=1}^{I} v'_{ij'(i)} &= \sum_{i=1}^{i'-2} v'_{ij'(i)} + v'_{i'-1,j(i'-1)} + \sum_{i=i'}^{I} v'_{ij'(i)} \\
&= \sum_{i=1}^{i'-2} v_{ij(i)} + (I-i'+2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I}(w_{iJ} - w_{ij(i'-1)}) \\
&\geq \sum_{i=1}^{i'-2} v_{ij(i)} + (I-i'+2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I}(w_{ij(i)} - w_{ij(i'-1)}) \\
&= \sum_{i=1}^{i'-2} v_{ij(i)} + (I-i'+2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I}\sum_{i''=i'}^{i}(w_{ij(i'')} - w_{i,j(i''-1)}) \\
&\geq \sum_{i=1}^{i'-2} v_{ij(i)} + (I-i'+2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I}\sum_{i''=i'}^{i}(w_{i''j(i'')} - w_{i''j(i''-1)}) \\
&= \sum_{i=1}^{i'-2} v_{ij(i)} + (I-i'+2)w_{i'-1,j(i'-1)} + \sum_{i=i'}^{I}(I-i+1)(w_{ij(i)} - w_{ij(i-1)}) \\
&= \sum_{i=1}^{I} v_{ij(i)},
\end{aligned}$$

wherein the equalities are because of either reorganization or by invoking the definitions; the first inequality is because $w_{iJ} \geq w_{ij(i)}\ \forall i$; and the second inequality is due to SCP. ■

A.11 Proof of Proposition 12

Proof. From Equation (17), $w_{ij} = \sum_{i'=i}^{I} v_{i'j}$. We only need to fix v_{Ij} for all j to define the WTPs, w_{ij} for all i and j. We will show that for large enough v_{Ij}, the corresponding WTPs are non-decreasing in j and satisfy SCP. Observe that:

$$w_{ij+1} - w_{ij} = \frac{1}{I-i+1} \sum_{i'=i}^{I} (v_{i'j+1} - v_{i'j}).$$

Therefore, $w_{1j+1} - w_{1j} \geq 0$ is equivalent to $v_{Ij+1} - v_{Ij} \geq -\sum_{i'=1}^{I-1}(v_{i'j+1} - v_{i'j})$. We define $\bar{v}_{j+1} = \sum_{i'=1}^{I-1}(v_{i'j+1} - v_{i'j})$. Further, in order that w satisfy SCP, we require that $(w_{i+1j+1} - w_{i+1j}) - (w_{ij+1} - w_{ij}) \geq 0$. This simplifies to:

$$0 \leq (I-i+1) \sum_{i'=i+1}^{I} (v_{i'j+1} - v_{i'j}) - (I-i) \sum_{i'=i}^{I} (v_{i'j+1} - v_{i'j}) = \sum_{i'=i+1}^{I} (v_{i'j+1} - v_{i'j}) - (I-i)(v_{ij+1} - v_{ij}).$$

In other words,

$$v_{Ij+1} - v_{Ij} \geq (I-i)(v_{ij+1} - v_{ij}) - \sum_{i'=i+1}^{I-1} (v_{i'j+1} - v_{i'j}).$$

We define $v'_{j+1} = \max_i \left\{ (I-i)(v_{ij+1} - v_{ij}) - \sum_{i'=i+1}^{I-1}(v_{i'j+1} - v_{i'j}) \right\}$. Then, we may define $v_{Ij} = \sum_{j'=1}^{j} \max\{\bar{v}_j, v'_j\}$ to ensure that WTPs satisfy SCP and are non-decreasing in j. ■

A.12 Why CBPcyd is a valid reformulation of CBPcy with $0 \leq y \leq Y$

Let $v_i(y_i) - c(y_i) = f_i(y_i)$ for all i. Let each $f_i(y_i) \leq M$. Then, it follows that $l_i(y_i) = iM$ is a feasible solution to CBPcyd. Therefore, the optimal value of CBPcyd is no more than IM. We may therefore, assume that this constraint is present in the formulation without altering the optimal value. Consequently, each $l(y)$ can be assumed to be bounded over $[0, Y]$. Assume that y' is feasible to CBPcy. We show by induction that $l_i(y'_i) \geq \sum_{i'=1}^{i} f_{i'}(y'_{i'})$. For $i = 1$, $l_1(y'_1) \geq f_1(y'_1)$ because of Constraints (23) and (24).

Assume $l_i(y'_i) \geq \sum_{i'=1}^{i} f_{i'}(y'_{i'})$. Then

$$l_{i+1}(y'_{i+1}) \geq l_i(y'_{i+1}) + f_{i+1}(y'_{i+1}) \geq l_i(y'_i) + f_{i+1}(y'_{i+1}) \geq \sum_{i'=1}^{i+1} f_{i'}(y'_{i'}),$$

where the first inequality is by (23), second by (25), and the third by induction. Since $l_I(Y) \geq l_I(y'_I)$, the optimal value of CBPcyd is at least that of CBPcy. Assume now that the optimal value of the former exceeds that of the latter by an $\epsilon \geq 0$. Given $\bar{y} \in \mathbb{R}$, let $\lambda_i^k(\bar{y})$ be the k^{th} element of a sequence that converges monotonically to the optimal value in $l_i(y) = \sup\{l_{i-1}(y'') + f_i(y'') \mid y'' \leq \bar{y}\}$. Let $\lambda_{i \cdots i'}^k(\bar{y})$ denote $\lambda_i^k \circ \cdots \circ \lambda_{i'}^k(\bar{y})$. Then, let $y^k = (\lambda_{1 \cdots I}^k(Y), \ldots, \lambda_I^k(Y))$, and observe that y^k is feasible to CBPcy. Therefore, $l_I(Y) - \epsilon \geq \sum_{i=1}^{I} f_i(y_{i-1}^k)$. Taking $k \to \infty$,

$$\lim_{k \to \infty} \sum_{i=1}^{I} f_i(y_i^k) \leq l_I(Y) - \epsilon.$$

For any $\delta > 0$ and k, we can find $k'(k)$ nondecreasing in k, such that for all i,

$$l_i(y_i^k) \leq l_{i-1}(y_{i-1}^{k'(k)}) + f_i(y_{i-1}^{k'(k)}) + \delta.$$

Summing for all i, we obtain

$$\sum_{i=1}^{I} l_i(y_i^k) \leq \sum_{i=1}^{I-1} l_i(y_i^{k'(k)}) + \sum_{i=1}^{I} f_i(y_{i-1}^{k'(k)}) + I\delta.$$

Observe that

$$\lim_{k \to \infty} l_I(y_I^k) \geq \lim_{k \to \infty} \left(l_{I-1}(y_I^k) + f_I(y_I^k) \right) = l_I(Y) \geq \lim_{k \to \infty} l_I(y_I^k),$$

where the first inequality is by the definition of $l_I(\cdot)$, the first equality is by the definition of y_I^k, and the second inequality is because $l_I(\cdot)$ is non-decreasing and $y_I^k \leq Y$. Therefore, equality holds throughout.

Taking the limit as $k \to \infty$,

$$l_I(Y) \leq \lim_{k\to\infty} \sum_{i=1}^{I} f_i(y_{i-1}^{k'(k)}) + I\delta \leq l_I(Y) - \epsilon + I\delta.$$

Since δ was chosen arbitrarily, the above implies $\epsilon \leq 0$, proving that the optimal value of CBPcyd matches that of CBPcy.

A.13 Proof of Lemma 15

Proof. To show that $S' = \mathrm{conv}(S)$, we show that $\mathrm{vert}(S') \subseteq S \subseteq S'$, where $\mathrm{vert}(S')$ are the vertices of S'. Then, the result follows because S' is bounded, $\mathrm{conv}(S') = \mathrm{conv}(\mathrm{vert}(S')) \subseteq \mathrm{conv}(S) \subseteq \mathrm{conv}(S') = S'$, and $\mathrm{conv}(S') = S'$. We first show that $S \subseteq S'$. Let $(y', x') \in S$. Then, y' satisfies (22) and x' is as defined in (26). We show that $(y', x') \in S'$. For that, we need to show that $\sum_{j'=j}^{J} x'_{ij} \leq \sum_{j'=j}^{J} x'_{i+1j}$ for all j. Let j_1 be such that $k_{j_1} \leq y'_i \leq k_{j_1+1}$. Because of (26), for $0 \leq j \leq j_1$, $\sum_{j'=j}^{J} x'_{i+1j} = 1 \geq \sum_{j'=j}^{J} x'_{ij}$. For $j > j_1 + 1$, $\sum_{j'=j}^{J} x'_{ij} = 0 \leq \sum_{j'=j}^{J} x'_{i+1j}$. Therefore, we only need to consider $j = j_1 + 1$. Then,

$$\begin{aligned} k_{j_1} + (k_{j_1+1} - k_{j_1})x'_{ij_1+1} = \sum_{j=0}^{J} k_j x'_{ij} \leq \sum_{j=0}^{J} k_j x'_{i+1j} = k_{j_1} + \sum_{j=j_1+1}^{J} (k_j - k_{j-1}) \sum_{j'=j}^{J} x_{i+1j'} \\ \leq k_{j_1} + (k_{j_1+1} - k_{j_1}) \sum_{j=j_1+1}^{J} x_{i+1j_1+1} \end{aligned}$$

Since $k_{j_1+1} - k_{j_1} > 0$, it follows that

$$\sum_{j=j_1+1}^{J} x'_{ij} = x'_{ij_1+1} \leq \sum_{j=j_1+1}^{J} x_{i+1j_1+1}.$$

Now, we show that $\mathrm{vert}(S') \subseteq S$. Let $(y', x') \in \mathrm{vert}(S')$. Obviously, $x' \in \mathrm{vert}(\mathrm{proj}_x(S'))$. However, by Proposition 8, the constraint matrix defining $\mathrm{proj}_x(S')$ is totally unimodular. Therefore, x' is binary-valued.

Then, it follows from $\sum_{j=0}^{J} x_{ij} = 1$ that $x_{ij}x_{ij'} = 0$ for all $j \neq j'$. Finally,

$$\sum_{j=0}^{J} k_j x_{ij} = \sum_{j=1}^{J}\sum_{j'=1}^{j}(k_{j'} - k_{j'-1})x_{ij} = \sum_{j'=1}^{J}(k_{j'} - k_{j'-1})\sum_{j=j'}^{J} x_{ij} \leq \sum_{j'=1}^{J}(k_{j'} - k_{j'-1})\sum_{j=j'}^{J} x_{i+1j}$$
$$= \sum_{j=1}^{J}\sum_{j'=1}^{j}(k_{j'} - k_{j'-1})x_{i+1j} = \sum_{j=0}^{J} k_j x_{i+1j},$$

where the inequality follows because $k_{j'} \geq k_{j'-1}$ and $\sum_{j=j'}^{J} x_{ij} \leq \sum_{j=j'}^{J} x_{i+1j}$.

We now show that $\text{proj}_y(S') = \text{proj}_y(S) = A$. Towards this end, we prove that $\text{proj}_y(S) \subseteq A$. Let $(y, x) \in S$. It follows that $0 \leq y_i \leq Y$ because $0 = k_0 \sum_{j=0}^{J} x_{ij} \leq \sum_{j=0}^{J} k_j x_{ij} \leq k_J \sum_{j=0}^{J} x_{ij} = Y$. Also, $y_i \leq y_{i+1}$ follows directly from $\sum_{j=0}^{J} k_j x_{ij} \leq \sum_{j=0}^{J} k_j x_{i+1j}$. Since $A \subseteq \text{proj}_y(S)$ follows directly from (26), it follows that $\text{proj}_y(S) = A$. Then, $\text{proj}_y(S') = A$ follows from

$$\text{proj}_y(S') = \text{proj}_y(\text{conv}(S)) = \text{conv}(\text{proj}_y(S)) = \text{conv}(A) = A,$$

where the second equality because a linear transformation commutes with convexification, the third equality because $\text{proj}_y(S) = A$ and the last equality because A is convex. The last statement in the lemma follows from $\text{conv}(\text{proj}_x(S)) = \text{proj}_x(\text{conv}(S)) = \text{proj}_x(S')$. ∎

A.14 Proof of Theorem 18

Proof. We first show that $\Pi^k \leq \Pi^c$. Define $w_{ij} = w_i^k(k_j)$, $c_j = c^k(k_j)$, and $v_{ij} = w_{ij} - (I-i)(w_{i+1j} - w_{ij})$. Then, we solve CBP2 to find Π^k and the optimal solution x_{ij} for all i, j. The prices p_j are assumed to satisfy Proposition 5. Now, for any $y' \in [0, Y]$, define $p'(y') = \min\{p(k_j) \mid k_j \geq y', j = 0, \ldots, J\}$. Observe that since $y \leq Y = k_J$, the minimum in the formula is attained. Let $y_i = \sum_{j=0}^{J} k_j x_{ij}$. We claim that (y, p') is feasible to CBPc1 and has an objective value of Π^k. Consider Constraints (20). Let $k_{j'-1} < y \leq k_{j'}$ for some j'. Then, since (x, p) is feasible to CBP1, it follows that

$$w_i(y_i) - p'(y_i) = \sum_{j=0}^{J}(w_{ij} - p_j)x_{ij} \geq w_{ij'} - p_{j'} = w_i(k_{j'}) - p'(k_{j'}) \geq w_i(y) - p'(y).$$

The objective function value of (y, p') is then

$$\sum_{i=1}^{I}(p'(y_i) - c(y_i)) = \sum_{i=1}^{I}\sum_{j=0}^{J}(p_j - c_j)x_{ij} = \Pi^k.$$

Since (y, p') is feasible to CBPc1 and has an objective value of Π^k, it follows that the optimal value Π^c to CBPc1 is at least Π^k.

Now, we show that $\Pi^c \leq \Pi^k + \epsilon$. Let (y', p') be the optimal assignment and price for CBPc1. Now consider CBPc1 where the $w_i(\cdot)$ and $c(\cdot)$ functions are replaced with $w_i^k(\cdot)$ and $c^k(\cdot)$ and call this problem Q. Since $w_i^k(\cdot)$ and $c^k(\cdot)$ are piecewise-linear with breakpoints in $\{k_1, \ldots, k_J\}$, it follows from Theorem 16 that the optimal value of Q is Π^k. Now, we define $p''(y) = \min\{p(y_i') - i\delta \mid y_i' \geq y, i = 0, \ldots, I+1\}$, where y_0' and y_{I+1}' are assumed to be 0 and Y respectively and δ will be fixed later. Assume $p''(\cdot)$ is the price in Q. We show that there is a feasible solution (y'', p'') to Q, where $y_i'' \in \{y_1', \ldots, y_I'\}$ for each i. Instead, let y_i be an allocation to Consumer i such that $y_{i'-1}' < y_i < y_{i'}'$. However,

$$w_i^k(y_i) - p''(y_i) \leq w_i^k(y_{i'}') - p''(y_{i'}'),$$

where the inequality follows since $y_i < y_{i'}'$ implies that $w_i^k(y_i) \leq w_i^k(y_{i'}')$ and the definition of $p''(\cdot)$ implies that $p''(y_i) = p''(y_{i'}')$. Therefore, the consumer may substitute $y_{i'}'$ for y_i without loss of surplus. Now, observe that the choice set of each consumer is finite, therefore there exists a bundle size that provides maximum surplus to the consumer. Now, we show that, by suitably choosing δ, we can ensure that there exists a feasible solution that satisfies $y_i'' \geq y_i'$ for all i. Assume otherwise and consider a Consumer i who purchases $y''_i = y'_{i'} < y'_i$. First, observe that Lipschitz continuity of $w_i(\cdot)$ and $c(\cdot)$ guarantees that for any y,

$$\begin{aligned} |w_i(y) - w_i^k(y)| &\leq \max\{w_i^k(k_{j+1}) - w_i(y), w_i(y) - w_i^k(k_j)\} \\ &= \max\{w_i(k_{j+1}) - w_i(y), w_i(y) - w_i(k_j)\} \leq k\beta \end{aligned} \tag{40}$$

$$\begin{aligned} |c(y) - c^k(y)| &\leq \max\{|c(y) - c^k(k_j)|, |c^k(k_{j+1}) - c(y)|\} \\ &= \max\{|c(y) - c(k_j)|, |c(k_{j+1}) - c(y)|\} \leq k\beta, \end{aligned} \tag{41}$$

where j is chosen such that $k_j \leq y < k_{j+1}$. The first inequality follows since w^k is non-decreasing and the first equality because $w_i(\cdot)$ (resp. $c(\cdot)$) match $w^k(\cdot)$ (resp. $c^k(\cdot)$) at all $y \in \{k_1, \ldots, k_J\}$. Then, choosing $\delta = 2k\beta$ it follows that:

$$\begin{aligned} w_i^k(y'_{i'}) - p''(y'_{i'}) &= w_i^k(y'_{i'}) - (p(y'_{i'}) - i'\delta) \leq w_i(y'_{i'}) + k\beta - (p(y'_{i'}) - i'\delta) \\ &\leq w_i(y'_i) + k\beta - (p(y'_i) - i'\delta) \leq w_i^k(y'_i) + 2k\beta - (p(y'_i) - i'\delta) \\ &\leq w_i^k(y'_i) + 2k\beta - (i - i')\delta - (p(y'_i) - i\delta) \leq w_i^k(y'_i) - p''(y'_i). \end{aligned}$$

Therefore, no consumer purchases a smaller sized bundle and so, for any $i' > i$:

$$p''(y'_{i'}) - c^k(y'_{i'}) \geq p(y'_{i'}) - c(y'_{i'}) - i'\delta - k\beta \geq p(y_i) - c(y_i) - i'\delta - k\beta \geq p(y_i) - c(y_i) - (2I+1)k\beta,$$

where the first inequality follows from the definition of p'' and (41) and the second inequality from Proposition 5 and $i' > i$, and the third inequality because $\delta = 2k\beta$. Therefore,

$$\Pi^k \geq \Pi^c - I(2I+1)k\beta = \Pi^c - \epsilon.$$

Because $J = \lceil \frac{Y}{k} \rceil$ and CBP2 can be solved in $\mathrm{O}(IJ)$ time, CBPc1 can be approximated within ϵ in $\mathrm{O}\left(\frac{I^2(I+2)\beta Y}{\epsilon} + I\right)$ time. ■

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