Due date: Friday, February 19, 2016 (before class).

1. (8 pts) Prove or disprove the following statements:

(a) For all positive integers $n$, if $n$ is a perfect square then $n + 3$ is not a perfect square. (Recall the definition of a perfect square: an integer $n$ is a perfect square if and only if there exists an integer $a$ such that $a^2 = n$.)

Solution: The claim is false.

Proof: Let $n = 1$. Then, $n + 3 = 4$ and $4 = 2^2$. Therefore, 4 is a perfect square. Thus, the claim is false.

(b) For every real number $x$, there is a nonzero real number $y$ such that $x \cdot y = x + y$.

Solution: The claim is false.

Proof: Let $x = 0$. Then for any nonzero real number $y$, we have: $x \cdot y = 0$ and $x + y = y \neq 0$. Thus, the claim is false.

(c) There is a real number $x$ such that for every integer $n$ we have $\frac{n}{x} > 0$.

Solution: The claim is false.

Proof: Let $x$ be any real number. Then, let $n = 0$ and we have $\frac{n}{x} = 0 \leq 0$ for any $x$. Thus, the claim is false.

(d) The following statements are equivalent for all nonnegative integers $a$ and $b$:

- $a < b$
- $(a + b)^2 < 4b^2$
- $4a^2 < (a + b)^2$

Solution: The claim is true.

Proof: First we show that $a < b \rightarrow (a + b)^2 < 4b^2$. Assume $a < b$. Then, consider $(a + b)^2$:

$$(a + b)^2 = a^2 + 2ab + b^2 < b^2 + 2bb + b^2 \text{ (since } a < b \text{ and are nonnegative)}$$
$$= 4b^2$$

Thus we have shown the first result. Next, we show that $(a + b)^2 < 4b^2 \rightarrow a < b$ by contrapositive. Assume that $a \geq b$. Then consider $(a + b)^2$:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$\leq b^2 + 2bb + b^2 \text{ since } a \geq b \text{ and are nonnegative}$$
$$= 4b^2$$
Thus the contrapositive is true and thus we have shown the second result. Next, we show that \( a < b \rightarrow 4a^2 < (a + b)^2 \). Assume \( a < b \). Then,

\[
(a + b)^2 = a^2 + 2ab + b^2 \text{ since } a < b \text{ and are nonnegative} \\
> a^2 + 2aa + a^2 \\
= 4a^2
\]

Thus we have shown the third result. Finally, we show that \( 4a^2 < (a+b)^2 \rightarrow a < b \) by contrapositive. Assume that \( a \geq b \). Then,

\[
(a + b)^2 = a^2 + 2ab + b^2 \\
\geq a^2 + 2aa + a^2 \text{ since } a \geq b \text{ and are nonnegative} \\
= 4a^2
\]

Thus we have shown the contrapositive and thus the final result is true. Thus, the three statements are equivalent.

2. (6 pts) List all the elements of the following sets:
   (a) \( S = \{i \mid i \in \mathbb{Z} \land i^2 \leq 4\} = \{-2, -1, 0, 1, 2\} \)
   (b) \( S = \{p \mid p \in \mathbb{Q}, \ 0 < p < 1, \ p \text{ is even}\} = \emptyset \)
   (c) \( S = \{x \mid x \in \mathbb{C}, x \text{ is a root of } x^4 - 1\} = \{1, -1, i, -i\} \) (where \( i = \sqrt{-1} \))

3. (10 pts) What are the cardinalities of the following sets?
   (a) \( \emptyset \) Answer: 0
   (b) \( \{\emptyset, 1\} \) Answer: 2
   (c) \( \{1, 2, \{3, 4\}, \emptyset\} \) Answer: 4
   (d) \( \{\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}\} \) Answer: 4
   (e) \( \mathcal{P}(\emptyset) \) Answer: 1

4. (14 pts) Let \( A = \{1, 2, 4, 6, 7\} \) and \( B = \{3, 4, 5\} \), and let our universe be \( U = \{n \mid n \in \mathbb{Z}, 1 \leq n \leq 10\} \).
   (a) List the elements of \( A \cup B = \{1, 2, 3, 4, 5, 6, 7\} \)
   (b) List the elements of \( A \cap \overline{B} = \{1, 2, 6, 7\} \)
   (c) List the elements of \( \overline{A} - B = \{8, 9, 10\} \)
   (d) List the elements of \( \overline{A} - (A \cup B) = \{3, 5\} \)
   (e) List the elements of \( \mathcal{P}(B) = \{\emptyset, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}\} \)
(f) List the elements of $A \times B =$
\{(1,3), (2,3), (4,3), (6,3), (7,3),
(1,4), (2,4), (4,4), (6,4), (7,4),
(1,5), (2,5), (4,5), (6,5), (7,5)\}

(g) List the elements of $B \times A =$
\{(3,1), (3,2), (3,4), (3,6), (3,7)
(4,1), (4,2), (4,4), (4,6), (4,7)
(5,1), (5,2), (5,4), (5,6), (5,7)\}

5. (8 pts) Let $A$ and $B$ be sets.

(a) Use a venn diagram to show that $A \cap B \subseteq A$.
Solution: Red = $A$, Green = $A \cap B$

Since the green section is within the bounds of $A$, the green section is a subset of $A$.

(b) Use a venn diagram to show that $A \cap B \subseteq B$.
Solution: Pink = $B$, Blue = $A \cap B$

Since the blue section is within the bounds of $B$, the blue section is a subset of $B$.

(c) Use a venn diagram to show that $(A \cup \overline{B}) = (\overline{A} \cap B)$
Solution:
Note that in the diagram for $A \cap B$, we can see that $A \cap B$ is represented by the white portion of the diagram, and this white portion is the same as the diagram for $A \cup B$.

(d) Use a venn diagram to show that $B \subseteq (A - B)$

Solution:

Note that in the diagram for $A - B$, the white area represents $A - B$ and note that the area of $B$ is within this white area. Thus, $B \subseteq A - B$.

6. (4 pts) Let $A$ and $B$ be two sets. Define the symmetric difference of $A$ and $B$ as $A \oplus B = \{ s \mid s \text{ is in } A \text{ or } B \text{ but not both} \}$ Prove that $A \oplus B = (A - B) \cup (B - A)$.

Proof: We use the definitions to prove.

$$A \oplus B = \{ s \mid s \text{ is in } A \text{ or } B \text{ but not both} \}
= \{ s \mid (s \in A \lor s \in B) \land (s \notin A \cap B) \}
= \{ s \mid (s \in A \land s \notin B) \lor (s \notin A \land s \in B) \}
= \{ s \mid (s \in A \land s \notin B) \} \cup \{ s \mid (s \notin A \land s \in B) \}
= (A - B) \cup (B - A)$$

Thus, we have shown the result.

Note: Another way to show this result is to show that each side is a subset of the other.
7. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Given the following definitions of $f$, state whether or not it is a function. If it is a function, state the domain, codomain, and range. If it is not a function, state which domain (if any) will make it a function.

(a) $f(x) = 1/x$.
(b) $f(x) = x^2 + 1$.
(c) $f(x) = 0$
(d) $f(x) = \pm \sqrt{x}$
(e) $f(x) = \log_e(x)$, where $e$ is Euler’s Number.

Solutions:

(a) Not a function. Domain to make it a function: $\mathbb{R} - \{0\}$
(b) Is a function. Domain: $\mathbb{R}$, Codomain: $\mathbb{R}$; Range: $\{r \mid r \in \mathbb{R}, r \geq 1\}$
(c) Is a function. Domain: $\mathbb{R}$, Codomain: $\mathbb{R}$; Range: $\{0\}$
(d) Is not a function. A domain change to $\{0\}$ will make it a function.
(e) Not a function. Domain to make it a function: $\{r \mid r \in \mathbb{R}, r > 0\}$

8. (6 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.

(a) If $f(x) = x^2 + 1$ and $g(x) = 2x + 3$, find $(f + g)(x)$ and $(fg)(x)$.
(b) If $f(x) = x^3 + 2x$ and $g(x) = -x + 2$, find $(fg + f^2)(x)$.
(c) If $f(x) = 2x$ and $g(x) = 3x^2$, find $(f + g)^2(x)$.

Solutions:

(a) $(f + g)(x) = f(x) + g(x) = x^2 + 2x + 4$; $(fg)(x) = f(x)g(x) = 2x^3 + 3x^2 + 2x + 3$
(b) $(fg + f^2)(x) = f(x)g(x) + f(x)f(x) = x^6 + 3x^4 + 2x^3 + 2x^2 + 4x$
(c) $(f + g)^2(x) = (f + g)(f + g)(x) = f^2(x) + 2f(x)g(x) + g^2(x) = 4x^2 + 12x^3 + 9x^4$

9. (8 pts) Let $f$ be a function with domain and codomain defined below. For each $f$, show whether $f$ is injective, surjective, both, or neither.

(a) $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $f(x) = x + 1$. (Note: $\mathbb{Z}^+$ is the set of all positive integers.)
(b) $f : \mathbb{R} \to \mathbb{Z}$ and $f(x) = \lfloor x \rfloor$.
(c) $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x + 1$.
(d) $f : \mathbb{Z}^+ \to \{0, 1, 2\}$ and $f(x) = (x \mod 2) + 1$. (Note: $x \mod n$ is the remainder of $x$ when divided by $n$.)

Solutions:
(a) This function is injective but not surjective. Proof: We show injective: Let $a, b \in \mathbb{Z}^+$ such that $f(a) = f(b)$. Then,
\[
f(a) = f(b) \\
\rightarrow a + 1 = b + 1 \\
\rightarrow a = b
\]
Thus the function is injective. We now show not surjective: Take $b = 1$ in the codomain $\mathbb{Z}^+$. Then, there is no such positive integer $a$ such that $a + 1 = 1 = b$. Thus, the function is not surjective.

(b) The function is surjective but not injective. Proof: We show surjective. Let $b \in \mathbb{Z}$. Then, $b \in \mathbb{R}$ and $f(b) = [b] = b$. So the function is surjective. We now show not injective. Take $a = 0 \in \mathbb{R}$ and $b = 1/2 \in \mathbb{R}$. Then, $f(a) = 0$ and $f(b) = 0$ but $a \neq b$. Thus the function is not injective.

(c) The function is injective and surjective (it is a bijection). Proof: We show injective: Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. Then,
\[
f(a) = f(b) \\
\rightarrow a + 1 = b + 1 \\
\rightarrow a = b
\]
Thus it is injective. We now show surjective: Take any $b \in \mathbb{R}$ and let $a = b - 1$. Then, $f(a) = a + 1 = (b - 1) + 1 = b$. Thus, the function is surjective.

(d) The function is neither injective nor surjective. Proof: We show not injective: Let $a = 1$ and $b = 3$. Then, $f(a) = 1 + 1 = 2$ and $f(b) = 1 + 1 = 2$. Thus, $f(a) = f(b)$ but $a \neq b$ and thus the function is not injective. We now show not surjective: let $b = 0 \in \{0, 1, 2\}$. Then, for any positive integer $a$, $a \mod 2$ is either 0 or 1. Thus, $f(a)$ is either $0 + 1$ or $1 + 1$. Thus, there is no such $a \in \mathbb{Z}^+$ such that $f(a) = b$. Thus, the function is not surjective.

10. (12 pts) For each of the following $f$ and $g$, give $f^{-1}$ (if it exists), $g^{-1}$ (if it exists), $f \circ g$ (if it exists), and $g \circ f$ (if it exists), otherwise state that it does not exist (Note: $f^{-1}$ denotes $f$ inverse, not $1/f$):

   (a) $f(x) = x^3$, $g(x) = x + 2$, where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$

   (b) $f(x) = x/2$, $g(x) = x + 3$, where $f : \mathbb{Q} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{Q}$

   (c) $f(x) = x^2$, $g(x) = [x] + 1$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{Z}$

Solutions:

   (a) $f^{-1}(x) = \sqrt[3]{x}$, $g^{-1}(x) = x - 2$, $(f \circ g)(x) = (x + 2)^3$, $(g \circ f) = x^3 + 2$

   (b) $f^{-1}(x) = 2x$, $g^{-1} = x - 3$, $(f \circ g)(x) = (x + 3)/2$, $(g \circ f)(x) = (x/2) + 3$

   (c) $f^{-1}(x) = \sqrt{x}$, $g^{-1}$ does not exist, $(f \circ g)(x)$ does not exists, $(g \circ f)(x) = [x^2] + 1$