1. (4 pts) Prove that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite. Proof:

Consider the grid of $\mathbb{Z}^+ \times \mathbb{Z}^+$:

We follow the dotted red line to find our bijection with $\mathbb{Z}^+$:

$$
1 \rightarrow (1,1) \\
2 \rightarrow (1,2) \\
3 \rightarrow (2,1) \\
4 \rightarrow (3,1) \\
5 \rightarrow (2,2) \\
6 \rightarrow (1,3) \\
7 \rightarrow (1,4) \\
8 \rightarrow (2,3) \\
9 \rightarrow (3,2)
$$

and so on. This mapping takes every positive integer to a unique ordered pair of $\mathbb{Z}^+ \times \mathbb{Z}^+$, and will map to every element of $\mathbb{Z}^+ \times \mathbb{Z}^+$. Thus this map is a bijection and $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

2. (6 pts) Consider the set $\mathbb{R}$ consisting of all real numbers. Give an example of a subset $S$ of $\mathbb{R}$ with $S \neq \mathbb{R}$ such that:

(a) $S$ is countably infinite

(b) $S$ is finite

(c) $S$ is uncountably infinite

Solutions:
(a) Possible answers:

\[ S = \begin{cases} 
  \mathbb{Q} \\
  \mathbb{Q}^+ \\
  \mathbb{Z} \\
  \mathbb{Z}^+ \\
  \text{the set of all even integers} 
\end{cases} \]

(b) \( S = \{ \pi \}, S = \{ 0, 1, \ldots, 100 \}, S = \emptyset \), and so on.

(c) \( S = \mathbb{R} - \mathbb{Q}, S = \{ r : r \in \mathbb{R}, 0 < r < 1 \} \) are just some examples.

3. (4 pts) Let \( B \) be the set of all possible bit strings. That is, \( B = \{ b_0 b_1 \ldots b_n \mid n \in \mathbb{Z}^+ \cup \{ 0 \} \text{ and } b_i \in \{ 0, 1 \} \} \). Show that \( B \) is countably infinite.

Proof: Note that we can order \( B \) as follows: \( B = \{ 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \ldots \} \).

So we can define a mapping from \( \mathbb{Z}^+ \) to \( B \) as

\[
1 \rightarrow 0 \\
2 \rightarrow 1 \\
3 \rightarrow 00 \\
4 \rightarrow 01 \\
5 \rightarrow 10 \\
6 \rightarrow 11 \\
7 \rightarrow 000 \\
\vdots
\]

continuing to infinity. This mapping is a bijection and thus \( B \) is countably infinite.

**Alternative:** If leading zeros were ignored (i.e., \( 01 = 1, 000 = 0 \), etc.), we can define the mapping from \( \mathbb{Z}^+ \cup \{ 0 \} \) to \( B \) as \( B \) is the binary representation of elements of \( \mathbb{Z}^+ \cup \{ 0 \} \). That is

\[
0 \rightarrow 0 \\
1 \rightarrow 1 \\
2 \rightarrow 10 \\
3 \rightarrow 11 \\
4 \rightarrow 100 \\
\vdots
\]

which is a bijection.
4. (8 pts) For each of the following sets, state the cardinality (finite, countably infinite, or uncountably infinite) (you do not need to justify your answer):

(a) The set $\mathbb{R} - \mathbb{Q}$ \textbf{Uncountably Infinite}
(b) The set $\mathbb{Z} \cap \mathbb{Q}$ \textbf{Countably Infinite}
(c) The set $\mathcal{P}(\mathbb{Z})$ \textbf{Uncountably Infinite}
(d) The set $\mathbb{Z} \cap \{r \mid r \in \mathbb{R}, 0 \leq r \leq 1\}$ \textbf{Finite}

5. (8 pts) List the next 5 terms in each of the following sequences:

(a) $a_0 = 7$, $a_n = 2 \cdot a_{n-1} - 5$
(b) $b_0 = 1$, $b_1 = 2$, $b_n = 2 \cdot b_{n-1} + b_{n-2}$
(c) $c_0 = 2$, $c_n = (c_{n-1})^n$
(d) $a_0 = 2$, $a_1 = 3$, $a_n = \frac{a_{n-1}}{a_{n-2}}$

Solutions:

(a) $a_1 = 9$, $a_2 = 13$, $a_3 = 21$, $a_4 = 37$, $a_5 = 69$
(b) $b_2 = 5$, $b_3 = 12$, $b_4 = 29$, $b_5 = 70$, $b_6 = 169$
(c) $c_1 = 2$, $c_2 = 4$, $c_3 = 64$, $c_4 = 16777216$, $c_5 = (16777216)^5$
(d) $a_2 = \frac{3}{2}$, $a_3 = \frac{1}{2}$, $a_4 = \frac{1}{3}$, $a_5 = \frac{2}{3}$, $a_6 = 2$

6. (12 pts) Evaluate the following sums (note: page 166 in the book can be helpful for these):

(a) $\sum_{i=1}^{m} i^2 + i$

(b) $\sum_{j=17}^{n} 6j^2$

(c) $\sum_{k=20}^{\infty} \left(\frac{1}{2}\right)^k$

(d) $\sum_{h=10}^{20} h^3 - 2h^2$

Solutions:

(a)

\[
\sum_{i=1}^{m} i^2 + i = \sum_{i=1}^{m} i^2 + \sum_{i=1}^{m} i = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}
\]
(b) \[
\sum_{j=17}^{n} 6j^2 = 6 \sum_{j=17}^{n} j^2 \\
= 6 \left( \sum_{j=1}^{n} j^2 - \sum_{j=1}^{16} j^2 \right) \\
= 6 \left( \frac{n(n+1)(2n+1)}{6} - \frac{16(17)(33)}{6} \right) \\
= n(n+1)(2n+1) - 8976
\]

(c) \[
\sum_{k=20}^{\infty} \left( \frac{1}{2} \right)^k = \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k - \sum_{k=0}^{19} \left( \frac{1}{2} \right)^k \\
= 2 - \frac{(1/2)^{20} - 1}{1/2 - 1} \\
= 2 + (1/2)^{19} - 2 = (1/2)^{19}
\]

(d) \[
\sum_{h=10}^{20} h^3 - 2h^2 = \sum_{h=10}^{20} h^3 - 2 \sum_{h=10}^{20} h^2 \\
= \left( \sum_{h=1}^{20} h^3 - \sum_{h=1}^{9} h^3 \right) - 2 \left( \sum_{h=1}^{20} h^2 - \sum_{h=1}^{9} h^2 \right) \\
= \frac{20^2(21)^2}{4} - \frac{9^2(10)^2}{4} - \frac{20(21)(41)}{3} + \frac{9(10)(19)}{3} \\
= 44100 - 2025 - 5740 + 570 \\
= 36905
\]

7. (6 pts) Find a solution to each of these recurrence relations with their given initial conditions:

(a) \( a_n = -a_{n-1}, \ a_0 = 6 \)
(b) \( a_n = (n+1)a_{n-1}, \ a_0 = 2 \)
(c) \( a_n = a_{n-1} - n, \ a_0 = 3 \)

Solutions:

(a) \( a_n = (-1)^n 6 \)
(b) \(a_n = 2(n + 1)!\)

(c) \(a_n = 3 - \frac{n(n + 1)}{2}\)

8. (10 pts) Let \(A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & -1 \end{bmatrix}\) and let \(B = \begin{bmatrix} 2 & 3 \\ -2 & 5 \\ -1 & 7 \end{bmatrix}\)

(a) What is the size of \(A\)?

(b) What is the size of \(B\)?

(c) What is \(A^T + B\)?

(d) What is \(AB\)?

(e) What is \(BA\)?

Solutions:

(a) \(2 \times 3\)

(b) \(3 \times 2\)

\[
(c) \begin{bmatrix} 3 & 6 \\ 1 & 9 \\ -1 & 6 \end{bmatrix}
\]

(d) \[
\begin{bmatrix} -4 & 18 \\ -1 & 22 \end{bmatrix}
\]

(e) \[
\begin{bmatrix} 11 & 18 & -3 \\ 13 & 14 & -5 \\ 20 & 25 & -7 \end{bmatrix}
\]