Assignment Problems with Complementarities*

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November 2015

Abstract

The problem of allocating *bundles* of indivisible objects without transfers arises in many practical settings, including the assignment of courses to students, of siblings to schools, and of truckloads of food to food banks. In these settings, the complementarities in preferences are small compared with the size of the market. We exploit this to design mechanisms satisfying efficiency, envy-freeness, and asymptotic strategy-proofness. We introduce two mechanisms, one for cardinal and the other for ordinal preferences. When agents do not want bundles of size larger than k, these mechanisms over-allocate each good by at most $k-1$ units, *ex-post*. These results are based on a generalization of the Birkhoff-von Neuman theorem on how probability shares of bundles can be expressed as lotteries over *approximately* feasible allocations, which is of independent interest.

1 Introduction

This paper studies the problem of allocating *bundles* of indivisible objects without transfers. The problem arises naturally when there are complementarities in preferences—for example, in the assignment of courses to students (Budish [2011]); of CPU time, memory and disk space to computing tasks (Gutman and Nisan [2012]); of truck loads of food to food banks (Houlihan [2006]); of siblings to schools (Abdulkadiroğlu et al. [2006]); and of couples to hospital residency positions (Kojima et al. [2013], Ashlagi et al. [2014]).

Well-known methods for allocating indivisible goods studied in the literature, such as the probabilistic serial mechanism (PS) of Bogomolnaia and Moulin [2001], the competitive equilibrium with equal income mechanism (CEEI) of Hylland and Zeckhauser [1979], and random serial dicta-

*This paper is a merger of Peivandi [2012] and Nguyen and Vohra [2013].

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torship (RSD), were designed for unit demand agents. Existing generalizations of these methods to the multi-unit demand setting fail to preserve the attractive features of their antecedents.[1]

This paper considers multi-unit demand preferences and proposes an alternative approach to design mechanisms for both cardinal and ordinal preferences. Our mechanisms exhibit strong efficiency and equity properties and are asymptotically strategy-proof. However, this come at the cost of feasibility. These mechanisms are approximately infeasible, equivalently, approximately wasteful (some resources are withheld from consumption). The degree of infeasibility (or wastefulness) is determined by the degree of complementarity exhibited in the preferences. The smaller the degree of complementarity in preferences, the smaller the degree of infeasibility (or wastefulness) in the resulting allocations.

Our first mechanism, called MAXCU,[2] works with cardinal utilities. We assume agents are interested in bundles of, at most, size k, a number significantly smaller than the available supply of any good. Course allocation is a representative application of this setting. Each good is a course, and the available supply of each good is the number of seats in the classroom in which the course will be held. Typically, a student is not able to take more than 5 courses in any term, so $k = 5$. MAXCU is ex-ante envy-free, asymptotically strategy-proof, efficient, and approximately feasible. We show that ex-post, MAXCU might over allocate, at most $k - 1$ units of each good. This means, ex-post, we over-allocate at most 4 seats per class. In a classroom with 50 seats, this can easily be accommodated by adding 4 seats. Alternatively, before running the mechanism, one could reduce the available number of seats in each class by 4. The resulting ex-post allocations would be feasible.

Unlike the generalizations of the CEEI due to Budish [2011], which uses ordinal preferences and is based on computing a competitive equilibrium with equal income, our mechanism is based on a linear programming approach that maximizes social welfare. In the absence of transfers, a welfare-maximizing mechanism lacks a device to discourage agents from claiming an excessively large utility for their most preferred bundle of objects. To see this, consider an example with two agents, two diamonds, and two rocks. Each agent can consume at most two items and prefers diamonds to rocks. Any mechanism that implements the unconstrained welfare-optimal solution

[1] These generalizations are discussed in Section 5.
[2] MAXCU stands for maximizing cardinal utilities.

encourages each agent to report a utility of zero for rocks and the largest possible utility for diamonds. We circumvent this difficulty by imposing ex-ante envy-freeness. This forces an equal division of diamonds and rocks between the two agents, negating the incentive to exaggerate utilities.

However, when agents have multi-unit demands, the envy-free welfare-optimal solution might be fractional and not implementable as a lottery. Second, the solution of the welfare-maximizing problem may be sensitive to the reported valuations of the agents. To overcome the first difficulty, we derive a generalization of the Birkhoff-von Neuman theorem about approximate implementation of fractional solutions. This generalization is of independent interest and can be used as a component of other mechanisms. This is illustrated in our paper by generalizing the PS mechanism to a setting where agents have multi-unit demands. To overcome the second difficulty, we perturb the reported utilities before computing the envy-free welfare-optimal solution. With these ideas, we show that MAXCU is near welfare-maximizing and asymptotically strategy-proof.[3]

The chief virtue of this method is that it allows the designer to specify the outcome in terms of probability shares in bundles. Specifically, it gives one greater control over the outcomes.[4] Second, it allows for a succinct description of the mechanism (recall that the set of possible outcomes is significantly larger than the number of possible bundles that an agent can receive). Subject to a restriction on preferences, the method has a complexity that is polynomial in the $|N|$ and $|G|$.

Our second mechanism, called bundled probabilistic serial (BPS), is a natural extension of the PS mechanism of Bogomolnaia and Moulin [2001]. Here, agents select the best available bundle and "eat" that bundle at the same rate until one of the items in the bundle is no longer available. We again use our implementation result to convert the solution into a near feasible lottery. Similar to Bogomolnaia and Moulin [2001], we show this mechanism to be weakly strategy-proof, envy-free, and approximately Pareto optimal. Unlike earlier extensions of the PS mechanism to multi-unit settings that rule out complementarities altogether (as in Che and Kojima [2010] and Kojima [2009]), our mechanism accommodates limited complementarities in preferences.

The next section introduces notation, the setting and precise restrictions on preferences we

[3]The relation between envy-freeness and strategy-proofness has been observed, for example, in Jackson and Kremer [2007].

[4]For example, we show how to use this mechanism to implement a competitive equilibrium with equal incomes.

impose, and the approximate implementation result. Section 3 introduces and analyzes our first mechanism. Section 4 describes the second mechanism, which is the generalization of probabilistic serial mechanism. Section 5 discusses related literature. Section 6 concludes, and the appendix contains technical proofs.

2 Notation and Approximate Implementation

As noted earlier, the equivalence between probability shares and lotteries relies on the Birkhoff-von Neuman theorem. This section introduces an approximate generalization of the Birkhoff-von Neuman theorem that accommodates complementarities in preferences.

In the combinatorial assignment problem, we have a set N of agents and a set G of goods. For each $j \in G$, the available supply of good j is an integer s_j. A bundle is represented by a non-negative vector $B \in \mathbb{N}^{|G|}$, where the j^{th}-coordinate B_j indicates the number of copies of good j in the bundle B. The size of a bundle B, denoted $size(B)$, is defined as the total number of items in B, i.e., $\sum_{j \in G} B_j$. Agent i is interested in obtaining at most one bundle. Here we will assume that the maximum size of a single bundle is at most k. In the course allocation problem, for example, students are agents, and each good j corresponds to a course, with the number of available seats being s_j. Each student requires at most 1 seat in each class. In practice, students can only consume a bundle of size at most 5, so $k = 5$. In the problem of assigning couples to hospital residency positions, $k = 2$. Each bundle consists of 2 positions in the same hospital or in two different but nearby hospitals.

We assume that each agent receives at most one bundle of goods. To describe the set of feasible allocations of objects to agents, we introduce the variables $x_i(B) \in \{0, 1\}$ for each agent i and a bundle B such that $x_i(B) = 1$ if agent i obtains bundle B and $x_i(B) = 0$ otherwise.

The set of feasible *integral* allocations[5] can be described by the following constraints. First, each agent is only interested in bundles of size at most k; thus, we assign $x_i(B) = 0$ for all B of

[5]We use the term "integral allocation" to distinguish it from an allocation produced by a lottery.

size larger than k:

$$x_i(B) \in \{0,1\} \;\; \forall i, B \qquad\qquad (Integral)$$
$$x_i(B) = 0 \;\text{ if }\; size(B) > k$$

Second, each agent receives at most one bundle of goods:

$$\sum_B x_i(B) \leq 1 \; \forall i \in N. \qquad\qquad (Demand)$$

Third, for each type of good j, we do not allocate more than its available supply:

$$\sum_{i \in N} \sum_{B \ni j} B_j \cdot x_i(B) \leq s_j \; \forall j \in G. \qquad\qquad (Supply)$$

We call x a feasible *fractional* allocation if it satisfies (*Demand*) and (*Supply*) and

$$0 \leq x_i(B) \leq 1 \;\; \forall i, B \qquad\qquad (Fractional)$$
$$x_i(B) = 0 \;\text{ if }\; size(B) > k.$$

Every lottery over feasible integral allocations corresponds to a feasible fractional allocation by setting $x_i(B)$ equal to the probability that agent i obtains bundle B. The opposite, however, is not always true, except when $k = 1$ (this is the Birkhoff-von Neuman theorem).

When $k > 1$, the Birkhoff-von Neuman theorem fails. Our main result in this section shows that for general k, every feasible fractional allocation can be expressed as a lottery over integral allocations that might over-allocate at most $k - 1$ units of each good. Specifically, to define approximate supply constraints:

$$\sum_{i \in N} \sum_{B \ni j} B_j \cdot x_i(B) \leq s_j + k - 1 \; \forall j \in G. \qquad\qquad (Supply+k\text{-}1)$$

Our result is the following:

Theorem 2.1 *Any (fractional) solution of (Fractional–Demand–Supply) can be implemented as a lottery over (integral) allocations that satisfy (Integral–Demand–Supply+k-1).*

The proof of Theorem 2.1 is given in Appendix A. Note that in Theorem 2.1 the over-allocation amount is *independent* of market size and only depends on the size of the largest bundles.

To obtain an intuition for this result, consider the case $k = 1$, which yields the Birkhoff-von Neuman theorem. Specifically, when $k = 1$, any (fractional) solution x of (*Fractional–Demand–Supply*) can be implemented as a lottery over allocations that satisfy (*Integral–Demand–Supply*).

To prove a generalization of this theorem, it is useful to view the Birkhoff-von Neuman theorem in a different but equivalent way. Namely, given any (fractional) x satisfying (*Fractional–Demand–Supply*) and *any* cost vector u, there is an (integral) $\bar{x}$ satisfying (*Integral–Demand–Supply*) such that $u \cdot \bar{x} \geq u \cdot x$. (See Figure 1 for an illustration.)

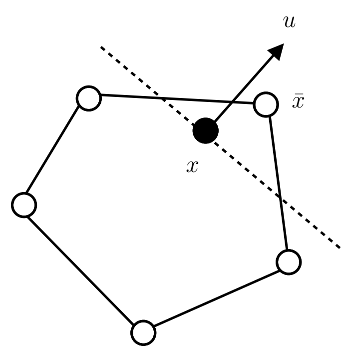

Figure 1: x can be expressed as a lottery over integral allocations

The statement above no longer holds for $k > 1$. However, it can be used to generalize the Birkhoff-von Neuman theorem by allowing $\bar{x}$ to satisfy a relaxed supply constraint. In particular, consider the following example.

Example 1 *Consider an economy consisting of one copy each of three goods,* a, b, c*, and 3 agents,* $1, 2, 3$*. Consider a fractional allocation* $x_1(\{a,b\}) = x_2(\{b,c\}) = x_3(\{a,c\}) = \frac{1}{2}$*. It is straightforward to check that* x *satisfies both* (*Demand*) *and* (*Supply*)*, but it cannot be implemented as a lottery over feasible integral allocations. This is because there exists a cost vector* u *such that for all feasible integral allocation* $\bar{x}$*,* $u \cdot \bar{x} < u \cdot x$*. Specifically, let* $u_1(\{a,b\}) = u_2(\{b,c\}) = u_3(\{a,c\}) = 1$ *and* $u_i(B) = 0$ *otherwise. We have* $u \cdot x = \frac{3}{2}$*. However, because all three goods are in unit supply, we can allocate at most one of the bundles* $\{a,b\}, \{b,c\}$ *or* $\{a,c\}$ *to the agents. That is,* $u \cdot \bar{x} \leq 1 < \frac{3}{2} = u \cdot x$ *for all integral allocation* $\bar{x}$*.*

On the other hand, if we allow over-allocating a good by at most 1 item, then the allocation *that assigns $\{a,b\}$ to 1 and $\{b,c\}$ to 2 will have a cost of 2, which is larger than $u \cdot x = 3/2$. In fact, the fractional allocation x can be implemented by the following lottery: with 1/2 probability assign $\{a,b\}$ to 1 and $\{b,c\}$ to 2, with 1/2 probability assign $\{a,c\}$ to 3.*

Our extension of the Birkhoff-von Neuman theorem relies on the following lemma.

Lemma 2.2 *Given any (not necessarily non-negative) utility vector $u_i(B)$ and any fractional vector x satisfying Fractional, (Demand), and (Supply), we can find in polynomial time an* integral *vector $\bar{x}$ satisfying (Integral), (Demand), and (Supply+k-1) such that $u \cdot \bar{x} \geq u \cdot x$.*

The proof of Lemma 2.2 is provided in Appendix A.1.[6]

The main idea in the proof of Lemma 2.2 is to take an **extreme point solution**, x^*, of (*Fractional*), (*Demand*), and (*Supply*) that maximizes $u \cdot x$ and round it into an integer solution $\bar{x}$. Now $\bar{x}$ will satisfy (*Integral*) and (*Demand*) but violate (*Supply*), but not by too much. This last claim follows from the fact that each component of x^* can appear in at most k inequalities of the form (*Supply*) and that x^* is an extreme point. Hence, no row of (*Supply*) can contain many non-zero components of x^*. Thus, in rounding up these components, we limit the magnitude of the violation of the corresponding constraint.

Given Lemma 2.2, the proof of Theorem 2.1 is as follows.

Proof of Theorem 2.1: For ease of exposition, let Q be the set consisting of all *real* vectors satisfying (*Fractional*), (*Demand*), and (*Supply*); let E_k be the set of *integral* solutions to (*Integral*), (*Demand*), and (*Supply+k-1*).

Suppose Theorem 2.1 does not hold. Then, there is an $x \in Q$ that is not in the convex hull of E_k. Hence, there exists a hyperplane that separates x from E_k. Let u be the vector of coefficients of that hyperplane. We can choose it so that $ux > uz$ for all $z \in E_k$, which contradicts Lemma 2.2. ■

The proof of Theorem 2.1 can be converted into an an algorithm to implement x as a lottery over integral solutions of E_k in polynomial time using standard arguments in optimization (see Grötschel et al. [1981]). However, this algorithm can be impractical for large markets. In

[6] It is an extension of a recent result in Combinatorial Optimization by Király et al. [2012]. In Király et al. [2012], it is assumed that B_j is either 0 or 1, but our proof does not require such an assumption.

Appendix A.2, we provide a practical polynomial time algorithm to construct a lottery with an expectation that is arbitrarily close to the given vector x in Q. Note that the ϵ error, in this implementation can be chosen arbitrarily small, independent of the problem parameters. However, ϵ influences the running time of the algorithm that constructs the lottery. A smaller ϵ implies a longer running time.

3 Maximizing Social Welfare

In this section, we introduce a general mechanism (called MAXCU, which stands for Maximizing Cardinal Utility) for allocating bundles of objects when agents have cardinal preferences over bundles. As discussed in the introduction, in the absence of transfers, identifying an integer allocation in (*Demand-Supply*) that maximizes a weighted sum of utilities subject to fairness and incentive compatibility is difficult. The absence of a numeraire, like money, makes it difficult to discourage agents from claiming an excessively large utility for their most preferred bundle of objects. To overcome this, we introduce (interim) envy-freeness as a constraint into the program (*Demand-Supply*). We then show how Theorem 2.1 can be used to find an approximately feasible allocation so as to maximize a weighted sum of utilities. Using this result, we show that when the economy is large, MAXCU is almost efficient and asymptotically strategy-proof.

k-demand Preferences

We first formally define k-demand preferences. If $u(B)$ is an agent's utility for the bundle $B \in \mathbb{N}^{|G|}$, we require $u(\emptyset) = 0$ and that $u(\cdot)$ have one of the following properties. The first is that no agent has preferences for bundles that are too large; that is,

$$u(B) = 0 \ \forall B \text{ such that } size(B) \geq k + 1. \tag{1}$$

The second is the monotone cover of the first that allows for free disposal and ensures monotonicity; that is,

$$u(B) = \max_{A \subseteq B:} \{u(A) : size(A) \leq k\} \quad \text{for } B \in \mathbb{N}^{|G|}. \tag{2}$$

One setting where (1) or (2) will hold is course assignment. Each good $j \in G$ is a course, s_j is the number of seats in the course, and each $i \in N$ is a student. There is an upper limit, k, on the number of courses any student can take. k is usually small relative to s_j for each $j \in G$. For example, k will be at most 4, while s_j is usually at least 20 and is frequently much larger.

Our mechanism is based on three ideas. First, find a fractional allocation that is envy-free and maximizes a weighted sum of utilities. Second, approximately implement the fractional allocation as a lottery. Third, perturb each agent's utility slightly before finding the optimal fractional allocation to ensure asymptotic strategy-proofness.

In Section 3.1 we formally introduce the mechanism, MAXCU, and the main result. Section 3.2 establishes the approximate efficicency of MAXCU. Section 3.3 discusses how to approximately implement the competitive equilibrium outcome as a special case of MAXCU. In Section 3.4, we prove that by perturbing the objective function in MAXCU, the mechanism is asymptotically strategy-proof.

3.1 Mechanism MAXCU

To implement MAXCU, choose a positive weight w_i for each agent i and solve the linear program of maximizing $\sum_{i\in N}\sum_B w_i \cdot u_i(B)x_i(B)$ subject to (*Demand-Supply*) and the envy-free condition defined below.

An allocation x satisfying (*Demand-Supply*) is envy-free if

$$\sum_B u_i(B)x_i(B) \geq \sum_B u_i(B)x_j(B) \;\forall i \;\forall j \neq i. \qquad \textit{(EnvyFree)}$$

In words, agent i prefers the lottery he or she is faced with to the lottery offered to any other agent. Moulin [1995] (page 166) lists envy-freeness as one of "the two most important tests of equity.".

Formally, the mechanism MAXCU is defined as follows.

Definition 3.1 *Given positive weights w, let x^* be an optimal solution of*

$$\max\{\sum_{i\in N}\sum_B w_i \cdot u_i(B)x_i(B): \;\; s.t. \;\; (\textit{Demand}), (\textit{Supply}), (\textit{EnvyFree})\}. \qquad (3)$$

(If there are multiple solutions, select one with a fixed tie-breaking rule. Note, $x_i(B) = 0$ for all i and B is a feasible solution to 3.)

By Theorem 2.1, x^* can be implemented as a lottery over integral assignments satisfying (Demand) and (Supply+k-1). The mechanism MAXCU takes as input a report of each agent's utility function, returns the optimal (fractional) solution to program (3), and implements it as a lottery.

To illustrate this mechanism, consider the following example.

EXAMPLE 2 *Consider an economy consisting of three goods, a, b, c, each with unit supply, and 3 agents, $1, 2, 3$. Each agent can consume at most 2 items. Each agent's utility for a single item is \$1. Agent 1's utility for $\{a, b\}$ is \$4; agent 2's utility for $\{b, c\}$ is \$4, and agent 3's utility for $\{a, c\}$ is \$4. All other bundles of size 2 give utility of \$2 to each agent.*

Take the weight vector w to be the vector of all 1s. Then, MAXCU will find a fractional allocation x^* that maximizes total utility subject to (Demand), (Supply), and (EnvyFree). This gives $x_1^*(a, b) = x_2^*(b, c) = x_3^*(a, c) = 1/2$. As in Example 1, x^* can be implemented as a lottery over near-feasible allocations as follows: with $1/2$ probability assign $\{a, b\}$ to 1 and $\{b, c\}$ to 2 with $1/2$ probability assign $\{a, c\}$ to 3.

Mechanism MAXCU has several attractive properties. First, it gives the designer control over the outcome through selection of the weights w_i. For example, by choosing $w_i = 1$ for all i, mechanism MAXCU selects an allocation of maximum total utility. We show in Section 3.2 that this mechanism is almost optimal among the envy-free randomized mechanisms. For another choice of the w_is, mechanism MAXCU can implement a generalization of the CEEI mechanism of Hylland and Zeckhauser [1979]. This is discussed in Section 3.3.

Second, mechanism MAXCU is incentive compatible in large markets. Specifically, in Section 3.4 we show that if (3) has a unique optimal solution, one can implement x^* in a way that is asymptotically strategy-proof.[7] Thus, under this condition, mechanism MAXCU is approximately efficient subject to ex-ante envy-free and is asymptotically strategy-proof.

[7]The assumption that (3) has a unique optimal solution is a mild one, because we can always guarantee this by perturbing w_i slightly.

3.2 Approximate Efficiency

In this section, we examine the efficiency of MAXCU with $w_i = 1$ for all i. Recall that the mechanism implements a lottery over allocations that may slightly violate the supply constraints. Our goal is to compare the total utility of MAXCU with the total utility of all envy-free randomized mechanisms that satisfy (*Supply+k-1*). Our main result in this section is as follows.

Theorem 3.2 *If the supply of each good j, $s_j \geq K$, then MAXCU with $w_i = 1$ achieves a total utility of at least $\frac{K}{K+k-1}$ times the total utility of any envy-free randomized mechanism that satisfies (Supply+k-1).*

In the course allocation problem, K tends to be 50, while $k = 5$; this means that our mechanism achieves at least 92.6% of the maximum possible total utility. As K gets larger and k stays constant, $\frac{K}{K+k-1}$ approaches 1, and thus, the MAXCU mechanism achieves almost the maximum total utility.

Also, as a corollary of Theorem 3.2, in the case of hard capacity constraints, one might choose to reduce the capacities of goods to $s'_j := s_j - (k-1)$ before applying MAXCU. MAXCU under this new capacity vector will generate an allocation that does not violate any resource constraint, and thus we obtain the following result.

Corollary 3.3 *If each capacity $s_j \geq K$, then MAXCU with $w_i = 1$ and the reduced capacities s' achieves a total utility of at least $\frac{K+1-k}{K}$ times the total utility of any envy-free, randomized, and feasible mechanism.*

Proof of Theorem 3.2 Observe that with the weights $w_i = 1$, MAXCU implements the solution of the following linear optimization problem.

$$Opt_1 = \max_x u \cdot x \text{ such that } x \text{ satisfies}(\textit{Supply}), (\textit{Fractional}) \textit{ and } (\textit{EnvyFree}). \tag{4}$$

Hence, MAXCU achieves total utility Opt_1. Notice that the total utility of all envy-free randomized mechanisms that satisfy (*Supply+k-1*) is at most

$$Opt_2 = \max_x u \cdot x \text{ such that } x \text{ satisfies}(\textit{Supply+k-1}), (\textit{Fractional}) \textit{ and } (\textit{EnvyFree}) \tag{5}$$

Thus, to prove our theorem, it is enough to show that:

$$Opt_1 \geq \frac{K}{K+k-1} Opt_2. \tag{6}$$

To see (6), let x^* be the optimal solution of (5). Thus, x^* satisfies (*Supply+k-1*), (*Fractional*), and (*EnvyFree*). Consider $\frac{K}{K+k-1}x^*$. It is straightforward to verify that $\frac{K}{K+k-1}x^*$ continues to satisfy (*Fractional*) and (*EnvyFree*). Furthermore, because $s_j \geq K$:

$$\frac{s_j}{s_j+k-1} \geq \frac{K}{K+k-1}.$$

Therefore, $\frac{K}{K+k-1}(s_j+k-1) \leq s_j$. This shows that if we scale x^* down by a factor of $\frac{K}{K+k-1}$, the new solution, $\frac{K}{K+k-1}x^*$, satisfies (*Supply*). Recall that Opt_1 is the maximum total utility for all allocations satisfying (*Supply*), (*Fractional*), and (*EnvyFree*). Hence,

$$\frac{K}{K+k-1}x^* \cdot u \leq Opt_1.$$

This implies $Opt_1 \geq \frac{K}{K+k-1} Opt_2$. ■

3.3 Implementing the CEEI mechanism with MAXCU

In the original CEEI mechanism, agents are restricted to unit demands and are endowed with equal budgets of fictitious money. Each agent i has a utility u_{ij} for good j. We extend each agent's utility function to probability shares in each good in the obvious way:

$$u_i(x) = \sum_j u_{ij} x_{ij},$$

where x_{ij} is the probability share or fraction of good j allocated to agent i. A competitive equilibrium allocation of probability shares in the economy just described is determined. Under the unit demand restriction, the Birkhoff-von Neuman theorem implies that the competitive equilibrium allocation can be implemented as a lottery. From the properties of a competitive equilibrium, it follows that the CEEI mechanism is ex-ante Pareto optimal and ex-ante envy-free. Recall that there exist positive weights $\{w_i\}_{i\in N}$ (Negishi weights) such that this competitive equilibrium allocation maximizes weighted social welfare subject to ex-ante envy-freeness. Hence, MAXCU with the correct Negishi weights can implement the CEEI mechanism.

The idea extends to the case of k-demand preferences as well. Namely, we extend each agent's utility function over bundles to a utility function over bundles of divisible goods. This extension is concave. Treating the goods as divisible, we give each agent the same fictitious budget. Market-clearing prices (one price for each good rather than each bundle) exist (see Appendix B.1 for a short proof via the celebrated results of Arrow-Debreu-McKenzie). This fractional allocation, while feasible, may not be implementable. Under Theorem 2.1 it can be implemented so that it over-allocates each good by at most $k-1$ units.[8] This is a generalization of the CEEI mechanism, which we call the bundled competitive equilibrium from equal income B-CEEI mechanism.

Notice that an equilibrium allocation of bundles can always be obtained by maximizing a suitable weighted sum of utilities subject to (*Demand*), (*Supply*); furthermore, an outcome of a competitive equilibrium with equal budgets is envy-free. Thus, our mechanism MAXCU with the proper weights w_i can implement the outcome of the B-CEEI mechanism.

We continue Example 2 to illustrate how one can use MAXCU with a suitable weight vector w to implement the B-CEEI mechanism.

Example 3 *Consider Example 2, provide each agent with \$1. In Appendix B.1, we show that there exists item prices of* individual *goods a, b, and c such that market clears. In this example, $p_a = p_b = p_c = 1$. With a budget of \$1, to maximize his or her utility, agent 1 would purchase 1/2 of the bundle $\{a, b\}$; similarly, agent 2 would purchase 1/2 of the bundle $\{b, c\}$ and agent 3 would purchase 1/2 of the bundle $\{a, c\}$. This gives us the fractional allocation x^*.*

From the property of a competitive equilibrium with equal budgets, we know that x^* satisfies (EnvyFree), (Demand), and (Supply); furthermore, x^* is Pareto optimal. Thus, x^* can be obtained as an optimal solution of

$$\max\{\sum_{i\in N}\sum_{B} w_i \cdot u_i(B)x_i(B) : \ s.t. \ (Demand), (Supply), (EnvyFree)\},$$

[8]Equivalently, we find prices at which the excess demand for each good is at most $k-1$.

for a proper positive weight vector w. In fact, as in Example 2, choosing $w_i = 1$ will suffice to select x^*. x^*, then, is implemented as a lottery over near-feasible allocations, with $1/2$ probability assigns $\{a,b\}$ to 1 and $\{b,c\}$ to 2; and with $1/2$ probability assigns $\{a,c\}$ to 3.

It is instructive to compare the B-CEEI mechanism to Budish's (Budish [2011]) generalization of the CEEI mechanism—call it the A-CEEI mechanism. A-CEEI is a randomized mechanism for the combinatorial assignment problem based on computing an *approximate* competitive equilibrium from *approximately* equal incomes. B-CEEI is a randomized mechanism for the combinatorial assignment problem based on computing an *exact* competitive equilibrium from equal incomes. The preference information required of agents by A-CEEI is ordinal but of B-CEEI is cardinal. A-CEEI and B-CEEI are both approximately Pareto optimal, asymptotically strategy-proof, and violate the resource constraints. Budish bounds the violation in terms of the Euclidean distance between the supply vector and the vector of the number of goods allocated. That bound is $\frac{\sqrt{\min\{2k,|G|\}|G|}}{2}$. In B-CEEI, the maximum violation in each type of good is at most $k-1$. The two bounds are not comparable. A-CEEI is ex-post approximately envy-free while B-CEEI is ex-ante envy-free. The notion of approximate envy-free used in Budish [2011] can be described in the following way. Suppose agent i is awarded the bundle B and agent i' awarded the bundle B'. If agent i prefers B' to B, then, there is a $j \in B'$ such that agent i prefers B to $B' \setminus j$. When complementarities in preferences are strong, this is a weak requirement. To see why, suppose agent i assigns positive utility to bundle B' but zero utility to any strict subset of B'.

In the next subsection, we introduce notation to define precisely what is meant by asymptotic strategy-proofness as well as prove that the mechanism just described is asymptotically strategy-proof.

3.4 Asymptotic Strategy-Proofness

In this section, we define the notion of asymptotic strategy-proofness when agents have *cardinal* utilities. First, we specify how the economy will "grow". In this we follow Che and Kojima [2010]. For each $q \in \mathbb{N}$, denote the set of objects in the $q-$economy by G and the set of agents by N_q. Each object $j \in G$ in the $q-$economy has $s_j^q \geq k$ copies. Furthermore, $\lim_{q\to\infty} s_j^q = \infty$ for all $j \in G$. The preference of each agent in N^q is determined by the agent's type. Let Θ be a finite set of types, and for each $\theta \in \Theta$, let n_θ^q be the number of agents of type θ in the $q-$economy. The type of an agent encodes his or her preferences, which are represented by a von-Neumann Morgenstern **utility function** defined on bundles of goods. For an agent of type $\theta \in \Theta$, let $u^\theta(B) \geq 0$ be his utility for bundle $B \in \mathbb{N}^{|G|}$. We will also use the notation $u_i^\theta(B)$ (or for short $u_i(B)$) for the utility of agent i for bundle B when his type is θ. We assume that an agent's utility depends exclusively on his type and outcome. Furthermore, we assume for each type, the utility function satisfies either (1) or (2). Without loss of generality, we also assume $u^\theta(\emptyset) = 0$ for all types θ. Given a **lottery** (a probability distribution) over a set of bundles, an agent's utility is his expected utility from the lottery.

We assume that the number of copies of each object and the number of agents of each type grow at the same rate as q.

Assumption 3.1 *There exist positive real numbers $(s_j^*)_{j\in G}$ and $(n_\theta^*)_{\theta\in\Theta}$ such that:* $\lim_{q\to\infty} \frac{s_j^q}{q} = s_j^*$ *and* $\lim_{q\to\infty} \frac{|n_\theta^q|}{q} = n_\theta^* \in \mathbb{R}$, $\forall \theta \in \Theta$.

Given a type profile, an allocation is **envy-free** if all agents weakly prefer the lottery assigned to them to any lottery assigned to another. That is,

$$u^{\theta_i}(x_i) \geq u^{\theta_i}(x_j).$$

Let A denote the set of (approximately) feasible allocations. For every $|N^q| > 0$, a mechanism $\Phi^{(N^q)}$ is a mapping from a profile of agents' types to a lottery over (approximately) feasible allocations. More precisely,

$$\Phi^{(N^q)} : \Theta^{|N^q|} \to \Delta(A).$$

We sometimes use Φ instead of $\Phi^{(N^q)}$ for short.

It will be useful to consider a mechanism from the perspective of an agent i. Let

$$\Phi_i^{|N^q|} : \Theta \times \Theta^{|N^{q-1}|} \to \Delta(A_i),$$

where A_i denotes the possible bundles that agent i obtains, and $\Phi_i(\theta_i, \theta_{-i})$ denotes the lottery over bundles that agent i receives when he or she reports θ_i and other agents report θ_{-i}.

A mechanism Φ is **strategy-proof** if it is optimal for each agent to truthfully report his or her type given any vector of type reports of the other agents, that is

$$u^{\theta_i}(\Phi_i(\theta_i, \theta_{-i})) \geq u^{\theta_i}(\Phi_i({\theta'}_i, \theta_{-i})) \; \forall {\theta'}_i \neq \theta_i. \tag{7}$$

A mechanism is **ϵ-strategy-proof** if it is "almost" optimal for each agent to report truthfully given any vector of reports by the other agents, that is

$$u^{\theta_i}(\Phi_i(\theta_i, \theta_{-i})) \geq u^{\theta_i}(\Phi_i({\theta'}_i, \theta_{-i})) - \epsilon \; \forall {\theta'}_i \neq \theta_i.$$

Finally, we define asymptotic strategy-proofness.

Definition 3.4 *Under Assumption 3.1, Φ is* **asymptotically strategy-proof** *if for any $\epsilon > 0$ there is a q_0 sufficiently large such that Φ is ϵ-strategy-proof whenever $q \geq q_0$.*

Our definition of asymptotic strategy-proofness is similar in spirit to the notion of "strategy-proofness in the large" introduced in Jackson and Kremer [2007] and Azevedo and Budish [2012]. To define it, assume agents' reports are drawn *independently* from a distribution over the type set Θ with full support. A mechanism is strategy-proof in the large if it is ϵ-strategy-proof when the number of agents is sufficiently large. In fact, any mechanism that is asymptotically strategy-proof in our sense will also be strategy-proof in the large.

Theorem 3.5 *Set w_i to be the same for each type of agent and suppose (3) has a unique optimal solution x^* implementable by a lottery $\bar{x}$. Then, the mechanism MAXCU that takes as input a report of each agent's type and returns $\bar{x}$ is asymptotically strategy-proof.*

Proof: See Appendix B.2.

A well-known method to guarantee the uniqueness of x^* in Theorem 3.5 is to perturb each w_i by a small independent random amount (see, for example, Ziegler [1995]). Thus, to obtain an asymptotically strategy-proof mechanism from MAXCU, we first perturb each w_i slightly to get $w'_i = w_i + \epsilon_i$, then run MAXCU on w'.

4 Generalizing the Probabilistic Serial Mechanism

Mechanism MAXCU requires that agents communicate cardinal preferences, which is sometimes impractical. Hence, we turn our attention to mechanisms that rely on ordinal information alone. We generalize the well-known probabilistic serial (PS) mechanism for allocating indivisible goods when agents have strict preferences and unit demands (introduced by Bogomolnaia and Moulin [2001]). The PS mechanism begins with each agent consuming, at the same constant rate, his or her most preferred object. When the supply of an object is exhausted, agents consuming that object switch to consuming the next available object on their preference list. At termination, the fraction of each object an agent has consumed determines the probability shares in that object. These probability shares can be implemented as a lottery over feasible allocations. It is well known that the PS mechanism is envy-free, ordinally efficient, and asymptotically strategy-proof. We define ordinal efficiency for cases when agents have preferences over bundles rather than single objects. Recall that a bundle is represented by a non-negative vector $B \in \mathbb{N}^{|G|}$, where the j^{th}coordinate indicates the number of copies of good j in the bundle B. A bundle can also be represented as a multi-set, and in some of the examples below it will be convenient to do so.

Assume that agents have strict preferences over bundles. Let $\prec_i$ be agent i's ordinal preference ranking over bundles. As each agent receives a lottery over allocations, we extend a preference ordering over bundles to a *partial* ordering over lotteries of bundles via stochastic dominance. Recall that a lottery over allocations induces probability shares x over bundles that satisfy (*Demand*) and (*Supply*). Thus, we may identify each lottery with a solution of (*Demand*) and (*Supply*).[9] An allocation x satisfying (*Demand*) and (*Supply*) *weakly* stochastically dominates an allocation y for agent i, if for all $B \subseteq G$:

$$\sum_{S \succ_i B} x_i(S) \geq \sum_{S \succ_i B} y_i(S).$$

Allocation x stochastically dominates y for agent i, if the above inequality holds strictly for some bundle B. A mechanism is *ordinally efficient* if there is no other random assignment that weakly stochastically dominates the mechanism's allocation with respect to all agents' preferences over bundles.

[9]Recall, each feasible fractional solution of (*Demand*) and (*Supply*) does not correspond to a lottery over feasible integral allocations.

As preferences in this section are ordinal, the notion of strategy-proofness and envy-freeness from Section 3.4 must be modified. An ordinal mechanism is *strategy-proof* if for any agent, the allocation resulting from misreporting is weakly stochastically dominated by the allocation from truthful reporting, with respect to an agent's true preference. A mechanism is envy-free if for all agents, the allocation assigned to them weakly stochastically dominates all other agents' assignments with respect to his preference. A mechanism is *weakly strategy-proof* if for each agent, his or her allocation from truthful reporting is not stochastically dominated by the allocation produced by a misreport, with respect to the agent's true preference. A mechanism is *weakly envy-free* if no agent's allocation is stochastically dominated by the allocation of another agent.

Ours is not the first paper to extend the PS mechanism beyond the unit demand case. See, for example, Kojima [2009] and Budish et al. [2013]. Our generalization differs from these papers in the kind of complementarities in preferences we allow. Those papers assume that agents rank lotteries over assignments based on first order stochastic dominance on *single* objects. This assumption allows these papers to abstract away from the implementability problem caused by complementarities in ordinal preferences. However, as example 4 below shows, in their setting, an agent with responsive cardinal preferences may prefer a utility-dominated (as defined in those papers) lottery.

EXAMPLE 4 *There are two copies each of objects a and b. Agent i has the following cardinal preference for bundles (that we will represent as multi-sets for convenience):*

$$u_i(\{a,a\}) = 6, u_i(\{a,b\}) = 5, u_i(\{b,b\}) = 2, u_i(\{a\}) = 1, u_i(\{b\}) = 0.5.$$

The ordinal preference associated with this cardinal preference is responsive. Consider the following two lotteries:

- *Lottery I: Agent i receives bundle $\{a,a\}$ with probability half and bundle $\{b,b\}$ with probability half.*
- *Lottery II: Agent i receives bundle $\{a,b\}$ with probability 0.99 and $\{b,b\}$, with probability 0.01.*

Under the preferences defined in Kojima [2009] and Pycia [2011], agent i prefers lottery I to lottery

II, since under lottery I agent i has a higher chance of receiving copies of object a. However, agent i has a higher expected utility for lottery II.

4.1 Bundled Probabilistic Serial Mechanism

A natural generalization of the PS mechanism for multi-unit demand is to have agents consume bundles rather than individual objects. When the supply of an object is exhausted, agents switch to their most preferred bundle not containing objects whose supply has been exhausted. This mechanism, which we call the Bundled Probabilistic Serial (BPS) mechanism, returns probability shares in bundles. Therefore, it produces outcomes that, in general, cannot be implemented. Recall that Kojima [2009] and Budish et al. [2013] circumvent this difficulty by assuming preferences satisfy first order stochastic dominance on *single* objects. In this restricted case, the BPS mechanism reduces to the PS mechanism. However, if we assume preferences satisfy the k-demand condition, we may invoke Theorem 2.1. Under these conditions, the BPS mechanism is envy-free, ordinally efficient, strategy-proof, and over-allocates each good by at most $k-1$ units. These results, along with a formal description of the BPS mechanism, are stated below.

To define the BPS mechanism formally, let $t(0) = 0 \leq t(1) \leq t(2) \leq t(v) \leq t(v+1)...$ be the instances in time when the supply of at least one good falls to zero. Also, let $BPS(v)$ be the fractional allocation that the algorithm produces up to time $t(v)$. Initially $BPS(0) = 0$, which is an allocation in which no agent receives an object. At time $t(v)$, the set of goods in non-zero-supply is denoted $G(v)$. Initially, $G(0) = G$. Let z^v be the integral allocation, not necessarily feasible, where each agent is assigned his or her most preferred bundle consisting only of objects in $G(v)$. Let $m_j(v)$ be the total number of copies of object j that appear in z^v.[10] Denote by r_j^v the measure of object j consumed at time $t(v)$, therefore $r_j^0 = 0$. The BPS mechanism consists of the following steps:

- Starting with available supply of $G(v-1)$, the latest time at which the current supply of good j would be exhausted is

$$t_j(v) = sup\{t \in [0,1] | r_j^{v-1} + m_j(v-1)(t - t(v-1)) \leq s_j\}.$$

[10] Agents are allowed to consume multiple copies of an object in $G(v)$.

- Therefore, the first instance at which any good is exhausted is $t(v) = \min_{j \in G(v-1)} t_j(v)$.

- At time $t(v)$, the set of goods with non-zero supply is

$$G(v) = G(v-1) \setminus \{j \in G(v-1) | t_j(v) = t(v)\}.$$

- The measure of object j consumed up to time $t(v)$ is

$$r_j^v = r_i^{v-1} + m_j(v-1)(t_j(v) - t_j(v-1)).$$

- The allocation returned by the BPS mechanism at time $t(v)$ is

$$BPS(v) = BPS(v-1) + (t(v) - t(v-1))z^{v-1}.$$

  The term $(t(v) - t(v-1))$ represents the fraction of the bundle in z^{v-1} that each agent receives in the time interval $(t(v-1), t(v)]$.

- BPS terminates at time $t(v)$ where v is the smallest index such that $t(v) = 1$.

Theorem 4.1 *The BPS mechanism is ordinally efficient, envy-free, weakly strategy-proof, and under k-demand preferences can be implemented so that it over-allocates each good by at most $k-1$ units.*

As the first three items admit a proof similar to the proof of Theorem 1 and Proposition 1 in Bogomolnaia and Moulin [2001], we content ourselves with a sketch. (See Appendix C for a formal proof.)

The greedy nature of BPS implies that a fractional assignment that stochastically dominates the BPS assignment must be infeasible. Similarly, for weak strategy-proofness, if a misreport stochastically dominates truth telling, the feasibility constraints would be violated. Envy-freeness follows from the fact that once BPS stops allocating probability shares of a bundle to agents, then at least one of the objects in that bundle is unavailable. Therefore, no other agent can be allocated probability shares from that bundle afterwards.

The last item follows from Theorem 2.1 as the probability shares produced by BPS satisfy (*Demand*) and (*Supply*).

4.2 Limited Complementarities

We extend BPS to environments in which agents are interested in larger bundles with limited complementarities. We assume complementarities are present only in bundles with size at most k. This extension is analogous to the extensions of unit-demand preferences considered in Kojima [2009], Pycia [2011], and Budish et al. [2013].

In particular, we define limited complementary preferences (LCP) as follows. There is an underlying partition $P_1, P_2 \ldots, P_t$ of G such that $|P_r| \leq k$ for all $r = 1, \ldots, t$. We assume this partition is common knowledge. Furthermore, we assume agents are interested in at most one copy of each good. The preferences of agents have complementarities among the goods inside each P_r, but not across different elements of the partition. Namely, each agent i has preferences, $\succ_i$, on the following set of bundles

$$\mathcal{B} = \{B : \exists r \text{ s.t } B \subset P_r\}.$$

As an example, suppose four goods $\{1, 2, 3, 4\}$. The given partition is (1,2); (3,4). Preferences of an agent, for example, can be $(1, 2) \succ (2) \succ (3, 4) \succ (1) \succ (3) \succ (4) \succ \emptyset$.

Based on this linear preference order, we can define preferences over expected assignments that possibly assign a positive probability over large bundles. Specifically, given an expected assignment x, let $\hat{x}_i(B) = \sum_{S:S \cap P_t = B} x_i(S)$. Thus, $\hat{x}$ is an expected assignment that assigns positive probability to subsets of the elements of the partition and $B \subset P_t$. As an example, assume x assigns 1/2 probability to the bundle (1,2,3) and probability 1/2 to the bundle (1,2), then, under the partition (1,2); (3,4), $\hat{x}$ assigns probability 1 to (1,2) and probability 1/2 to (3). Next, we define preferences over expected assignments. Namely, agent i prefers x to y if he prefers $\hat{x}$ to $\hat{y}$, where the preference between $\hat{x}$ and $\hat{y}$ is given by the stochastic dominance relation.

LCP contains the preferences defined in Kojima [2009], Pycia [2011], and Budish et al. [2013] as a special case. It suffices to choose as the underlying partition one that places each object into its own set.

LCP preferences arise in the bandwidth allocation problem. Blocks within the same spectrum band are complements up to a certain maximum number of blocks, and blocks from different bands are substitutes (see Schweitzer [2014]). Another example is consumption over time. Each element of the partition represents the set of available objects in a time period. Within a time period, there is no restriction on preferences; however, objects from different time periods are substitutes.

Preferences with both complementarities and substitutes have have been studied before; see Sun and Yang [2006] and Teytelboym [2014]. Their set-up is different in that goods in the same element are substitutes while they are complements across elements.

The extension of PS is straightforward. Each agent, at the speed of 1, is allocated probability shares of his or her most preferred available bundle consisting of objects from of a single element of the partition. There are two constraints in this mechanism. First, the total time an agent spends consuming subsets of an element of a partition is 1. Second, the mechanism stops when there is no available bundle in which an agent is interested. We call this extended mechanism the bundled probabilistic serial mechanism under limited complementarity assumption(BPSLC). Example 5 illustrates the workings of BPSLC.

Example 5 *There are two copies each of six objects, $\{a, b, c, d, e, f\}$, and three agents, 1, 2, and 3. Objects are partitioned into two sets, $\{a, b, c\}$ and $\{d, e, f\}$, and the complementarities are only between objects in the same set. Agents are interested in up to 4 objects, two from each element of the partition. Agents have the following preferences over bundles (which we will represent as sets for convenience):*

$\{a, b\} \succ_1 \{d, e\} \succ_1 \{e, f\} \succ_1 \{a, c\} \succ_1 \{b, c\} \succ_1 \{d, f\} \succ_1 \{c\} \succ_1 \emptyset \succ_1$ *all other bundles.*

$\{a, c\} \succ_2 \{d, f\} \succ_2 \{a, b\} \succ_2 \{e, f\} \succ_2 \{e, d\} \succ_2 \{b, c\} \succ_2 \{c\} \succ_2 \emptyset \succ_2$ *all other bundles.*

$\{a, b\} \succ_3 \{a, c\} \succ_3 \{b, c\} \succ_3 \{e, f\} \succ_3 \{d, f\} \succ_3 \{e, d\} \succ_3 \emptyset \succ_3$ *all other bundles.*

Step 1: *BPSLC starts by allocating probability shares from bundle $\{a, b\}$ to agent 1, $\{a, c\}$ to agent 2, and $\{a, b\}$ to agent 3. The supply of object a will be exhausted at time $t = \frac{2}{3}$, and that is when the first step ends.*

Step 2: *In the second step, agents 1 and 2 are allocated probability shares of their second best bundles, namely $\{d, e\}$ and $\{d, f\}$. However, agent 3 cannot be allocated $\{a, c\}$ since object a was* *exhausted in step 1. Therefore, agent 3 is allocated probability shares in* $\{b, c\}$*. Note that all three agents are allocated bundles from* $\{a, b, c\}$ *for* $\frac{2}{3}$ *unit of time, and this mechanism allocates bundles from a partition to agents for at most one unit of time. Therefore, in this step, probability shares are allocated for* $\frac{1}{3}$ *period of time, and this step terminates at time* $\frac{2}{3} + \frac{1}{3}$*.*

Step 3: *The mechanism continues to allocate probability shares from* $\{d, e\}$ *and* $\{d, f\}$ *to agents 1 and 2. However, the mechanism starts allocating agent 3 probability shares from* $\{e, f\}$*. After* $\frac{2}{3}$ *period of time the supply of object* d *is exhausted. Therefore, this step of the mechanism stops at time* $\frac{2}{3} + \frac{1}{3} + \frac{2}{3}$*.*

Step 4: *Note that agents 1 and 2 will not be allocated probability shares from* $\{e, f\}$*, since both were allocated probability shares of bundles from the partition* $\{d, e, f\}$ *for one period of time. In this step, agents 1 and 2 are allocated probability shares from* $\{b, c\}$*, and agent 3 is allocated a probability share from* $\{e, f\}$*. This step ends after* $\frac{1}{6}$ *unit of time, when the supply of object* b *is exhausted.*

Step 5: *The mechanism continues to allocate probability shares from bundle* $\{e, f\}$ *to agent 3. Agents 1 and 2 receive probability shares from* $\{c\}$*. The mechanism stops after* $\frac{1}{6}$ *unit of time, since the supply of both objects* e *and* f *are exhausted.*

The mechanism terminates as all agents have received probability shares of bundles from each partition for 1 unit of time. The mechanism returns the following probability shares: Agent 1 receives $\{a, b\}$ *with probability* $\frac{2}{3}$*,* $\{d, e\}$ *with probability 1,* $\{b, c\}$ *with probability* $\frac{1}{6}$*, and* $\{c\}$ *with probability* $\frac{1}{6}$*. Agent 2 receives* $\{a, c\}$ *with probability* $\frac{2}{3}$*,* $\{d, f\}$ *with probability 1,* $\{b, c\}$ *with probability* $\frac{1}{6}$*, and* $\{c\}$ *with probability* $\frac{1}{6}$*. Agent 3 is allocated* $\{a, b\}$ *with probability* $\frac{2}{3}$*,* $\{b, c\}$ *with probability* $\frac{1}{3}$*, and* $\{e, f\}$ *with probability 1.*

Similar to the BPS mechanism in Section 4.1, the expected assignment may not be implementable. However, it is approximately implementable as given in the following result.

Theorem 4.2 *Let x be a vector that satisfies the following*

$$0 \le x \le 1; x_i(B) = 0 \;\; if \;\; B \notin \mathcal{B} = \{S : \exists r \;\; s.t \; S \subset P_r\}$$

$$\sum_{B:B\subset P_r} x_i(B) \le 1 \; \forall r = 1,..,t; \;\; \forall i \in N \tag{8}$$

$$\sum_{i\in N}\sum_{B\ni j} x_i(B) \le s_j \; \forall j \in G. \tag{9}$$

Then x can be expressed as a lottery over deterministic *allocations satisfying*

$$z \in \{0,1\}; z_i(B) = 0 \;\; if \;\; B \notin \mathcal{B} = \{S : \exists r \;\; s.t \; S \subset P_r\}$$

$$\sum_{B:B\subset P_r} z_i(B) \le 1 \; \forall r = 1,..,t; \;\; \forall i \in N$$

$$\sum_{i\in N}\sum_{B\ni j} z_i(B) \le s_j + k - 1 \; \forall j \in G. \tag{10}$$

The proof of Theorem 4.2 is analogous to that of Theorem 2.1.[11] Here, instead of the constraint that each agent can consume at most 1 bundle, (8) requires that agents consume at most 1 bundle from each partition. (10) implies that the amount of over-allocation continues to be at most $k-1$. With this, we obtain the following result.

Theorem 4.3 *BPSLC has the following properties:*

1. *It produces ordinally efficient probabilistic allocations.*
2. *It is weakly envy-free.*
3. *It is weakly strategy-proof.*
4. *It is implementable in the relaxed economy.*

The proof is similar to the proof of Theorem (4.1). (See Appendix C for a formal proof.) The difference is that BPSLC is weakly envy-free. This is because an agent may stop consuming a bundle whose objects are still available. Note that if an agent assignment is FOSD dominated

[11]For example, one can obtain Theorem 4.2 from Theorem 2.1 by considering t different economies, where the set of objects are the elements of the partition.

by the assignment of another agent, then because of the symmetry of this mechanism and the feasibility constraints, one of the feasibility constraints would be violated.

The BPS mechanism under the k-demand preference assumption is suitable for environments where agents are interested in only a few objects. If k is large, the BPS mechanism may allocate to agents bundles that are too large, resulting in large ex-post envy. This problem is also present in the RSD mechanism. Example 6 below shows why BPSLC is fairer compared to RSD and BPS.

EXAMPLE 6 *Suppose there are 2 agents and 4 objects, numbered* $\{1,2,3,4\}$*. Objects are partitioned into two subsets,* $\{1,2\}$ *and* $\{3,4\}$*. Agent 1 ranks objects as 1, 2, 3, 4, but agent 2 ranks objects as 3, 4, 1, 2.*

RSD as well as BPS will allocate one agent his best bundle, $\{1,3\}$*, and the other his worst bundle,* $\{2,4\}$*. However the BPSLC allocates agent 1 the bundle* $\{1,4\}$*, and agent 2 gets* $\{2,3\}$*. Clearly this is an improvement in terms of fairness.*

4.3 Asymptotic properties of BPS

While the BPS mechanism is not strategy-proof, it is, as we show in this section, asymptotically strategy-proof. The notion of asymptotic strategy-proofness differs from that given in section 3.3. There we had defined asymptotic strategy-proofness for cardinal preferences; see equation (7) and definition 3.4. Here we must modify the definition to account for ordinal preferences.

Recall that agent is endowed with a type $\theta \in \Theta$, and there are a finite number of types. In this case, a type θ determines an agent's ranking over bundles.

Definition 4.4 *Allocation* x $\epsilon-$*stochastically dominates* y *for agents* i *with respect to ordinal preference* $\succ_i$ *if*

$$\sum_{S \succeq_i B} x_i(S) + \epsilon > \sum_{S \succeq_i B} y_i(S).$$

An ordinal mechanism is asymptotically strategy-proof if for all $\epsilon > 0$ *there is a* q *sufficiently large such that the allocation returned from truthful reporting* $\epsilon-$*stochastically dominates any allocation resulting from misreporting in the* $q-$*economy.*

Theorem 4.5 *BPS and BPSLC are asymptotically strategy-proof.*

Proof: Asymptotic strategy-proofness follows from characterizing the limit of the BPS mechanism, which generalizes Theorem 1 in Che and Kojima [2010]. To prove this, we first define a continuum economy as the limit of the q—economy as $q \to \infty$. Let BPS* be the BPS mechanism in the continuum economy. We show that the BPS mechanisms converge to BPS*. The difficulty in our case, compared with Che and Kojima [2010], arises because of the complementarities in agents' preferences. In the unit demand case, once consumption of an object begins, it continues until its supply is exhausted. In our case, consumption of an object occurs in fits and starts. Therefore, a simple adaptation of their proof is not possible. We show this result by proving that the available supply of each object at each step of the mechanism converges to the available supply in the continuum model. For a complete proof, see Appendix D.[12] ■

5 Related Literature

The literature on the allocation of indivisible goods is extensive and can be divided into two strands. The first strand of literature develops and studies mechanisms that specify a lottery over outcomes, while the second strand focuses on mechanisms that specify probability shares in objects. Examples of the first type of mechanism are random serial dictatorship (RSD) and top-trading with random endowments (TTC).[13] Neither explicitly randomizes over each possible outcome given the large number of possible outcomes. Instead, they specify a procedure for assigning goods to agents from a randomly chosen starting point.[14] These methods are typically strategy-proof and Pareto optimal but lack other desirable properties like ordinal efficiency and envy-freeness. Indeed, strategy-proofness and ex-post Pareto optimality rule out most mechanisms except some form of dictatorship (Pápai [2001], Ehlers and Klaus [2003] and Hatfield [2009]).

Hashimoto [2013] has proposed a generalization of the RSD mechanism that is feasible and strategy-proof and converges to the outcomes of a version of the CEEI mechanism (for bundles) in

[12]In the unit demand case, the PS and RSD mechanisms were shown to be asymptotically equivalent by Che and Kojima [2010]. Liu and Pycia [2013] generalized this result to show that any two mechanisms that satisfy ordinal efficiency, symmetry, and asymptotic strategy-proofness coincide asymptotically. PS and RSD are two examples of mechanisms in this class. Pycia [2011] generalized this result to the case of agents with multi-unit demand, assuming that agents rank lotteries over assignments based on first-order stochastic dominance on *single* objects.

[13]See, for example, Abdulkadiroğlu and Sönmez [1998] and references therein.

[14]In RSD, for example, agents are randomly assigned a priority ordering that determines who gets to choose first. In TTC, agents are randomly assigned a good, which they can then trade with others.

large markets. To obtain feasibility, Hashimoto's mechanism withholds some goods. Specifically, a fraction ϵ of goods that are in "excess" demand are withheld, but ϵ goes to 0 while the number of agents goes to infinity. The rate at which ϵ converges to 0 is sensitive to the type-space and the prior distribution. The drawback of this approach is that the economy may be implausibly large before the proposed allocations are acceptably close to meeting the desired criteria. In our setting, one can also withhold $k-1$ units from each good to ensure feasibility. However, unlike Hashimoto [2013], our "excess" demand is measured by an additive error and does not depend on market size or prior distributions.

A drawback of dictatorship mechanisms is that they may perform poorly in terms of ex-ante efficiency and fairness, especially when agents have multi-unit demands (see, for example, Budish and Cantillon [2012]). This particular deficiency of dictatorship mechanisms naturally increases interest in more ex-ante efficient and equitable mechanisms, regardless of their strategy-proofness. These mechanisms specify probability shares in objects rather than lotteries over feasible outcomes. Examples are PS and CEEI (Bogomolnaia and Moulin [2001], Hylland and Zeckhauser [1979]). Under the unit demand assumption, there is an equivalence between probability shares and lotteries over feasible outcomes. As probability shares are in a sense "easier" to specify, these mechanisms produce outcomes with many more desirable properties than either RSD or TTC. Moreover, several recent papers (Kojima and Manea [2010], Azevedo and Budish [2012]) show that when the economy is large, these mechanisms are essentially strategy-proof.

Generalizations of these mechanisms to multi-unit demand have been proposed (see Kojima [2009] and Budish [2011]). There are significant differences between these mechanisms and ours, and these are described in Section 3.3 and Section 4. Our work is also related to Budish et al. [2013], which considers a bi-hierarchical constraint structure and obtains exact implementation.[15] In the context of course allocation, the bi-hierarchical constraint structure severely restricts the set of bundles that can be allocated to students. Our paper does no more than limit the size of the bundle students can be assigned. In the context of course allocation and similar settings in which agents desire bundles of limited size, such a restriction is natural.

Recently, Akbarpour and Nikzad [2014] generalized Budish et al. [2013] to consider feasibility

[15] In the combinatorial optimization literature, this structure is called an intersecting laminar system.

constraints more complex than simple capacity constraints. They do not consider complementarities in preferences, and their approximation result holds only probabilistically rather than ex-post. Specifically, they bound the probability that a given good is not over-allocated by a certain amount, while our result guarantees that ex post *all* goods are not over-allocated by more than $k-1$.[16]

Our paper also contributes to the literature on approximate competitive equilibria with indivisibilities and non-convex preferences (Starr [1969], Broome [1972], Emmerson [1972], Mas-Colell [1977], and Garratt [1995]). These approximation results bite only when the size of the economy as measured by the number of goods becomes very large—usually too large for applications like the course allocation problem. This is a consequence of the generality of the constraint structure. Our approach, on the other hand, takes advantage of the "limited" complementarity property in the preferences to get a stronger bound that is independent of market size.

6 Conclusion

Attractive mechanisms for the allocation of indivisible goods when there are complementarities in preferences are rare. The positive results known have usually been obtained for continuum markets. This paper introduces two new mechanisms for allocating indivisible goods that are approximately feasible. The novel feature of these mechanisms is that the error bounds do not depend on the size of the market but on the degree of complementarity exhibited by the preferences.

Acknowledgements

We thank the referees, Eric Budish and Roger Myerson, for their many useful suggestions.

[16]For example, in the context of course allocation, the bounds in Akbarpour and Nikzad [2014] are too large to be applicable.

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A Appendix A: Iterative Rounding Algorithm

A.1 Proof of Lemma 2.2

The proof uses an algorithm called the Iterative Rounding Algorithm (IRA). The goal of the (IRA) is to identify an $x \in \arg\max\{c \cdot x : Dx \leq d, x \geq 0\}$ that is integral. Here D is an arbitrary matrix and d an arbitrary vector. The (IRA) begins by choosing an extreme point $x^* \in \arg\max\{c \cdot x : Dx \leq d, x \geq 0\}$. If x^* is integral, we are done. If not, the (IRA) will eliminate one or more constraints and resolve the linear program.

Step 0: Initiate $x^{opt} := x^*$.

Step 1: If x^{opt} is integral, stop and output x^{opt}; otherwise, continue to either Step 2a or 2b.

Step 2a: If any coordinate of x^{opt} is integral, fix the value of those coordinates, and update the linear program.

To describe the updated linear program, let C and $\overline{C}$ be the set of columns of D that correspond to the non-integer and integer valued coordinates of x^{opt}, respectively. Let D_C, and $D_{\overline{C}}$ be the sub-matrix of D that consists of columns in C and the complement $\overline{C}$, respectively. Similarly, for vector x, let x_C and $x_{\overline{C}}$ be the sub-vector of x that consists of all coordinates in C and $\overline{C}$. The updated linear program is:

$$\max\{c_C \cdot x_C : \text{ s.t. } D_C \cdot x_C \leq d - D_{\overline{C}} \cdot x_{\overline{C}}^{opt}\}.$$

In other words, we replace c by c_C; x by x_C; D by D_C and d by $d - D_{\overline{C}} \cdot x_{\overline{C}}^{opt}$ and move to Step 3.

Step 2b If all coordinates of x^{opt} are fractional, delete *certain* rows of D (to be specified later) and the corresponding constraints from the linear program. Update the linear program and move to Step 3.

Step 3 Solve the updated linear program $\max\{c \cdot x \text{ s.t. } Dx \leq d\}$ to get an extreme point solution. Let this be the new x^{opt} and return to Step 1.

Lemma A.1 *Assume that whenever the algorithm passes Step 1 and has not terminated, it can either enter Step 2a or will find at least one row of the current D to delete in Step 2b. Then, the algorithm will terminate in a finite number of steps and output a 0-1 vector. Furthermore, if x^{OUT} is the output, then, $c \cdot x^{OUT} \geq c \cdot x^*$.*

Proof: In Step 2a, we fix at least one coordinate and update the linear program, thus at least one column of the matrix is eliminated. In Step 2b, on the other hand, we delete at least one row. As D is a finite matrix, the algorithm can only execute Step 2a and Step 2b a finite number of times. Therefore, if the assumption in Lemma A.1 holds, then, the algorithm has to terminate.

Observe that after each iteration of the algorithm, we eliminate some constraints and resolve the linear program, thus the objective function cannot decrease. Therefore, $c \cdot x^{OUT} \geq c \cdot x^*$. ■

We apply the (IRA) to the following system:

$$x \geq 0$$

$$x_i(B) = 0 \text{ if } size(B) > k \qquad (Demand)$$

$$\sum_B x_i(B) \leq 1 \ \forall i \in N.$$

$$\sum_{i \in N} \sum_{B \ni j} B_j \cdot x_i(B) \leq s_j \ \forall j \in G. \qquad (Supply)$$

Recall that B_j denotes the number of copies j contained in bundle B.

Define approximate supply constraints:

$$\sum_{i \in N} \sum_{B \ni j} B_j \cdot x_i(B) \leq s_j + k - 1 \ \forall j \in G. \qquad (Supply+k-1)$$

Let P be the polytope defined by (*Demand*) and (*Supply*). The (IRA) takes as input an extreme point, $x^* \in \arg\max\{u \cdot x : x \in P\}$ where $u \geq 0$. It then rounds x^* into a 0-1 vector $\bar{x}$ that satisfies (*Demand*) and (*Supply+k-1*).

Beginning with x^*, execute step 2(a). Remove from (*Demand-Supply*) all variables $x_i(B)$ for which $x_i^*(B) = 0$. In other words, a variable that is zero in x^* will be rounded down to zero and fixed at that value in all subsequent iterations. Remove from (*Demand-Supply*) all variables

$x_i(B)$ for which $x_i^*(B) = 1$ and adjust the right hand sides of (*Supply*) accordingly. In other words, a variable set to 1 (or 0) by x^* is fixed at 1 (or 0) in all subsequent iterations. In the system that remains, pick a non-negative extreme point that optimizes the vector u and repeat. At some iteration, when the remaining supply of good j is s'_j, we may obtain an extreme point, with each variable taking a value strictly less than 1. Call it y. At this point we must execute step 2(b)—that is, identify a constraint to be omited. We show in Lemma A.2, below that in this case, there exists a $j \in G$ such that

$$\sum_{i \in N} \sum_{B \ni j} B_j \cdot \lceil y_i^*(B) \rceil \leq s'_j + k - 1.$$

For each such j, remove the corresponding constraint (*Supply*) and proceed to Step 3. Suppose the (IRA) terminates in a 0-1 vector $\bar{x}$.

There are three observations to be made about $\bar{x}$.

1. At each iteration, inequality (*Demand*) holds. Thus, $\bar{x}$ satisfies (*Demand*).

2. By Lemma A.1, $u \cdot \bar{x} \geq u \cdot x^*$.

3. Because $\bar{x}_i(B) = 1$ only if $x_i^*(B) > 0$, it follows that for the inequalities in (*Demand*) discarded, $\sum_{i \in N} B_j \cdot \sum_{B \ni j} \bar{x}_i(B) \leq s_j + k - 1$.

We now show that the constraints in Step 2(b) to be discarded do indeed exist.

Lemma A.2 *Let $u_i(B)$ be any utility function. Let $\mathcal{S}_0(i), \mathcal{S}_1(i)$ be the set of bundles that $x_i(B)$ for all $B \in \mathcal{S}_0(i)$ have been fixed to be 0, and $x_i(B)$ for all $B \in \mathcal{S}_1(i)$ have been fixed to be 1, respectively.*

Let x^* be an extreme point of the linear program

$$\max\{u \cdot x : x \text{ satisfies } (Demand) \text{ and } (Supply), x_i(B) = 0 \forall B \in \mathcal{S}_0(i); x_i(B) = 1 \forall B \in \mathcal{S}_1(i)\}.$$

Assume that $x_i^*(B) < 1$ for all $i \in N$ and B such that $B \notin \mathcal{S}_0(i) \cup \mathcal{S}_1(i)$. (In other words, $x_i^*(B)$

has not been fixed). Then, there exists a $j \in G$ such that

$$\sum_{i \in N} \sum_{B:j \in B} B_j \lceil x_i^*(B) \rceil \leq s_j + k - 1.$$

To prove Lemma A.2, assume for a contradiction that $0 < x_i^*(B) < 1$ for all i, B and $\sum_{i \in N} \sum_{B:j \in B} B_j \lceil x_i^*(B) \rceil > s_j + k - 1$ for all $j \in G$. Because $\sum_{i \in N} \sum_{B:j \in B} B_j \lceil x_i^*(B) \rceil$ is an integer, we have

$$\sum_{i \in N} \sum_{B:j \in S} B_j \lceil x_i^*(B) \rceil \geq s_j + k \ \forall j \in G. \tag{11}$$

We use (11) to contradict the fact that x^* is an extreme point. Recall that an extreme point of a linear program has the following property:

> The number of non-zero variables in an extreme point x^* is equal to the number of linearly independent and binding constraints in (*Demand*) and (*Supply*).

The contradiction arises from trying to reconcile two things. First, each column of (*Demand*) and (*Supply*) that correspond to a positive component of x^* consists of exactly two non-zero entries. Second, each row corresponding to $j \in G$ intersects at least $s_j + k$ columns that correspond to positive components of x^*. By a counting argument, we show that it is impossible to achieve both without violating the extreme point property of x^*.

Given the extreme point x^*, we credit each non-zero variable $x_i^*(B)$ with a single token. We then redistribute these tokens to the binding, linearly independent constraints in a particular way. We show that if (11) holds, then each binding constraint will get at least one token with one token left over. This proves that the number of non-zero variables $x_i^*(B)$ is larger than the number of binding, linearly independent constraints, which is a contradiction.

We redistribute the tokens as follows. Credit $x_i^*(B)$ fraction of the tokens to the constraint corresponding to agent i (*Demand*). Credit $B_j \frac{1 - x_i^*(B)}{k}$ to each constraint corresponding to each good $j \in B$. Notice that this is feasible because the size of each bundle is $\sum_{j \in G} B_j \leq k$.

If the constraint corresponding to agent i binds, then the number of tokens this constraint is credited with is $\sum_S x_i(B) = 1$. For a binding constraint corresponding to good j, we have:

$$\sum_{i \in N} \sum_{B \ni j} B_j x_i(B) = s_j. \tag{12}$$

The total quantity of tokens that this constraint obtains is

$$\sum_{B, i \in N : x_i^*(B) > 0} B_j \frac{1 - x_i^*(B)}{k} = \frac{1}{k} \sum_{B, i \in N : x_i^*(B) > 0} B_j - \frac{1}{k} \sum_{i \in N} \sum_{B \ni j} B_j x_i(B).$$

From (11) and (12) this number of tokens is at least

$$\frac{1}{k}(s_j + k - s_j) = 1.$$

Thus, any binding constraint j (*Demand*) is credited with at least 1 token.

Hence, we have shown that the amount of tokens given at the beginning (which is the number of non-zero x^* variables) has been redistributed to the binding constraints, so that each is credited with at least 1 token. Thus, the number of non-zero x^* variables is at least the number of binding constraints.

Now, the equality obtains only if for every nonzero $x_i^*(B)$, the size of bundle B is exactly k—that is, $\sum_{j \in G} B_j = k$. Furthermore, the constraint corresponding to agent i as well as all the constraints corresponding to all $j \in B$ binds. However, in this case, one can show that the set of binding constraints is not linearly independent. To see this, consider the sum of all the binding constraints in *Supply*:

$$\sum_{j \in G} \sum_{i, B : B \ni j} B_j x_i^*(B) = \sum_{j \in G} s_j.$$

Because for each $x_i^*(B) > 0$, $\sum_{j \in B} B_j = k$, this sum can be rewritten as

$$k \cdot \sum_{i, B} x_i^*(B) = \sum_j s_j.$$

This last expression is the sum of all the constraints in (*Demand*), contradicting linear independence of the binding constraints. By this we have shown that the number of non-zero variables in an extreme point solution is larger than the number of linearly independent binding constraints. ■

A.2 An Algorithm to Construct a Lottery

Recall that Theorem 2.1 shows that any $x \in Q$ can be expressed as a convex combination of points in E_k. In this section, we show how to (approximately) decompose any $x \in Q$ into a convex combination of points in E_k.

Assume E_k is bounded with diameter D. Denote by $|x - y|$ the Euclidean distance between x and y. Recall that we have a subroutine that will, for any fractional $x \in Q$ and any cost vector c, return an integral $\bar{x} \in E_k$ such that $c\bar{x} \geq cx$.

Given this subroutine, we exhibit a polynomial time algorithm that, for a given point $x \in Q$, finds at most $d + 1$ integral points in E_k, the convex hull of which is arbitrarily close to x. The algorithm also returns a lottery over these $d + 1$ integral vectors, the expectation of which is close to x.

Given a fractional solution $x \in Q$, let

$$B(x, \delta) = \{z : \text{ satisfying } (\textit{Demand}) \text{ and } |z - x| \leq \delta\}.$$

We assume there exists $\delta > 0$ such that $B(x, \delta) \subset Q$. Notice that for our purpose, this assumption is without loss of generality, because otherwise we can always choose x' in the interior of Q close to x.

Given an allowable error $\epsilon > 0$, the algorithm is the following.

Algorithm In each step, maintain a subset B of points in E_k. Each iteration consists of the following steps.

1. Compute $y \in conv(B)$ that is closest to x. If $|y - x| < \epsilon$, the algorithm terminates.

2. Otherwise, because y is the closest point to x in B, y lies in a hyperplane of $conv(B)$. Thus, there exists a subset $B' \subset B$ of size at most d such that $y \in conv(B')$. (Recall d is the dimension).

Consider $z = x + \delta \frac{x-y}{|x-y|}$. Notice, $z \in Q$ because $B(x, \delta) \in Q$. Use the rounding algorithm to find an integral $z' \in E_k$, such that

$$< z, x - y > \leq < z', x - y > .$$

3. Update $B := B' \cup \{z'\}$ and repeat.

To show that the algorithm terminates in polynomial time, we show that after each iteration, the distance $|x - y|$ is reduced by at least a constant factor. To prove this, let y' be the point in the interval $[z', y]$ that is closest to x. We will prove the following.

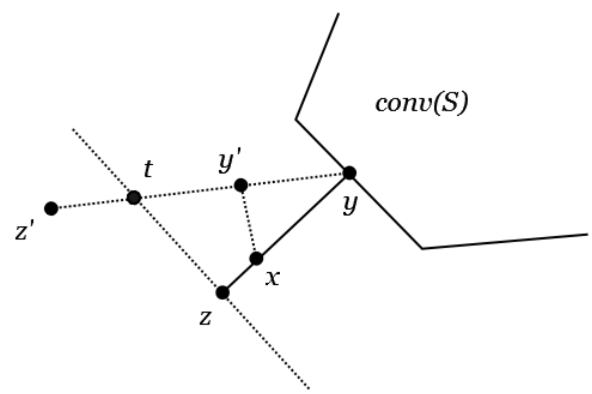

Figure 2:

CLAIM A.3 *There exists* $0 < \gamma < 1$ *that depends on* D *and* δ *such that* $|x - y'| < (1 - \gamma)|x - y|$

Proof: Let t be the point in the interval (z', y) such that $< t - z, x - y >= 0$. Because $< z, x - y >\leq< z', x - y >$, such a t exists. See Figure 2.

Now,

$$\frac{|x-y|^2}{|x-y'|^2} = \frac{|t-y|^2}{|t-z|^2} = \frac{|t-z|^2 + (|x-z| + |x-y|)^2}{|t-z|^2} \geq \frac{|t-z|^2 + \delta^2}{|t-z|^2}$$

We have $|t - z| \leq |z' - z|$. Furthermore, because the diameter of E_k is D, $|z' - z| \leq D$. Thus, $|t - z| \leq D$.

Hence, we obtain

$$\frac{|x-y|^2}{|x-y'|^2} \geq \frac{D^2 + \delta^2}{D^2}.$$

Thus, there exists $0 < \gamma < 1$, depending on D and δ such that

$$|x - y'| < (1 - \gamma)|x - y|,$$

which is what we need to prove. ■

The claim above shows that after each iteration, the distance between x and y is reduced by at least a factor of $(1 - \gamma)$. Consider $K = \frac{\ln(D/\epsilon)}{\gamma}$; we have

$$D(1 - \gamma)^K \leq \epsilon.$$

Thus, after at most K iterations, the algorithm will terminate.

B Equilibrium & Asymptotic Strategy-Proofness

B.1 Equilibrium Existence in B-CEEI

In this section, we show how to use the standard Arrow-Debreu-Mckenzie argument to establish the existence of market-clearing item prices when agents have non-unit demands. As before, assume an agent $i \in N$ who consumes a fraction $x_i(B)$ of bundle B enjoys a utility of $\sum_B u_i(B)x_i(B)$, where $u_i(B)$ is the utility derived from bundle B. To invoke the Arrow-Debreu-McKenzie theorem, we need to express agent i's utility not as a function of bundles but as a concave function of the quantities of the individual goods consumed. If we give agent i the vector $z \in [0, 1]^{|G|}$, where z_j is interpreted to be a fraction of good j, agent i's utility can be represented as

$$U_i(z) = \max \sum_B u_i(B)x_i(B)$$

$$\text{s.t.} \sum_{B \ni j} B_j \cdot x_i(B) \leq z_j \ \forall j \in G$$

$$x_i(B) \geq 0 \ \forall i \in N, \forall \text{ bundle } B$$

Notice, $U_i(z)$ is concave in z.

If agent i has an income of W, the unit price of $j \in G$ is p_j, and agent i's consumption problem is

$$\max U_i(z)$$

$$\text{s.t.} \sum_{j \in G} p_j z_j \leq W$$

$$z_j \geq 0 \ \forall j \in G$$

Let $D_i(p)$ be the set of optimal solutions to the above program. It follows immediately from the usual arguments that there must exist a price vector p^* and $z^i \in D_i(p^*)$ for all $i \in N$ such that $\sum_{i \in N} z_j^i \leq s_j$ for all $j \in G$.

B.2 Proof of Theorem 3.5

Fix a q sufficiently large. In the sequel, we suppress the dependence on q. Recall that the set of types, Θ, is finite. Let θ_i be the type reported by agent i, and denote by n_θ the number of agents reporting type θ. Also let w_{θ_i} be the weight that the mechanism chooses for an agent of type θ_i.

Consider the following program for finding a utilitarian allocation that is envy-free:

$$\max \sum_{i \in N} \sum_{B} w_{\theta_i} \cdot u^{\theta_i}(B) x_i^{\theta_i}(B) \tag{13}$$

$$\sum_{B} x_i^{\theta_i}(B) \leq 1 \ \forall i \in N \tag{14}$$

$$\sum_{i \in N} \sum_{B \ni j} x_i^{\theta_i}(B) \leq s_j \ \forall j \in G \tag{15}$$

$$\sum_{B} u^{\theta_i}(B) x_i^{\theta_i}(B) \geq \sum_{B} u^{\theta_i}(B) x_j^{\theta_j}(B) \ \forall i, j \tag{16}$$

Recall that we can set $x_i^{\theta_i}(B) = 0$ whenever $size(B) > k$. Call (13-16) the disaggregate formulation.

Introduce variables $y^\theta(B)$ to denote the "aggregate" amount of bundle B that all agents reporting θ get. Namely, if we consider an anonymous solution of (13-16), that is $x_i^{\theta_i}(B) = x_j^{\theta_j}(B)$ whenever $\theta_i = \theta_j = \theta$, then $y^\theta(B) = n_\theta x_i^{\theta_i}(B)$. Now consider the following "aggregate" formulation:

$$\max \sum_{\theta \in \Theta} \sum_{B} w_\theta \cdot u^\theta(B) y^\theta(B) \tag{17}$$

$$\sum_{B} \frac{1}{n_\theta} y^\theta(B) \leq 1 \ \forall \theta \in \Theta \tag{18}$$

$$\sum_{t \in \Theta} \sum_{B \ni j} y^\theta(B) \leq s_j \ \forall j \in G \tag{19}$$

$$\frac{1}{n_\theta} \sum_{B} u^\theta(B) y^\theta(B) \geq \frac{1}{n_{\theta'}} \sum_{B} u^\theta(B) y^{\theta'}(B) \ \forall \theta, \theta' \in \Theta. \tag{20}$$

To show that our mechanism is asymptotically strategy-proof, we need to prove that for every $\epsilon > 0$, there exists q_0, such that if $q \geq q_0$, then, no agent can improve his utility by more than ϵ from misreporting. Notice that Assumption 3.1, $\lim_{q \to \infty} \frac{|n_\theta^q|}{q} = n_\theta^* > 0$, $\forall \theta \in \Theta$ implies that the number of agents for each type goes to infinity as q increases. Thus it will be enough for us to show that there exists n_0 large enough such that the number of agents for each type θ is at least n_0, i.e, $n_\theta \geq n_0$ for all $\theta \in \Theta$, then, no agent can improve his utility by more than ϵ from misreporting.

Suppose agent i of type τ pretends to be of type ν. We will show that the impact on the allocations of the other agents from this misreport can be computed by solving (17-20) with a perturbed right-hand side.

If agent i of type τ pretends to be of type ν, then the number of agents reporting τ is decreased by one and the number of agents reporting ν is increased by 1. Let n'_θ be the number of agents reporting type θ in this case, then

$$n'_\tau = n_\tau - 1; n'_\nu = n_\nu + 1; n'_\theta = n_\theta \quad \forall \theta \neq \tau, \nu.$$

Thus, the aggregate program becomes

$$\max \sum_{\theta \in \Theta} \sum_{B} w_\theta \cdot u^\theta(B) y^\theta(B) \tag{21}$$

$$\sum_{B} \frac{1}{n'_\theta} y^\theta(B) \leq 1 \ \forall \theta \in \Theta \tag{22}$$

$$\sum_{\theta \in \Theta} \sum_{B \ni j} y^{\theta}(B) \leq s_j \ \forall j \in G \tag{23}$$

$$\frac{1}{n'_{\theta}} \sum_{B} u^{\theta}(B) y^{\theta}(B) \geq \frac{1}{n'_{\theta'}} \sum_{B} u^{\theta}(B) y^{\theta'}(B) \ \forall \theta, \theta' \in \Theta. \tag{24}$$

Compare program (21)-(24) to program (17)-(20). If both n_τ and n_ν are large enough, then the objective function and the constraints of both programs are close to each other. Thus, as n_τ and n_ν go to infinity, the maximum value of (21)-(24) converges to the maximum value of (17)-(20). Furthermore, because of the assumption that (17)-(20) has an **unique** maximizer, the solution of (21)-(24) will converge to that unique maximizer, otherwise the maximum value of (21)-(24) would not converge to the maximum value of (17)-(20).

Thus, there exists n_0 such that if both n_τ and n_ν are at least n_0 then (21)-(24) also has a unique solution. Furthermore, as n_0 increases, the solution of (21)-(24) converges to the solution of (17)-(20). In other words, if n_0 is large enough, the agent who misreports his or her type can only change the allocation by $O(\epsilon)$. Thus, by the envy-free constraint, the utility changes by at most $O(\epsilon)$. This shows that the mechanism is asymptotically strategy-proof according to Definition 3.4. ∎

C Proof of Theorems 4.1 and 4.3

C.1 Proof of ordinal efficiency

It suffices to prove the result for BPSLC, since BPSLC is a generalization of BPS. If BPSLC is not ordinally efficient, there must exist an expected assignment x such that all agents weakly prefer x to the outcome of the mechanism and at least one agent strictly prefers x. Therefore, for some agent j and bundle B, the following inequality must hold:

$$\sum_{S \succeq_j B, S \subset P_r, 1 \leq r \leq t} x_j(S) > \sum_{S \succeq_j B, S \subset P_r, 1 \leq r \leq t} BPSLC_j(S). \tag{25}$$

Here $BPSLC_j(S)$ is the fraction of bundle S assigned to agent j under $BPSLC$. For each agent i and bundle $S \subset P_r, 1 \leq r \leq t$, let $v(i, B)$ be the last instance at which agent i is allocated bundle

B in the BPSLC mechanism. If i is not allocated bundle B, set $v(i,B) = \min_{S \succeq_i B} v(i,S)$.

Choose a pair j and B such that strict inequality (25) holds and $v(j,B)$ is minimal. By the choice of j and B, for all agents i and bundle $S \succeq_i B$, $x_i(S) = BPSLC_i(S)$. Note that a feasibility constraint in the following form

$$\sum_{i,S\colon v(i,S) \leq v(j,S)} BPSLC_i(S) a(i,S) \leq 1 \quad \forall S \; a(i,S) \geq 0 \;\; \& \;\; a(j,B) > 0, \tag{26}$$

binds; otherwise, BPSLC would allocate agent j bundle B. However, inequality (25) implies that x violates (26), a contradiction.

C.2 Proof of weak strategy-proofness

As before, it suffices to prove the result for BPSLC. Let $\succ_i'$ be a misreport for agent i when his or her true preference ranking is $\succ_i$. Let $BPSLC$ and $BPSLC'$ be the assignments when agent i reports $\succ_i$ and $\succ_i'$, respectively. If weak strategy-proofness is violated, then:

$$\sum_{S \succeq_i B, S \subset P_r, 1 \leq r \leq t} BPSLC_i(S) \leq \sum_{S \succeq_i B, S \subset P_r, 1 \leq r \leq t} BPSLC_i'(S) \tag{27}$$

with strict inequality for some bundle B. Choose B such that $v(i,B)$ is minimized and inequality (27) is strict. The choice of B implies that for all $S \succ_i B$, $BPSLC_i(S) = BPSLC_i'(S)$ and $BPSLC_i(B) > BPSLC_i'(B)$. Note that all agents consume the same bundle in $BPSLC'$ and $BPSLC$ up to time $v(i,B)$, since agent i is assigned the same expected allocation up to that point. However, a feasibility constraint must have stopped the mechanism from allocating bundle B to agent i at time $v(i,B)$. This feasibility constraint is violated by assignment BPSLC$'$ since $BPSLC_i(B) > BPSLC_i'(B)$. This contradiction proves the result.

C.3 Proof of weak envy-freeness of BPSLC

If BPSLC is not weakly envy-free, some agent i will envy another agent, say j. Therefore, for all bundles B, the following inequality holds:

$$\sum_{S \succeq_i B, S \subset P_r, 1 \leq r \leq t} BPSLC_i(S) \leq \sum_{S \succeq_i B, S \subset P_r, 1 \leq r \leq t} BPSLC_j(S) \tag{28}$$

with strict inequality for some bundle B. Choose B such that $v(i, B)$ is minimized and (28) holds strictly. The choice of B implies that for all $S \succ_i B$, $BPSLC_i(S) = BPSLC_j(S)$ and $BPSLC_j(B) > BPSLC_i(B)$. Note that a constraint in the form of (26) binds for agent i. But the same constraint would be violated by agent i, which contradicts feasibility.

C.4 Proof of envy-freeness for BPS

Let $BPS_j(S)$ be the fraction of bundle S assigned to agent j under $BPSLC$. We show for all agents i and j and bundle B the following inequality holds:

$$\sum_{S \succeq_i B, S \subset P_r, 1 \leq r \leq t} BPS_i(S) \geq \sum_{S \succeq_i B, S \subset P_r, 1 \leq r \leq t} BPS_j(S) \tag{29}$$

Note that before time 1, agents stop consuming a bundle only when one of the objects in that bundle becomes unavailable. Therefore, after $v(i, B)$ none of the bundles that were consumed before are available hence, agent j cannot consume those bundles. This proves inequality (29).

D Properties of BPSLC

We generalize the definition of limited complementarities to accommodate the case in which agents are interested in multiple copies of an object in a bundle. Fix a partition P_1, P_2, $P_3, ..., P_r$ of G. The size of an element of this partition may exceed k. Let Ω_l for $1 \leq l \leq r$ be the set of bundles of size at most k, the objects of which are from P_l. Given an expected assignment x, assume $\hat{x}$ is an expected assignment that assigns positive probability to bundles in $\cup_{1 \leq l \leq r} \Omega_l$ and given $B \in \Omega_t$, $\hat{x}_i(B) = \sum_{B: B \subseteq S \ \& \ \nexists \hat{B} \in \Omega_l,\ B \subset \hat{B} \subseteq S} x_i(B)$. With this new definition, one can view the limited complementarities preference assumption as a generalization of the k-demand preference assumption.

Under k-demand, there is only one partition consisting of all goods, G. The BPSLC mechanism under the limited complementarities preference assigns each agent, at each instant of time, his or her best available bundle from $\cup_{1\leq l\leq r}\Omega_l$. Each agent consumes bundles from a partition for at most 1 unit of time. Therefore, the output of this mechanism is an expected assignment that assigns positive probability to bundles in $\cup_{1\leq l\leq r}\Omega_l$. Moreover, the result satisfies (8) and (*Supply*).

The proof is adapted from Che and Kojima [2010]. First, we define a *continuum* economy, which is the natural candidate for what the limit of a q-economy might be as $q \to \infty$. For each object $j \in G$, there is a mass s_j^* of this object. The set of agents, N^*, is an interval of real numbers partitioned into d intervals $(N_\theta^*)_{1\leq\theta\leq d}$. Each point in N^* corresponds to an agent, and each point in N_θ^* is an agent with type θ. For each $1 \leq \theta \leq d$, the length of N_θ^* is n_θ^*. Agents have limited complementarities preferences. Let $\mathbf{B}^* = \cup_{1\leq l\leq r}\Omega_r$. For each $\theta \in \Theta$, agents in N_θ^* have the same preference ranking of bundles in $\mathbf{B}^*$ as do agents with type θ in a $q-$ economy.

An allocation in the continuum economy is a function $x : N^* \times \mathbf{B}^* \to \mathbb{R}^+$. The allocation satisfies (8) and (*Supply*) if the following two conditions are met:

$$\forall j \in G \int_{N^*} \sum_{B\in\mathbf{B}^*} x_i(B)B_j di \leq s_j^*,$$

$$\forall i \in N^* \sum_{B\in\mathbf{B}^*} x_i(B) \leq d_i.$$

Given a subset of objects $G' \subseteq G$, let $\Omega_l(G')$ be the set of bundles of size at most k with objects from the set $G' \cap P_l$. The extension of the BPSLC mechanism to the continuum economy with limited complementarities is called $BPSLC^*$ and is defined as follows: for step $v = 0$, let $G_\theta^*(0) = G, t_\theta^*(0) = 0\ \forall 1 \leq \theta \leq d$, and $t^*(0) = z^v = 0$. Given $G^*(v-1), t^*(v-1), z^{v-1}$, let $m^\theta(v)$ denote type θ agents' most preferred bundle from $\cup_{1\leq l\leq r}\Omega_l(G_\theta^*(v))$. Let $m_j(v) = \sum_{\theta\in\Theta} |m^\theta(v)|_j |N_\theta^*|$ be the mass of object j in agents' most preferred bundles. The new variable $t_{\theta,l}(v)$ denotes the total time that agents with type θ have consumed bundles from partition l; set $t_{\theta,l}(v) = 0$. The steps of the mechanism are as follows:

1. For all $1 \leq l \leq r$, $j \in P_l$ and $\theta \in \Theta$,

$$\tau^*_{j,\theta}(v) = \begin{cases} 1 & \text{if } m_j(v-1) = 0 \\ sup\{t | z_j^{v-1} + m^\theta(v-1)(t - t(v-1)) \leq s_j^*, t + t_{\theta,l}(v-1) \leq 1\} & \text{otherwise.} \end{cases}$$

2. The first instance at which any good is unavailable to some agents because it is exhausted or the bundles from an element of the partition were consumed for 1 unit of time is $t^*(v) = t^*(v-1) + \tau^*(v)$ where $\tau^*(v) = \min_{j,\theta} \tau_j^*(v)$.

3. The total amount of time that agents of type θ have consumed bundles from element l is $t_{\theta,l}(v) = t_{\theta,l}(v-1) + \tau^*(v) I_{\{m^\theta(v) \in \Omega_l(G^*(v))\}}$.

4. At time $t^*(v)$, the set of goods that are unavailable to agents with type θ consists of objects that are exhausted and also objects in elements of the partition that agents with type θ have consumed in their bundles for 1 unit of time. Therefore, the set of available objects for agents with type θ is $G^*_\theta(v) = G^*_\theta(v-1) \setminus \{j \in G | \tau^*_{\theta,j}(v) = \tau^*(v)\}$.

5. $r_j^v = r_j^{v-1} + m_j(v-1)(t^*(v) - t^*(v-1))$

6. The allocation returned by the BPSLC* mechanism at time $t(v)$ is

$$BPSLC^*(v) = BPSLC^*(v-1) + (t^*(v) - t^*(v-1)) z^{v-1}.$$

7. BPSLC terminates at time $t^*(v)$ where v is the smallest index such that $t^*_\theta(v) = 1$ for all θ satisfying $1 \leq \theta \leq d$ and outputs $BPSLC^*(v)$ as the final allocation.

BPSLC in the $q-$economy, denoted by BPSLC^q, is defined similarly by the following steps:

1. For all $1 \leq l \leq r$, $j \in P_l$ and $\theta \in \Theta$,

$$\tau^q_{j,\theta}(v) = \begin{cases} 1 & \text{if } m_j(v-1) = 0 \\ sup\{t | r_j^{v-1}(q) + \frac{m_j^q(v-1)}{q}(t - t^q(v-1)) \leq \frac{s_j}{q}, t + t^q_{\theta,l}(v-1) \leq 1\} & \text{otherwise.} \end{cases}$$

2. The first instance at which any good is exhausted is $t^q(v) = t^q(v-1) + \tau^q(v)$ where $\tau^q(v) = \min_{j,\theta} \tau^q_{j,\theta}(v)$.

3. The total amount of time that agents of type θ have consumed bundles from partition l is $t^q_{\theta,l}(v) = t^q_{\theta,l}(v-1) + \tau^q(v) I_{\{m_q^\theta(v) \in \Omega_l(G^q(v))\}}$.

4. At time $t^*(v)$, the set of goods that are unavailable to agents with type θ consists of objects that are exhausted and also objects in elements of the partition that agents with type θ have consumed their bundles for 1 unit of time. Therefore, the set of available objects for agents with type θ is:

$$G^q_\theta(v) = G^q_\theta(v-1) \setminus \{j \in G | \tau^q_{\theta,j}(v) = \tau^q(v)\}.$$

5. $r^v_j(q) = r^{v-1}_j(q) + \frac{m^q_j(v-1)}{q}(t^q(v) - t^q(v-1))$.

6. $BPSLC^q(v) = BPSLC^q(v-1) + (t^q(v) - t^q(v-1))z^{v-1}(q)$

7. BPSLC terminates at time $t^q(v)$ where v is the smallest index such that $t^q_\theta(v) = 1\ \forall \theta \in \Theta$ and outputs $BPSLC^q(v)$ as the final allocation.

D.1 Convergence of $BPSLC^q$ to $BPSLC^*$

We first prove the result with the following assumption:

$$n^q_\theta = 0 \text{ for all large enough } q \text{ and } \theta \text{ such that } n^*_\theta = 0 \tag{30}$$

As the supply of multiple objects can become unavailable to some agents at the same time, the set $\{t^*(1), t^*(2), t^*(3), ..., t^*(v^*)\}$ will contain some duplicates. Let the distinct values in the set $\{t^*(1), t^*(2), t^*(3), ..., t^*(v^*)\}$ be $\{t_1, t_2, ..., t_g\}$ and ordered so that $t_1 < t_2 < t_3 < ... < t_g$. Set $t^*(v_i) = t_i$. Note g may not be equal to v^*. Let $A^h_\theta \subseteq G$ be the set of objects that became unavailable to agents with type θ at time t_h in the $BPSLC^*$ mechanism. Let $s^q_j(t)$ and $s^*_j(t)$ be the supply of object j at time t in $BPSLC^q$ and $BPSLC^*$, respectively. Also, assume $BPSLC^*(t)$ and $BPSLC^q(t)$ are the allocations of the $BPSLC$ algorithm at time t in the continuum economy and the $q-$economy, respectively. Denote by $\theta^q_l(t)$ and $\theta^*_l(t)$ the times when agents with type θ have consumed bundles from partition l at time t. We prove the following by induction on h:

1. For all $1 \le h \le g$ and all $j \in G$, $\lim_{q\to\infty} \frac{s^q_j(t_h)}{q} = s^*_j(t_h)$.

2. For all $1 \le h \le g$, $1 \le l \le r$ and $1 \le \theta \le d$, $\lim_{q\to\infty} \theta^q_l(t_h) = \theta^*_l(t_h)$.

3. For all $1 \le h \le g$, $\lim_{q\to\infty} BPSLC^q(t_h) = BPSLC^*(t_h)$.

For $h = 1$, note that at the beginning, since all objects are available, all agents are allocated their most preferred bundle. Therefore, $t_1 = \max\{\min_{j \in G} \frac{s_j^*(0)}{m_j(G)}, 1\}$. Note that for all $j \in G$, the supply of object j at time t_1 in $BPSLC^q$ is $\max\{s_j^q(0) - t_1 m_j^q(G), 0\}$. Hence,

$$\lim_{q\to\infty} \frac{s_j^q(0) - t_1 m_j^q(G)}{q} = s_j^*(0) - t_1 m_j(G) = s_j^*(t_1) = \lim_{q\to\infty} \frac{\max\{s_j^q(0) - t_1 m_j^q(G), 0\}}{q}.$$

This proves the base case of the induction.

Assume statements 1, 2, and 3 are true for $h-1$; we prove them for h. For all $\epsilon > 0$, we show there exists a large enough $Q > 0$ such that for all $q > Q$, $1 \le \theta \le d$ and $1 \le l \le r$:

$$|\frac{s_j^q(t_h)}{q} - s_j^*(t_h)| < \epsilon \text{ and } |\theta_l^*(t_h) - \theta_l^q(t_h)| < \epsilon.$$

Given ϵ_1, let Q_1 be such that

$$|\frac{s_j^q(t_{h-1})}{q} - s_j^*(t_{h-1})| < \epsilon_1 \text{ and } |\theta_l^*(t_{h-1}) - \theta_l^q(t_{h-1})| < \epsilon_1 \tag{31}$$

for all $q > Q_1$, $j \in G$, $1 \le \theta \le d$ and $1 \le l \le r$. If object j became unavailable to agents with type θ at time t_τ for some $\tau < h$, then under the $BPSLC^q$ mechanism, either these agents can only consume bundles that include object j for ϵ_1 period of time after t_{h-1} or the availability of object j at time t_{h-1} is at most $\epsilon_1 q$. If the latter is the case, then this object will be allocated for at most $\frac{\epsilon_1 q}{\vartheta^q}$ period of time, where $\vartheta^q = \min_{\theta\in\Theta} n_\theta^q$. Note $\vartheta^q = O(q)$. Let $\vartheta^* = \lim_{q\to\infty} \frac{q}{\vartheta^q}$. Let Q_2 be such that $\frac{q\epsilon_1}{\vartheta^q} \le 2\vartheta^*\epsilon_1 = \epsilon_2$ for all $q > Q_2$. Choose ϵ_1 such that $\epsilon_1 + \epsilon_2 = \epsilon_1(2\vartheta^* + 1) < t_h - t_{h-1}$. For all $q > \max\{Q_1, Q_2\}$, the $BPSLC$ mechanism in the $q-$economy at $t_{h-1} + \epsilon_1 + \epsilon_2$ would allocate the same bundles as $BPSLC^*$. This allocation would end when a good becomes unavailable to some agents; call this date τ_h^q. This date is within the $\epsilon_1 + \epsilon_2$ neighborhood of

$$\hat{\tau}_h^q = \min\{\min_{a\in G^*(v_{h-1})} \frac{s_a^q(t_{h-1})}{m_a^q(G^*(v_{h-1}))} + t_{h-1}, \min\{t|\theta_l^q(t_{h-1}) + t - t_{h-1} \ge 1,\ m^\theta(G^*(v)) \in \Omega^l(G^*(v))\}\},$$

where $m^\theta(A)$ is type θ's most preferred bundle from the set A. Note that the following holds:

i) $\min\{t|\theta_l(t_{h-1}) + t - t_{h-1} \ge 1\} \ge t_h$ by definition and $\lim_{q\to\infty} \theta_l^q(t_{h-1}) = \theta_l(t_{h-1})$.

ii) Also note that $\lim_{q\to\infty} \frac{s_j^q(t_{h-1})}{m_a^q(G^*(v_{h-1}))} = \frac{s_a^*(t_{h-1})}{m_j^*(G^*(v_{h-1}))}$.

iii) $t_h =$

$\min\{\min_{a\in G^*(v_{h-1})} \frac{s_a^*(t_{h-1})}{m_a^*(G^*(v_{h-1}))} + t_{h-1}, \min\{t|\theta_l^*(t_{h-1}) + t - t_{h-1} \geq 1,\ m^\theta(G^*(v)) \in \Omega^l(G^*(v))\}\}$. Hence, for $\epsilon_3 > 0$ there exists Q_3 such that for all $q > Q_3$, $|\hat{\tau}_h^q - t_h| < \epsilon_3$. Therefore, if $q > max\{Q_1, Q_2, Q_3\}$, then $|\tau_h^q - t_h| < \epsilon_1 + \epsilon_2 + \epsilon_3$. Note that if $q > max\{Q_1, Q_2, Q_3\}$, then $BPSLC^q$ and $BPSLC^*$ allocate the same bundles in the interval of $[t_{h-1}, t_h]$ except for a period with length at most $\epsilon_1 + \epsilon_2 + \epsilon_1 + \epsilon_2 + \epsilon_3$. Hence,

$$|\frac{s_j^q(t_h)}{q} - s_j^*(t_h)| \leq \frac{k(\epsilon_1 + \epsilon_2 + \epsilon_1 + \epsilon_2 + \epsilon_3)|N^q|}{q}. \tag{32}$$

Since $\lim_{q\to\infty} \frac{|N^q|}{q}$ exists, choosing ϵ_is small enough for this case proves the inductive steps.

In the case that (30) does not hold, the proof follows from the fact that if there is a small measure $o(q)$ of agents added to the economy, since (as the proof shows) the stock of objects in the $q-$economy at all times is either $O(q)$ or $o(q)$, the date at which objects are exhausted stays the same in the limit.