The Value of Sharing Intermittent Spectrum*

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Abstract

We consider a model of Cournot competition with congestion motivated by recent initiatives in wireless spectrum policy to allow commercial use of spectrum assigned to government agencies. A key feature of such spectrum is that it may be intermittently available due to the incumbent government user's activity. In our model, wireless Service Providers (SPs) compete for a common pool of customers using their own proprietary (exclusively licensed) bands of spectrum along with access to an additional intermittent band. When the intermittent band is unavailable, any traffic carried on that band must be shifted to the proprietary bands. Customers are sensitive to both the price they pay and the average congestion they experience across the bands of spectrum from which they receive service. We further consider two different access policies for this intermittent band: one in which it is shared by all SPs and one in which it is licensed to a single SP. We also allow the band to be divided into multiple sub-bands where each sub-band is either shared or licensed. We show that the choice of access policy and assignment of sub-bands to providers that maximize social welfare is subtle and may depend on factors such as the number of SPs in the market and the amount of proprietary bandwidth owned by each SP. Further, these decisions may be counter-intuitive. For example, in some cases, licensing the intermittent spectrum to the SP that values it the most leads to less welfare than assigning it to another SP.

1 Introduction

The evolution of wireless networks for mobile broadband access has led to a proliferation of applications and services that have greatly increased the demand for electromagnetic spectrum. In response, regulatory agencies, such as the Federal Communications Commission (FCC) in the U.S., have introduced new initiatives for increasing the amount of spectrum that can be used to meet this demand. One approach in particular that has received much attention is to allow commercial wireless services to share spectrum assigned to government agencies. The motivations for this, as highlighted in the 2012 report by the President's Council of Advisors on Science and Technology (PCAST) [21], is the recognition that relocating the associated services (e.g., satellite) to other bands would be expensive and incur large delays, and that much of the federal spectrum is used sporadically and often only in isolated geographic regions. The PCAST report also put forward a framework for sharing this spectrum in which commercial usage would only be allowed at a given location when the incumbent federal users are not using the spectrum.[1] This framework has recently been used by the FCC as a basis for the new Citizens Broadband Radio Services (CBRS) band from 3550 to 3700 MHz [8]. Sharing spectrum using this or related approaches is also being considered to make several other bands of federal spectrum available for commercial use [9].

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[1]Access to this spectrum is to be controlled by a distributed database system.

A key feature of the PCAST framework is that due to the activity of the federal incumbents, the shared spectrum may be intermittently available for use by commercial wireless Service Providers (SPs). The main objective of this paper is provide insight into the potential impact of adding such an intermittently available resource into the market for wireless services. In the wireless market today, customers typically sign-up for service over time periods of one-month or longer with an expectation that the service will have fairly high availability over that period. Assuming that this expectation persists, then the only way a SP could utilize intermittently available spectrum would be to also have access to other spectrum bands that it can use to serve its customers when the intermittent band is not available. This makes the value of intermittent spectrum unclear. For example, if a SP uses the intermittent spectrum to expand the number of customers it serves, then during periods of unavailability, its other spectrum will see an increase in congestion making the service at those times less desirable.

The impact of adding intermittent spectrum into the market will in turn depend on how the spectrum is allocated. There are several dimensions to this. First, there are different options for how commercial usage can be licensed. One option is to give an SP a license that excludes other SPs from using the shared spectrum in given location when it is available, an approach we refer to as *shared licensed access* to emphasize that the spectrum is still shared with the federal incumbent. Alternatively, the shared band could be open for use by any SP when available, an approach we refer to as *shared open access*. Open access allows for more SPs to enter the market, which may benefit consumers, but also may lead to the band becoming more congested, decreasing its value. A second dimension is that the band can be partitioned into multiple sub-bands, where each sub-band is either licensed (to different SPs) or made available for open access. When licensed to different SPs, more partitions may lead to greater competition, but as each partition has less bandwidth it is more congestible. A final dimension is to which SP to assign a license. Again, there is a trade-off between increasing competition by assigning licenses to attempt to equalize the spectrum holding of different SPs versus the efficiency gains of a assigning licenses to larger SPs that may be better able to use the intermittent resource. Most of these issues arise in the CBRS policy. The rules adopted for this service allow for two tiers of commercial users, one tier, referred to as Priority Access (PA) users, correspond to our notion of shared licensed access, and a second tier referred to as generalized authorized access (GAA) users corresponds to our notion of shared open access. Further, this band is divided into 15 sub-bands (10 Mhz each); some bands can be for GAA users and some for PA users.

In this paper, we present a model to study the aforementioned issues. In our model, multiple SPs compete to provide service to a common pool of customers, modeled as non-atomic users. Each SP has its own *proprietary* band of spectrum for exclusive use, and in addition, there is a new intermittent band that can be allocated as either shared licensed access or shared open access and further can be partitioned into multiple sub-bands. Each SP then serves customers using a combination of its proprietary band and any portion of the intermittent band that it is licensed to use or which is open access. Each customer served on a given band of spectrum is assumed to experience a congestion or latency cost that is given by the ratio of the mass of customers served on that band and the bandwidth. Intermittency is modeled by assuming that the band is available with a given probability α.[2] As is common in the literature on competition with congestible resources, we assume that customers receive a dis-utility based on their *delivered price*, given by the sum of the service price charged by a

[2]In this paper we assume that the entire shared band is either available or not available, i.e., the federal incumbent users utilizes the entire band when active. Alternatively, one could consider models in which the incumbent may utilize different sub-bands with different probabilities. We leave such generalizations to future work.

SP and the congestion cost they face, where here the congestion cost is averaged over time to account for intermittency.

We adopt a model of Cournot or quantity competition in which each SP specifies the mass of customers it will serve on each band of available spectrum. Each SP seeks to maximize its profit given by the product of the customer mass it serves and its service price. The service prices in turn are determined using a given downward sloping customer inverse demand, which specifies the market-clearing delivered price as a function of the aggregate mass of customers served. This differs from much of the work on competition with congestible resources (e.g.[2]) which instead considers price (Bertrand) competition. As pointed out in [22, 7], Cournot models may be more appropriate for modeling the long-run behavior of competition among firms in the wireless industry. This is supported to some degree by [14] which showed that Cournot models can approximate a multi-stage model in which firms first specify capacity and then compete on price (though in our model there is no hard capacity limit but rather a soft limit due to the congestion cost).

Comparing the equilibria associated with licensed and open access sharing, we find that which method should be used depends on the structure of the market and the welfare objectives. If one's objective is consumer welfare and all the SPs are symmetric, then our results show a clear choice: open access sharing. Intuitively, with open access, SP's do not fully internalize the congestion externality in the shared band, leading them to serve more customers. Due to intermittency, this may be accompanied with a decrease in the number of customers they serve in their proprietary bands. For an arbitrary number of symmetric SPs we show that the net result is always an increase in the number of customers served, which translates into more consumer surplus.

However, if the objective is social welfare (consumer surplus plus SP revenue), the conclusions are more subtle. The increased number of users served under open access, leads to more congestion, which also limits the prices the SPs can charge, potentially reducing their revenue. For an example of two symmetric SPs, we show that open access again can result in more social welfare. However, as competition increases this conclusion changes. We find that if the market consists of a large number of SPs, no one being dominant in terms of the amount of their proprietary bandwidth, then the licensed regime generates more social welfare compared to open access. This suggests that for a market with many nearly symmetric SPs, the choice between open access and licensed access can be viewed as a choice between optimizing consumer surplus or social welfare. A "mixed" approach in which part of the shared band is open access and the rest is licensed offers one way of trading off these two objectives. As the fraction of licensed bandwidth increases, the SPs shift traffic from the open access band to their licensed bands, raising prices for the licensed bands. As licensed sharing is initially introduced, relative to full open access for the entire shared band, the *average* price (across proprietary and shared bands) *and* average latency initially *increase*. However, once a critical threshold on the amount of licensed bandwidth is exceeded, prices continue to increase but latency drops. The observed net effect is that consumer surplus decreases while social welfare increases.

The licensed regime generates greater social welfare provided there is sufficient competition. What if this is not the case? We model this possibility as a market that contains one or more SPs characterized by a relatively large amount of proprietary bandwidth compared to the other SPs in the market.. This leads to a trade-off: the larger SPs are better able to handle the intermittency associated with the shared band, but allocating more licensed bandwidth to the larger SPs places the smaller SPs at a disadvantage, compromising the benefits of competition. For two SPs (one large and one small) consumer surplus will benefit more from allocating a small amount of shared spectrum to the smaller SP compared to the allocating it to the larger one. However, as the amount of shared spectrum to be

allocated increases, this conclusion becomes more subtle. In particular, as the spectrum availability becomes smaller, allocating it to the larger SP gives more surplus. As the number of small SPs in the market increases, this also makes allocating the spectrum to the larger SP more attractive. Interestingly, if the large SP has sufficient proprietary bandwidth, allocating the shared band as open access achieves the same outcome as licensing it to the smaller SPs, namely, the large SP does not make use of the open access spectrum due to the congestion from the other SPs.

If the shared bandwidth is to be licensed, how should it be allocated among the SPs? Auctions are the standard response. We find that the natural auction rule for allocating licensed bandwidth, giving it all to the highest bidder, will be inefficient. The problem is that the marginal benefit of additional bandwidth increases with the amount of proprietary bandwidth that each bidder possesses prior to auction. This comes from the fact that such a bidder is better able to absorb the variability associated with intermittently available bandwidth. Thus, bidders endowed with a larger amount of initial bandwidth are willing to pay more for additional bandwidth. This produces a lop sided distribution of bandwidth which reduces consumer *and* social welfare. However, if the shared band is open access, a large SP may not use it, leaving it for smaller SPs. In fact, we give conditions under which given a choice between open access and licensed access allocated by auction, perhaps surprisingly, the bidders would strictly prefer open access.

We also consider the trade-off between the reliability of the shared band and the amount of shared bandwidth. Holding the expected quantity of shared bandwidth fixed (i.e., αW where W is the shared bandwidth), SPs would prefer a smaller amount of shared bandwidth with greater availability. The variation in how the shared band is valued can vary substantially with α, depending on the relative amounts of proprietary bandwidth. Less variability is also shown to lead to larger consumer surplus. For a regulator, this shows that simply looking at the average amount of spectrum that will be available is not the correct metric. In particular, it may be desirable for the government to "repack" the spectrum to create a smaller amount of average bandwidth with less variability.

1.1 Related work

This paper fits within a stream of work studying competition in industries where firms utilize congestible resources and customer utility depends on a delivered price given by a linear combination of a service price and a congestion cost (also sometimes called a total price). This work has been motivated by a variety of different industries. In transportation settings, work such as [6] and [27] consider competition among different toll roads. As another example, [15] and [5] consider competition in service industries in which congestion results in queueing of customers; see also [4] for a related model in which firm's may additionally commit to a given level of service. Telecommunications provides another motivation (e.g. competition between Internet service providers) in works such as [10, 2, 3]. Some key differences of our model from all of this work is that in these papers each resource is owned by only one firm (i.e., there is no sharing), competition is based on price instead of quantity, and the underlying resources are not intermittently available. There have been a number of papers which have individually relaxed one of these assumptions, which we discuss next.

Prior work that has considered resource sharing among multiple firms includes [17, 19, 18, 29]. The sharing in all of these papers is motived by unlicensed access to wireless spectrum, similar to the open access spectrum model considered here. Unlike our model, these papers focus on price competition and non-intermittent spectrum. Our results show that changing these two assumptions leads to significantly different outcomes. In particular, the model in [18] is closest to ours and will be used to highlight these differences. For example, in that paper, the price of open access spectrum

is always competed to zero, while with quantity competition a non-zero price is maintained. Other forms of sharing spectrum based on priority access have been studied in [16]. Here, we do not consider such possibilities.

Cournot competition with congestible resources was considered in [22]. In that work there was no intermittency or resource sharing. However, the authors do consider a utility model that allows for spillover effects in which a user's utility for a given firm's resource might depend on the consumption of another firms resource. We do not model such dependencies here. Further, a main focus of [22] was on the welfare loss due to Cournot competition, a topic that has also recently been studied, for example, in [12] and [13] without the presence of congestion. Here, instead our focus is on comparing the welfare between different ways of allocating the shared spectrum.

We are not aware of other papers that have directly modeled competition with intermittent congestible resources as we have here. Perhaps, closest to to this are papers such as [28] which have studied Cournot competition in the face of production uncertainty. In this work, the quantity a firm can produce is limited by a value which is randomly determined. If the chosen quantity exceeds this constraint, firms incur a penalty, as in the classic newsvendor framework [23]. In our case, when the intermittent spectrum is not available, this also limits the amount of customers that can be served in that band, however, here, the "penalty" for exceeding this limit is endogenous due to the offloading of the traffic onto other spectrum. Another line of work with ties to our model of intermittency is work such as [20] which studies traffic equilibria with stochastic link delays. This work follows in the spirit of the "selfish routing" literature (e.g. [25]) and considers models in which there are no prices on the links.

In addition to studying the price or quantity competition among firms with congestible resources, there have also been a number of papers that consider the firms' investment in capacity including [11] and [1]. Generally, this line of work considers models in which each resource is owned by one firm which decides how much to invest. Models with resource sharing have been considered in [30]. In this paper, we assume that any investment is sunk and focus only on the competition among the SPs given this investment. In addition, here, we are not concerned with the incentives needed for the incumbent federal agencies to make more efficient use of their spectrum, which could include sharing, but rather assume that a given amount of spectrum is available for sharing.[3]

The remainder of this paper is organized as follows. The next section presents the Cournot model with congestion. In Section 3 we analyze this model for the case of two competing SPs. Section 4 examines the opposite case with many SPs, each with a proportionately small share of the available proprietary bandwidth. This models the scenario in which there are low barriers to entry so that many operators may wish to set up competing networks within a local area. In Section 5, we consider scenario in which their is asymmetry among the SPs in terms of the amount of proprietary bandwidth. Section 6 presents extensions to concave decreasing demand and convex increasing latencies. Section 7 concludes and proofs of the main results are given in the appendices.

2 The Model

Suppose N SPs compete to offer wireless service to a common pool of customers. Each SP $i \in \{1, 2, \ldots N\}$ possesses an amount of proprietary (licensed) bandwidth, denoted B_i. In the status quo, this is the only resource the SPs can access. We are interested in the scenario where an amount W of

[3]For example, an incentive for providing shared access, proposed in the PCAST report, might be a 'scrip' system that rewards more efficient use of spectrum. Another approach, discussed in [26], is to assign overlay rights as an alternative to sharing.

new spectrum is made available that is to be shared with an incumbent user. When the incumbent is actively using the band, it is unavailable to carry traffic for the SPs. Otherwise, it is available for the SPs. We consider two different policies that govern the way a particular SP can access this band: *shared licensed access*, where a part of the band W_i is designated for exclusive use by SP i, and *shared open access* where all SPs can access the band. We will allow the shared band to be divided into several disjoint sub-bands, where each sub-band can be designated as either licensed to a particular SP, or as open access. To simplify the model description, we first assume that the entire shared band is either licensed to a single SP, or is open access. We subsequently consider the scenario in which parts of the shared band are allocated to different SPs for licensed and open access.

We assume a pool of infinitesimal customers or users with a downward slopping inverse demand curve

$$P(y) = 1 - y, \tag{1}$$

which gives the marginal utility obtained by the yth customer served, where all customers require the same amount of (average) service. As in [18], the price the yth customer is willing to pay for service is given by the difference between their marginal utility and the latency or congestion cost they experience. Following [18], we refer to the sum of the latency cost and the service price as the *delivered price*.

If SP i serves x_i customers on its proprietary band, the resulting latency cost is given by

$$\ell_i(x_i) = \frac{x_i}{B_i}, \tag{2}$$

which is increasing in the amount of traffic served and decreasing in the amount of bandwidth available to the SP. When each SP i serves w_i customers using the entire band of secondary spectrum, we model the latency by

$$\ell_w\left(\sum_i w_i\right) = \frac{1}{W}\sum_i w_i, \tag{3}$$

which is now increasing in the sum of the traffic from the SPs and decreasing in the available secondary spectrum W. Note, if the entire secondary band is licensed to a single SP i, this corresponds to constraining $w_j = 0$, $j \neq i$. We assume that the shared band is intermittently available with probability $\alpha \in [0, 1]$. When unavailable, the traffic designated by SP i for the secondary band, w_i, must be off-loaded onto SP i's proprietary band.[4] Thus, the expected latency of traffic served by SP i on its proprietary band is

$$\bar{\ell}_i = (1-\alpha)\cdot \ell_i(x_i + w_i) + \alpha \cdot \ell_i(x_i). \tag{4}$$

The expected latency of traffic experienced by SP i's traffic on the secondary spectrum will be

$$\bar{\ell}_w = \alpha \cdot \ell_w\left(\sum_i w_i\right) + (1-\alpha)\cdot \ell_i(x_i + w_i). \tag{5}$$

Proprietary spectrum is assumed to be available at all times.[5]

The SPs compete according to a Cournot model. Each SP i decides on a pair (x_i, w_i), which represents the amount of traffic it will carry. Given a choice of (x_i, w_i) by each SP i, the resulting

[4] This is a reasonable assumption when the customers have a high dis-utility for not receiving service.

[5] As our results will show, our model can also be applied to the case where some portion of the proprietary spectrum is intermittently available, provided that the remainder is always available.

price paid by the users will be the difference between their marginal utility and the resulting expected latency. Specifically, the delivered price for the user load is

$$p_d = P\left(\sum_i (x_i + w_i)\right) = 1 - \sum_i (x_i + w_i). \tag{6}$$

The actual price paid for service depends on the latency experienced by the traffic. For SP i's licensed band, the price is given by

$$p_i = p_d - (1-\alpha)\cdot \ell_i(x_i + w_i) - \alpha \cdot \ell_i(x_i), \tag{7}$$

and for the secondary band, the price paid by SP i's users is given by

$$p_i^w = p_d - \alpha \cdot \ell_w\left(\sum_i w_i\right) - (1-\alpha)\cdot \ell_i(x_i + w_i). \tag{8}$$

Each SP i seeks to maximize its revenue given by

$$R_i = p_i x_i + p_i^w w_i. \tag{9}$$

The model just described assumes that each SP serves two classes of customers: one using their proprietary band and the other with the shared spectrum, charging each class different prices. However, one can also interpret the model as one where there is only one class of customers and the SP decides whether to serve each customer via the proprietary or secondary bands. Formally, we think of $\frac{x_i}{x_i+w_i}$ and $\frac{w_i}{x_i+w_i}$ as the probability that a consumer is served via proprietary spectrum or secondary spectrum, respectively. The price that SP i charges is then

$$\bar{p}_i = \frac{x_i}{x_i + w_i} p_i + \frac{w_i}{x_i + w_i} p_i^w. \tag{10}$$

The revenue of SP i is still given by (9).

2.1 Shared Sub-bands

In the preceding model the entire band of shared spectrum is either licensed to one SP, or is open access. More generally, we allow this band to be divided into multiple disjoint sub-bands with bandwidths $W_k \geq 0$, $k = 0, 1, \ldots, N$, with

$$\sum_{i=0}^{N} W_k = W.$$

Here, W_i represents the part of the shared band allocated to user i as licensed bandwidth and W_0 represents any remaining bandwidth that is allocated for open access (where any of these terms may be zero if no bandwidth is allocated in that way). The resulting traffic load for a sub-band with W_i units of bandwidth is then given by y/W_i where y is the total traffic in that sub-band. We assume that when the incumbent is active, it claims the *entire* band so that all sub-bands must be vacated.

Following the preceding model, a SP would then specify an amount of traffic for each shared sub-band it is permitted to use, as well as for its proprietary band. However, the next result shows that we can 'pool' all of the licensed bands assigned to an SP and represent them as a single equivalent band that serves the aggregate traffic on these sub-bands. Formally, an SP with proprietary bandwidth B_i

and licensed shared bandwidth W_i can be viewed as having a single band having bandwidth $B_i + W_i$ with probability α and bandwidth B_i with probability $1 - \alpha$.

Lemma 2.1 *Suppose SP i has access to B_i units of proprietary spectrum, W_i units of licensed shared spectrum and W_0 units of open access spectrum; let x_i, $w_{i,L}$ and $w_{i,0}$ be the amounts of traffic served on each respective band in equilibrium. This is equivalent to a model where instead of allocating x_i and $w_{i,L}$ separately, SP i allocates the total traffic $\tilde{x}_i = x_i + w_{i,L}$ to a single band where the price is determined by*

$$p_i = p_d - (1-\alpha)\frac{\tilde{x}_i + w_{i,0}}{B_i} - \alpha\frac{\tilde{x}_i}{B_i + W_i}. \tag{11}$$

Proof: See Appendix A.1

Similarly, given multiple subbands of shared spectrum that are designated as open access, those subbands can also be pooled and treated as a single (intermittent) open access band with the combined bandwidth as could more than two bands of licensed shared spectrum. Hence, in the following, without loss of generality, we will focus on the scenario with one band of licensed shared spectrum per SP and at most one open access band.

2.2 Welfare Measures

We will focus on two basic welfare measures, consumer surplus and social welfare. The consumer surplus associated with an equilibrium allocation is the difference between the amount the customers receiving service would be willing to pay and the total cost they incur. Since all customers incur the same cost p_d, it follows that the consumer surplus is given by

$$CS(z) = \int_0^z (P(y) - P(z))\, dy = \frac{z^2}{2}. \tag{12}$$

where z denotes the total number of customers served over all bands. Note that $CS(z)$ is a strictly increasing function of z so that to compare the consumer welfare of different equilibria we need only compare the number of customers served. The social welfare of an equilibrium is the sum of the consumer surplus and the total revenue earned by all SPs.

3 Two Service Providers

In this section we focus on scenarios in which there are two competing SPs. This allows an illustration of the basic properties of the model. We then consider scenarios with more than two SPs in the subsequent sections.

3.1 Shared Licensed Access

We first examine the scenario in which all of the shared bandwidth is available for licensed access. We begin by defining a notion of *equivalent licensed spectrum* and examining some of its properties. We then use this to study the competition between the two firms.

3.1.1 Equivalent Spectrum

The next lemma shows that in the absence of open access spectrum we can simplify the model and represent each SP as though it has an equivalent amount of proprietary spectrum.[6]

[6]The results in this expression apply also when there are more than two SPs as long as there is no open access spectrum.

Lemma 3.1 *Suppose SP i has access to B_i units of proprietary spectrum and W_i units of licensed shared spectrum, and that there is no open access spectrum ($W_0 = 0$). In this case, SP i can be equivalently represented as an SP with T_i units of proprietary spectrum and no other licensed spectrum, where*

$$T_i = B_i \frac{B_i + W_i}{B_i + (1-\alpha)W_i}. \tag{13}$$

This follows from Lemma 2.1 by noting that when $w_{i,0} = 0$, the expression for p_i in (11) can equivalently be written as

$$p_i = p_d - \frac{\tilde{x}_i}{T_i},$$

where T_i is given by (13).

As expected, T_i is an increasing function of reliability α, with a minimum of B_i (proprietary bandwidth) when $\alpha = 0$, and maximum value $B_i + W_i$ when $\alpha = 1$. In other words, for a given bandwidth, higher availability translates into more equivalent spectrum for a SP, and if the shared spectrum is always available, the equivalent amount spectrum is simply the sum of the two bandwidths.

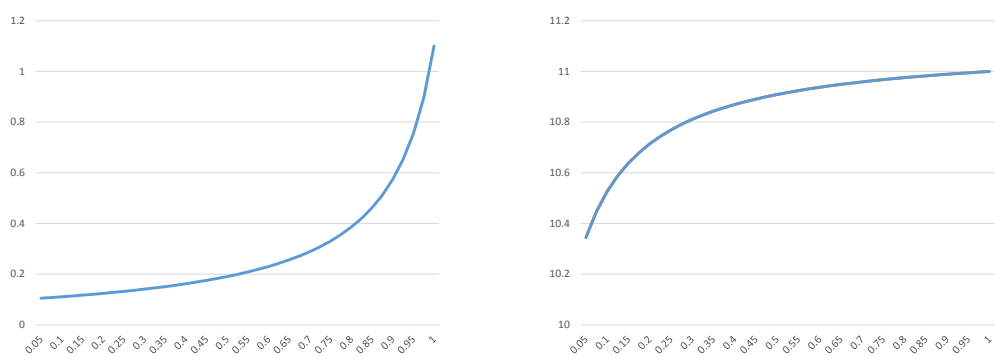
Let $m_i = \frac{\partial T_i}{\partial W_i}$ denote the marginal gain in equivalent spectrum per unit increase in licensed shared spectrum for SP i. The next lemma summarizes some properties of this quantity.

Lemma 3.2 *The marginal gain in equivalent spectrum has the following properties:*

1. *If $\alpha = 1$, $m_i = 1$ for all values of W_i and B_i.*
2. *For $\alpha < 1$, m_i is strictly decreasing in W_i.*
3. *If $W_i = 0$, then $m_i = \alpha$ for any value of B_i.*
4. *If $W_i > 0$ and $\alpha < 1$, then m_i is strictly increasing in B_i.*
5. *If for two SPs, i and j we have $\alpha < 1$, $T_i \leq T_j$, and $B_i > B_j$, then $m_i > m_j$.*

The first two properties highlight a key difference between non-intermittent and intermittent spectrum. Without intermittency, the marginal gain in equivalent spectrum for any unit of new spectrum is always equal to one (which is expected from the previous comment that in this case the equivalent spectrum is simply $B_i + W_i$). However, when spectrum is intermittent the gain decreases with each new unit of intermittent added, i.e., T_i is a strictly concave function of W_i. Further from (13), it can be seen that with $\alpha < 1$, as W_i increases, T_i approaches the limiting value of $\frac{B_i}{1-\alpha}$, i.e., even with an unbounded amount of intermittent resource the equivalent amount of non-intermittent resource is bounded. The third property shows that if only a small amount of intermittent spectrum was to be allocated, then the marginal gain in equivalent spectrum for any user would approximately be equal to the expected amount of available spectrum, i.e., αW. Since this marginal gain does not depend on B_i, it means that if two users had different amounts of proprietary spectrum, they would each see approximately the same gain in equivalent spectrum from allocating a small amount of intermittent spectrum. However, if a larger amount of intermittent spectrum was to be allocated, then the fourth property shows that with intermittency, there is a larger marginal gain in T_i from allocating this to a SP with a larger value of B_i. Comparing to the first property, this shows another difference between intermittent and non-intermittent spectrum. Intuitively, the SP with larger B_i is better able to absorb the fluctuations in traffic due to the intermittency. The final property shows a similar effect. For example, if two SPs i and j have the same equivalent spectrum but i has more proprietary spectrum, then SP i will have a larger marginal gain. This means that the sum of the equivalent spectrum between the two SPs could be increased by shifting some intermittent spectrum from SP j to SP i.

The preceding results also shed some light into the trade-offs between the amount of shared spectrum and the availability of that spectrum. Consider the case $W_i = W$, in which all bandwidth is allocated to SP i. We ask whether SP i would prefer W units of bandwidth available with probability α, or αW units of non-intermittent bandwidth. In other words, is it better to have a smaller amount of bandwidth always available, or a larger amount with intermittent availability, fixing the average? Property 3 in Lemma 3.2 show that for a small value of W, the SPs would essentially be indifferent between these options. However, for larger values of W, it follows from property 4 that $B_1 + \alpha W > T_1$, meaning that the SP would prefer the smaller amount of certain bandwidth. Figure 1 shows plots of T_i versus α with fixed αW_i, the average amount of shared bandwidth. The plots show that T_i is monotonically increasing with α, which implies that an SP always prefers higher reliability with less bandwidth. Furthermore, the left plot shows that the variation in T_i with α can be substantial. That corresponds to the scenario in which $W_i = 1$ and $B = 0.1$, so that the shared band greatly increases the amount of spectrum potentially available. The knee of the curve, however, occurs when $\alpha > 0.7$, indicating that the band must be relatively reliable in order to provide a significant enhancement of available spectrum. In contrast, the variation shown in the right plot is much smaller since $W << B$.



$B = 0.1,\ \alpha W = 1$ $\qquad$ $B = 1,\ \alpha W = 1$

Figure 1: Plots of the equivalent bandwidth T_i in (13) versus α with fixed average bandwidth αW.

3.1.2 Equilibrium

Next we turn to considering the equilibrium when two providers compete when the shared specturm is only used for licensed shared access. From Lemma 3.1, we can view each provider i as having T_i units of proprietary spectrum, which includes its portion of the shared spectrum. For $N = 2$, the conditions for Cournot competition reduce to:

$$
\begin{aligned}
p_d &= 1 - x_1 - x_2 \\
p_1 &= p_d - \frac{x_1}{T_1} = 1 - x_1\left(1 + \frac{1}{T_1}\right) - x_2 \\
p_2 &= p_d - \frac{x_1}{T_2} = 1 - x_2\left(1 + \frac{1}{T_2}\right) - x_1
\end{aligned}
$$

where the SPs choose x_1 and x_2, respectively. SP i's revenue is given by $R_i = p_i x_i$, which from these relations is a quadratic function of x_i. Using this we can determine the best response function of each SP and then use these directly to show that a Nash equilibrium exists and is unique. Further, we can explicitly characterize this equilibrium as given in the following result.

Theorem 3.3 *There is a unique Nash equilibrium given by*

$$x_1^* = \frac{T_1T_2 + 2T_1}{3T_1T_2 + 4 + 4(T_1 + T_2)}$$
$$x_2^* = \frac{T_1T_2 + 2T_2}{3T_1T_2 + 4 + 4(T_1 + T_2)}$$

with the equilibrium prices given by

$$p_1^* = \frac{T_1T_2 + 2T_1 + T_2 + 2}{3T_1T_2 + 4 + 4(T_1 + T_2)}$$
$$p_2^* = \frac{T_1T_2 + 2T_2 + T_1 + 2}{3T_1T_2 + 4 + 4(T_1 + T_2)}.$$

The existence of a Nash equilibrium in this setting also follows from Theorem 6.1, which shows that under more general assumptions, the underlying game is a potential game. Furthermore, since we have used Lemma 3.1 to reduce this to an equivalent model with no intermittent spectrum, this corresponds to a special case of the model studied in [22], which also shows existence and uniqueness of the Nash equilibrium.

This theorem enables us to deduce the following comparative statics.

Theorem 3.4 *Let $R_i(T_1, T_2)$ be the equilibrium revenue of SP $i \in \{1, 2\}$ given that each SP i has T_i units of equivalent proprietary spectrum.*

1. *$R_1(T_1, T_2)$ is strictly increasing and concave in T_1 holding T_2 fixed.*
2. *$\frac{\partial R_1}{\partial T_2} < 0$.*

Theorem 3.4 has two immediate implications. First, unsurprisingly, each SP would prefer to have larger amounts of the equivalent proprietary spectrum than not, other parameters held fixed. Second, an increase in the equivalent spectrum of one's rival results in a decrease in one's own revenue. Again, this is somewhat expected, as the two firms are competing and so increasing the rival's equivalent spectrum makes it a stronger competitor.

The second part of Theorem 3.4 shows that, as a result of their downstream competition, allocating spectrum to one firm can impose an externality on the other firm. As we show next, this externality has implications when a simple auction is used to allocate the shared spectrum. We illustrate this first with an example.

Example 1 *Consider two SPs, where SP 1 has an arbitrarily large amount of proprietary spectrum and SP 2 has no proprietary spectrum so that with no additional spectrum, SP 1's customers see no congestion and SP 2 can not serve any customers. Hence, in the status quo, SP 1 will be a monopolist and since there is no congestion, the outcome would be the same as in the standard Cornout setting (without production costs), i.e., SP 1 will select $x_1 = 1/2$ giving it a revenue of $R_1 = 1/4$.*

Now, suppose that there is an arbitrarily large amount of new shared spectrum with $\alpha = 1$ to be allocated via a second price auction as a single good to one of the two providers. If the spectrum is allocated to SP 1, it will continue to be a monopolist making the same revenue as in the status quo. However, if it is allocated to SP 2, then SP 2 can now also serve customers without congestion and so the downstream competition becomes the standard Cournot duopoly model, in which case each SP will select $x_i = 1/3$ and obtain a revenue of $R_i = 1/9$.

Compared to the status quo, obtaining the new spectrum allows SP 2 to increase its revenue, while SP 1's revenue is unchanged if it obtains the spectrum. This might lead one to conclude that SP 2

would win the auction. However, this is not correct, as SP 1's value for obtaining the new spectrum should be compared to its loss in value if the spectrum goes to SP 2, i.e., it should be willing to bid $1/4 - 1/9$*, which is larger than gain of* $1/9$ *obtained by SP 2 if it wins the auction. Hence, SP 1 would win such an auction. The outcome is driven by SP 1's desire to reduce the competition it faces, rather than by the increase in profit it may see.*

The following theorem shows that this type of outcome is not limited to this extreme example.

Theorem 3.5 *Consider allocating the shared spectrum to one of the two SPs using a second price auction. Then if* B_1 *is sufficiently large and* B_2 *and* W *are sufficiently small, SP* 1 *will submit the winning bid in this auction.*

If spectrum is allocated using such an auction, then the profit of a SP that wins the auction is given by its revenue less the price it pays to acquire the spectrum. We next consider this for the setting in Example 1.

Example 1 continued *After winning the second price auction SP 1 will still obtain a revenue of* $1/4$ *and in a second price auction, it would pay a price of* $1/9$*, the amount which SP 2 would bid. Hence, SP 1's profit is given by* $(1/4) - (1/9)$*, which is obviously less than it profit of* $1/4$ *prior to the introduction of the new spectrum. Hence, the introduction of the new spectrum has reduced SP 1's profit (while SP 2's profit in this case is unchanged).*

The next theorem generalizes this example.

Theorem 3.6 *Consider allocating the shared spectrum to one of two SPs using a second price auction. If the winning firm's change in revenue compared to the status quo is no greater than the change in revenue of the losing firm compared to the status quo if it would have been allocated the spectrum, then both firms will have no greater profits after the auction than in the status quo.*

Proof: In the status quo there is no shared spectrum and so for each SP i, we have $T_i = B_i$. Let $\hat{T}_i$ denote the equivalent spectrum of SP i if it wins the auction. The bid of SP 1 in a second price auction will then be given by

$$R_1(\hat{T}_1, B_2) - R_1(B_1, \hat{T}_2).$$

Without loss of generality, assume that this SP wins the auction. From property 2 in Theorem 3.4 it follows that SP 2's profit will decrease. SP 1's resulting profit after winning will be

$$R_1(\hat{T}_1, B_2) - (R_2(B_1, \hat{T}_2) - R_2(\hat{T}_1, B_2)).$$

Hence, SP 1 will have no greater profit after winning than in the status quo if and only if

$$R_1(B_1, B_2) \geq R_1(\hat{T}_1, B_2) - (R_2(B_1, \hat{T}_2) - R_2(\hat{T}_1, B_2)).$$

Rearranging, this is equivalent to

$$R_1(\hat{T}_1, B_2) - R_1(B_1, B_2) \leq R_2(B_1, \hat{T}_2) - R_2(\hat{T}_1, B_2). \tag{14}$$

Using property 2 in Theorem 3.4 and that $\hat{T}_1 > B_1$, we have that

$$R_2(B_1, \hat{T}_2) - R_2(\hat{T}_1, B_2) > R_2(B_1, \hat{T}_2) - R_2(B_1, B_2).$$

Hence, a necessary condition for (14) to hold is that

$$R_1(\hat{T}_1, B_2) - R_1(B_1, B_2) \leq R_2(B_1, \hat{T}_2) - R_2(B_1, B_2).$$

The terms on the left and right side of this inequality are the change in revenue compared to the status quo when SP 1 and SP 2 are allocated the spectrum, respectively, proving the theorem. ■

This theorem shows that whenever that the winning SP is determined more by one SP's desire to reduce competition rather than that SP's desire to increase profit over the status quo, then the outcome will be lower profits for both SPs. Note in particular that a symmetric model in which $B_1 = B_2$ will always satisfy the assumption in this theorem since in this case the change in revenue compared to the status quo will be the same for both SPs.

Next we turn to considering the equilibrium consumer surplus. Recall, from (12) this is given by

$$CS(x_1^* + x_2^*) = \frac{(x_1^* + x_2^*)^2}{2}, \tag{15}$$

and from Theorem 3.3 we have

$$\begin{aligned} x_1^* + x_2^* &= \frac{T_1T_2 + 2T_1}{3T_1T_2 + 4 + 4(T_1 + T_2)} + \frac{T_1T_2 + 2T_2}{3T_1T_2 + 4 + 4(T_1 + T_2)} \\ &= \frac{2T_1T_2 + 2(T_1 + T_2)}{3T_1T_2 + 4 + 4(T_1 + T_2)}. \end{aligned} \tag{16}$$

It can be seen that $x_1^* + x_2^*$ is an increasing function of T_1 and T_2. From the discussion at the end of Section 3.1.1, this means that if one could reduce intermittency, while keeping the average amount of shared spectrum for each SP, αW_i fixed, it would increase consumer welfare.

Suppose now that given a budget of W units of shared spectrum, a regulator seeks to allocate this so as to optimize the resulting consumer surplus in (15) (or equivalently to maximize $x_1^* + x_2^*$ in (16). Here, an allocation corresponds to a choice of W_1 and W_2 satisfying $W_1 + W_2 \leq W$. This then determines the value of T_i for each SP. The next result summarizes some properties of this allocation.

Theorem 3.7 *The consumer surplus maximizing allocation (W_1^*, W_2^*) has the following properties:*

1. $W_1^* + W_2^* = W$.
2. *If $\alpha = 1$, the optimal allocation equalizes the T_i if possible and if not allocates all of W to the SP with the smaller amount of proprietary bandwidth.*
3. *If $B_1 = B_2$, $W_i^* = W/2$ for each SP i.*
4. *For $\alpha < 1$, if $B_1 > B_2$, then the optimal allocation will always result in $T_1 > T_2$.*
5. *If $B_1 > B_2$ and an infinitesimal amount of shared spectrum is to be allocated, then it should be allocated to SP 2.*

The first property simply states that it is always beneficial to allocate all available shared spectrum. The second property shows that when the spectrum is not intermittent, it is desirable to try to equalize the equivalent spectrum holdings of the two SP's. The third property shows that with intermittency, if the SP's originally had the same amount of spectrum, then it is again optimal to keep their equivalent spectrum equal. Intuitively, in both of these cases, having more "equal" SPs results in greater competition, which improves consumer surplus. The fourth property shows, however, that with intermittency and unequal amounts of proprietary spectrum, it is no longer optimal

to equalize the T_i values and in fact the larger SP (i.e., the one with a larger amount of proprietary spectrum) will always have a larger value of T_i. The reason for this is that the larger SP, is better able to absorb the traffic fluctuations due to the intermittency, which counteracts to some degree the benefits of stronger competition. Though the optimal allocation will always make the larger SP have a larger value of T_i, the smaller SP may still get a larger amount of the new shared spectrum. The last property illustrates this and shows that when there is only a small amount of shared spectrum, it should all be given to the smaller SP.

Comparing the last property in Theorem 3.7 to the conclusion in Theorem 3.5, we can see that under the conditions in those theorems, the outcome of the simple auction is the opposite of the allocation that maximizes consumer surplus, i.e., the auction would result in the larger SP getting the spectrum, while to optimize consumer surplus it should be given to the smaller SP. A similar conclusion holds for the setting in Example 1, in that case, allocating the new spectrum to SP 2 would lead to both a larger consumer surplus and a larger social welfare than obtained in the auction outcome. Also, note that if we include the cost of winning the auction when calculating the social welfare, it follows from the discussion prior to Theorem 3.6 that in this case, auctioning the new spectrum may actually decrease social welfare. Examples in the subsequent sections will show that this behavior persists in more general settings, namely that such an auction gives new spectrum to the "better" provider, which is not always optimal in terms of welfare.

3.2 Shared Open Access

We now assume that each SP has its own proprietary bandwidth and that the additional shared bandwidth W is open access.

3.2.1 Symmetric Model

We begin with the symmetric case where each SP has the same amount of proprietary bandwidth, i.e., $B_1 = B_2 = B/2$.

Theorem 3.8 *When $B_1 = B_2 = B/2$, the unique Nash equilibrium is given by the quantities*

$$x_i^* = \frac{\frac{3}{W}}{\frac{9}{W} + \frac{12}{B} + \frac{12}{BW} + \frac{16(1-\alpha)}{B^2}} = \frac{3B^2}{9B^2 + 12BW + 12B + 16(1-\alpha)W}$$
$$w_i^* = \frac{\frac{4}{B}}{\frac{9}{W} + \frac{12}{B} + \frac{12}{BW} + \frac{16(1-\alpha)}{B^2}} = \frac{4BW}{9B^2 + 12BW + 12B + 16(1-\alpha)W}.$$

for $i = 1, 2$.

This theorem shows that in particular that there is a symmetric equilibrium in which both SPs use both their proprietary and the open access spectrum.

Direct computation using the preceding quantities shows that both SP revenue and consumer surplus increase with the following parameter variations:

1. B increases holding W and α fixed;
2. W increases holding B and α fixed;
3. α increases holding B and W fixed.

In other words, it is preferable to have more proprietary bandwidth, more shared open access bandwidth and less intermittency. We remark that for the analogous Bertrand model considered in [18],

the SP revenue generally *decreases* as W increases. In the Bertrand model in [18], the price in the open access spectrum is competed to zero, drawing more customers to that band and reducing the profits of the SPs. In contrast, for the Cournot model, the corresponding shift in traffic to the open access spectrum generally lowers the price but not to zero so that the SPs can offset the price decrease by an increase in the number of customers served. Comparing this to the conclusion in Theorem 3.6 immediately yields the following result:

Corollary 3.9 *Given two symmetric SPs, each SP's revenue if the new intermittent spectrum is open access will be strictly larger than their profits if it is allocated to one SP for shared licensed access using a second price auction.*

Essentially, the cost of winning a license will be large enough to make each SP prefer open access over having to bid for a license. An illustration of this result is shown in Table 1. This table shows the revenue obtained for a market with two SPs where $B_1 = B_2 = 1$ and $W = 1$. Each row corresponds to a different choice of α. The column labeled 'pre-allocation' lists the revenue of each SP before the allocation of additional bandwidth. The column labeled 'large' records the revenue of the SP that receives the entire one unit of additional bandwidth as shared licensed spectrum. The column labeled 'small' is the revenue of the SP who did not receive the additional bandwidth. The last column is the revenue of each SP when the additional unit of bandwidth is offered as open access. As shown in the proof of Theorem 3.6, when two symmetric SPs bid for the spectrum in a second price auction, their profit including the cost of acquiring the spectrum will be equal to the amount in the "small" column, which is always less than both the pre-allocation level and the open access case.

α	pre-allocation	large	small	open access
0.1	0.08	0.096	0.052	0.081
0.5	0.08	0.102	0.057	0.083
0.9	0.08	0.11	0.064	0.086

Table 1: Examples of Corollary 3.9.

We can also use these properties to again study the impact of changing the intermittency, keeping the average amount of shared spectrum αW fixed. As with licensed shared spectrum, it can again be seen that in this case consumer welfare is increasing the less intermittent the spectrum becomes.

Comparing the expressions for x_i^* and w_i^*, note that for each SP i, we have

$$\frac{x_i^*}{B} = \left(\frac{3}{4}\right) \cdot \left(\frac{w_i^*}{W}\right).$$

Using that $B_i = B/2$ and that $w_i = w_j$, this is equivalent to

$$\frac{x_i^*}{B_i} = \left(\frac{3}{4}\right) \cdot \left(\frac{w_1^* + w_2^*}{W}\right). \tag{17}$$

This shows that conditioned on the intermittent spectrum being available, the latency of the proprietary band will be lower than the latency on the shared band. For example, if W and B are equal, then both SPs will put more traffic on the shared band. Note that in the proof of Lemma 2.1, we showed that in the shared licensed case, in equilibrium an SP would equalize these two latency terms. In other words, with shared open access, SPs are more incentivized to congest the shared band compared to their proprietary band. When an SP increases traffic in any band, it increases the

congestion experienced by all other users of the band. In the licensed case, these other users are all that SP's customers, while in the open access case, some of these users are customers of other SPs, which lessens the "cost" to an SP of putting more traffic in the open band.

From the expressions in Theorem 3.8, it can also be seen that as W increases for a fixed B, the SPs will increasingly shift their traffic to the intermittent band. In the limit of large W, x_i^* will converge to zero and w_i^* will converge to

$$\frac{4B}{12B + 16(1-\alpha)}.$$

Taking a similar limit using the expressions from Theorem 3.3 for the licensed shared case when each SP is allocated $W/2$ it follows in that case that the limiting traffic served by each SP is again given by the previous expression. In other words, with a sufficiently large amount of new intermittent spectrum, whether it is licensed or not has little impact of consumer welfare. Intuitively, when W is large, the congestion costs goes to zero in either case.

We can also evaluate the limiting prices for each provider as $W \to \infty$. For the open access case, we have that p_i^w will converge to

$$\frac{B + 2(1-\alpha)}{3B + 4(1-\alpha)}$$

and for the licensed case, we have that p_i converges to the quantity. This shows that in this limit, the SPs receive the same revenue under either licensed or open access and hence both approaches also yields the same social welfare.

Alternatively, if we hold W fixed and let B increase, then from Theorem 3.8, the SPs will increasing shift their traffic to their proprietary band. In the limit of large B, w_i^* will converge to zero and x_i^* will converge to $1/3$, which is the same limiting value as in the licensed case. Likewise, in both cases the limiting price is $1/3$ and so again in this limit the choice of licensed or open access has no impact on either social or consumer welfare. In this case, as B increases SPs have little incentive to use the shared band and so its allocation has little impact.[7]

3.2.2 Asymmetric Model

Next we generalize our model in two ways. First, we allow for the SPs to have asymmetric amounts of proprietary spectrum, i.e., $B_1 \neq B_2$. Further, we allow the shared bandwidth to be partitioned so that $W_0 = \beta W$ is provisioned as open access, and the remainder is partitioned as W_i $i = 1, 2$, and allocated as licensed shared bandwidth to the respective SPs, where $\beta \in [0, 1]$ and $W_1 + W_2 = (1-\beta)W$. It follows from a more general result in Section A.11.4 that with a unique Nash equilibrium still exists. The main result we present here is to characterize this equilibrium as as B_1 increases relative to B_2.

Theorem 3.10 *For a given partition (β, W_1, W_2), there is a unique equilibrium with $w_1^* = 0$, $x_1^* > 0$ and $x_2^* > w_2^* > 0$ if and only if*

$$B_1 + W_1 + 2(1-\alpha)\frac{W_1}{B_1} \geq 2(\beta W + W_2) + 2B_2 + 2 + 4(1-\alpha)\frac{\beta W + W_2}{B_2}. \tag{18}$$

For $\beta = 1$, all the shared bandwidth is open access and condition (18) simplifies to $B_1 \geq 2W + 4(1-\alpha)\frac{W}{B_2} + 2B_2 + 2$.

[7]Note also that this limiting outcome is the same as the outcome of the standard Cournot model without congestion and with no production costs.

Hence, if an SP has an amount of proprietary bandwidth that greatly exceeds that held by the other SP, there is a range of W for which it will *not* make use of the shared bandwidth, leaving it for the smaller SP. Interestingly, as α decreases, i.e., the shared band is more likely to be pre-empted, the 'larger' SP is more likely to use it. This is because its proprietary bandwidth makes it better able to handle the traffic in the event of pre-emption.

The proof is given as part of Appendix A.11 (see section A.11.4). There it is also shown that conditioned on the shared spectrum being available,

$$\frac{x_i^*}{B_i + W_i} \leq \frac{w_i^* + \frac{w_{-i}^*}{2}}{\beta W} \leq \frac{w_1^* + w_2^*}{\beta W},$$

for $i \in \{1, 2\}$, and where $-i$ denotes the other SP. Suppose that $\beta = 1$, so that $W_i = 0$. The condition then states that when the shared band is available, the congestion in the proprietary bands is always less than the congestion in the open access band, generalizing the conclusion in (17) to the asymmetric setting. If SP i uses the shared band, it is shown that the first inequality is tight. Additionally, if in equilibrium $w_i^* > 0$ for both SPs, then for SP i the congestion in the open access band is strictly greater than the congestion in its licensed bands by $w_{-i}^*/(2W)$. Furthermore, if in equilibrium $w_i^* = 0$, then the open access band and proprietary band for the other provider $-i$ have the *same* congestion level, which is at least twice the congestion in SP i's proprietary band. It is this additional congestion which causes SP i to assign $w_i = 0$, and to use only its proprietary bands. The appendix also considers $N > 2$ asymmetric SPs, and gives a condition for when all but one SP assigns traffic to the open access band.

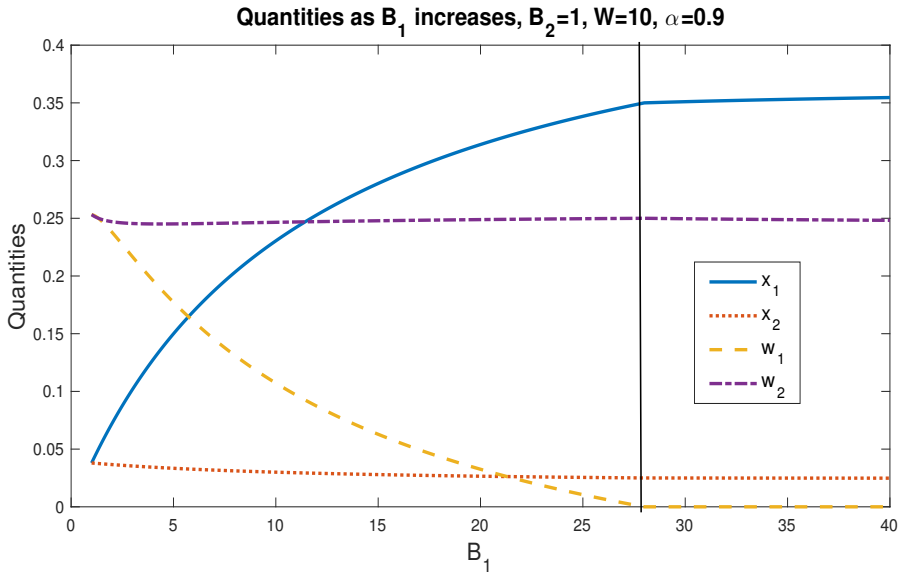
Figure 2 illustrates this result for a scenerio in which $\beta = 1$ so that all bandwidth is shared. These figures show how quantities and prices in the shared and proprietary bands change as the amount of proprietary bandwidth for SP 1 (B_1) increases. Figure 2a shows that as B_1 increases, SP 1 increases the quantity of customers x_1 served in its proprietary band, and decreases its allocation of customers w_1 to the shared band. SP 2 maintains nearly constant quantities in both bands. The vertical line shows the threshold B_1^* at which w_1 becomes zero. For $B_1 > B_1^*$ the quantities are nearly constant with only slight variations due to the limited competition with only two SPs.

Figure 2b shows that SP 1 charges higher prices than SP 2 in both bands since it is able to provide lower latency than SP 2. For $B_1 < B_1^*$, SP 1's prices increase, due to decreasing latency, whereas SP 2's prices decrease to maintain its quantity of customers. For $B_1 > B_1^*$, p_1 increases slowly, since latency in that band continues to decrease, whereas the remaining prices decrease to maintain the nearly constant quantities shown in Figure 2a.

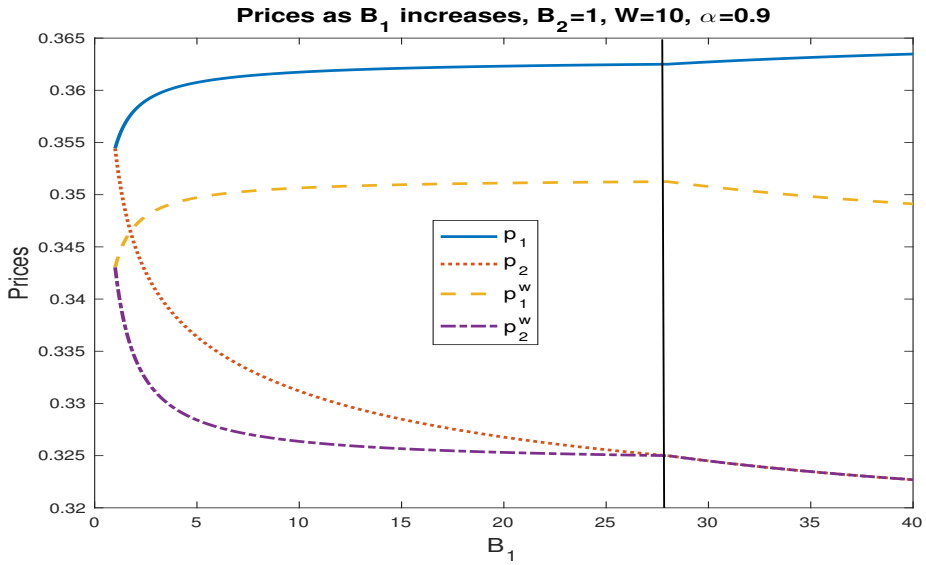
3.3 Welfare Comparisons

For $\beta = 1$, if B_1, B_2 and W satisfy the conditions in Theorem 3.5, then interestingly these parameters will also satisfy the conditions in Theorem 3.10. Recall, under these conditions, Theorem 3.5 showed that SP 1, with the larger amount of proprietary bandwidth would win the spectrum if it was auctioned with licensed shared access. However, Theorem 3.10 shows that in the same conditions with shared open access, SP 1 would *not* use the the spectrum and it would instead be used by SP 2. This further reinforces the point made earlier that in the auction SP 1's main incentive is to exclude SP 2 from utilizing the shared resource. Also, from Theorem 3.7, it follows that under these conditions, consumer welfare will be greater under shared open access as here that is equivalent to allocating it to SP 2.

Recall, for a symmetric model, Corollary 3.9 showed that both SPs would prefer shared open



(a) Quantities



(b) Prices

Figure 2: Equilibrium quantities x_i, w_i, and prices p_i, p_i^w, $i = 1, 2$, as B_1 increases with $B_2 = 1$, $W = 10$ and $\alpha = 0.9$.

access over having to bid for licensed access in a second price auction. The next corollary shows that in an asymmetric model this is not necessarily the case, in particular when the conditions in Theorem 3.10 are satisfied.

Corollary 3.11 *Consider allocating the shared spectrum to one of the two SP's using a second price auction. Suppose that SP 1 wins this auction and that if instead the spectrum was open access then the condition in Theorem 3.10 would be satisfied so that $w_1 = 0$. In this case, SP 1's profit from winning the auction will be larger than its profit if the spectrum was open access.*

Proof: Let $\hat{T}_i$ again be the equivalent spectrum if SP i wins the auction. Then SP 1's profit from winning is again given by

$$R_1(\hat{T}_1, B_2) - (R_2(B_1, \hat{T}_2) - R_2(\hat{T}_1, B_2)).$$

Likewise, under the conditions of Theorem 3.10, SP 1's revenue if the spectrum is open access will be the same as if the spectrum is allocation to SP 2, i.e., $R_1(B_1, \hat{T}_2)$. To prove the theorem we must show that

$$R_1(\hat{T}_1, B_2) - (R_2(B_1, \hat{T}_2) - R_2(\hat{T}_1, B_2)) > R_1(B_1, \hat{T}_2).$$

Re-arranging this is equivalent to

$$R_1(\hat{T}_1, B_2) - R_1(B_1, \hat{T}_2) > R_2(B_1, \hat{T}_2) - R_2(\hat{T}_1, B_2),$$

which is exactly the condition for SP 1 to win the auction. ■

Interestingly, this shows that in some cases SP 1 would prefer to have the spectrum licensed, even though if it was not licensed, that SP would not use the shared spectrum.

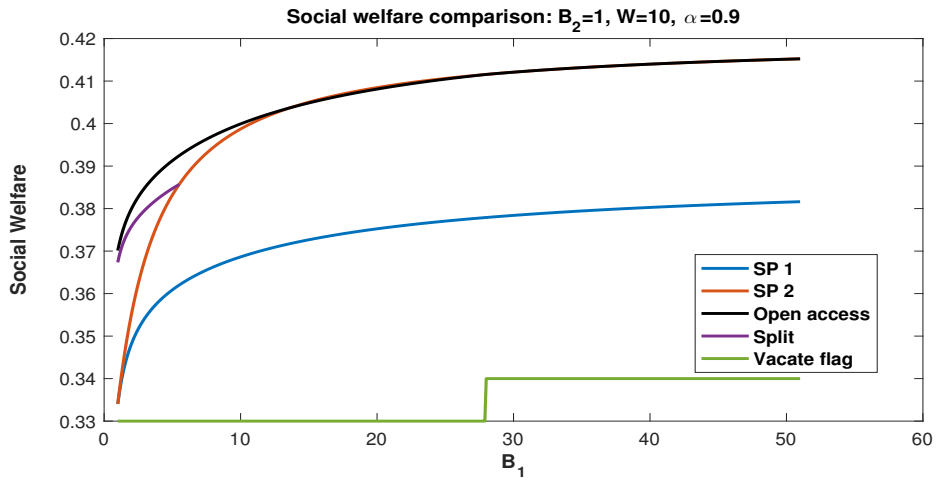
In Figure 3 we numerically compare the social welfare achieved by four different schemes for allocating the shared spectrum. The label "SP 1" in Figure 3 indicates all of the shared spectrum is allocated to SP 1, which always possesses the greater amount of proprietary spectrum. Similarly, the label "SP 2" allocates all of the shared spectrum to SP 2. We compare the social welfare for these outcomes with that obtained by allocating the shared spectrum as open access, labeled "Open access". Finally, the label "Split" allocates the shared spectrum to equalize the equivalent bandwidths T_1 and T_2, if possible, or otherwise allocates all of the shared bandwidth to SP 2.

Figure 3a depicts how social welfare changes as a function of $B_1 \geq B_2 = 1$ (with $\alpha = 0.9$ and $W = 10$). Assigning W to the smaller provider SP 2 always achieves higher social welfare than assigning it to SP 1, since this enables SP 2 to compete more effectively with SP 1. Recall in Section 3.1, we showed that in some cases this was also the allocation that maximized consumer surplus, but would not necessarily be the outcome of a second price auction. The resulting loss in social welfare is indicated in the figure.

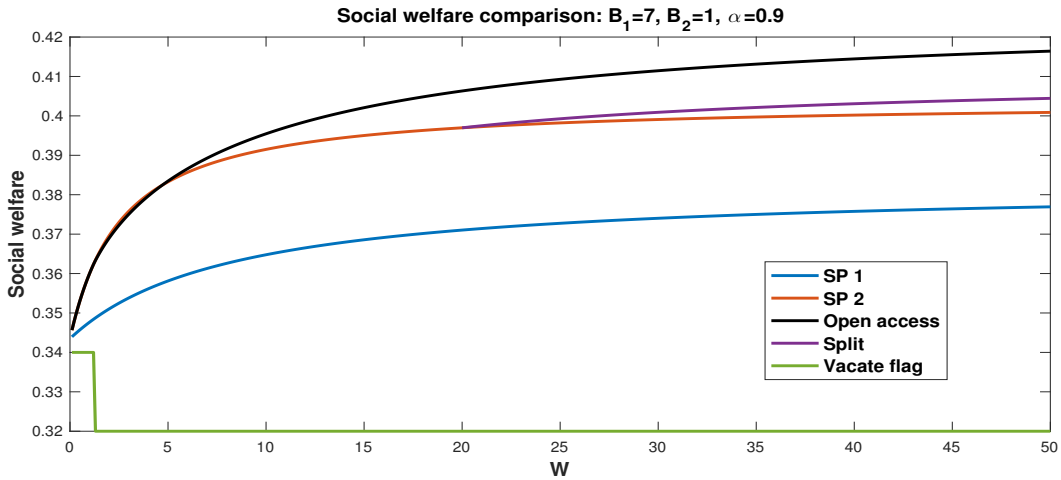
For the schemes considered in Figure 3, open access sharing yields the highest social welfare except for a small region where the scheme "SP 2" does marginally better. This shows that in general the choice between licensed access and open access depends on the setting. The "Vacate flag" indicates the values of B_1 for which SP 1 does not use the shared spectrum, so that it is effectively allocated to SP 2. Hence in that region the social welfare for open access coincides with that for scheme SP 2.

When it is possible to set $T_1 = T_2$, the "Split" scheme achieves a higher social welfare than either scheme SP 1 or SP 2. This indicates that among schemes that partition the shared spectrum between the two SPs, there is an optimal split that lies between the two extreme schemes SP 1 and SP 2.

Open access sharing can be thought of as a more flexible split between the two SPs. Figure 3b depicts how social welfare changes as W increases for the four allocation schemes considered. The relative differences observed previously also apply in this regime. However, for open access sharing, as W increases, in equilibrium, SP 1 always uses the shared spectrum.



(a) B_1 varies with $W = 1$



(b) W varies with $B_1 = 7$

Figure 3: Comparison of social welfare for various schemes for allocating the shared bandwidth W; $B_2 = 1$ and $\alpha = 0.9$.

4 Many Symmetric Service Providers

We now examine the scenario with an arbitrary number of SPs N, where the SPs are symmetric. That is, they each have the same amount of licensed bandwidth (including licensed shared bandwidth). Specifically, the shared band is split into an open access part with bandwidth βW and a licensed part with bandwidth $(1-\beta)W$ for a fixed $\beta \in [0, 1]$. Each SP has its own proprietary bandwidth B/N, and the licensed part of the shared band is split equally among the SPs. The shared band therefore adds $(1-\beta)W/N$ units of licensed bandwidth to each SP with availability α. We will compare the total welfare, total revenue, and consumer surplus when the shared band is allocated as licensed

versus open access. Analytical results are presented for $N \to \infty$, reflecting perfect competition.

Let $(\bar{x}(N), \bar{w}(N))$ denote the symmetric equilibrium allocation, *i.e.*, $\bar{x}(N)$ and $\bar{w}(N)$ are the same for all SPs. Here $\bar{x}$ and $\bar{w}$ refer, respectively, to the quantities allocated to the licensed bands, including the licensed part of the shared band, and the open access band.

Lemma 4.1 *For any finite N the equilibrium is symmetric and unique.*

Theorem 4.2 *As $N \to \infty$, the limiting equilibrium is specified by*

$$(x^*, w^*) = \lim_{N\to\infty} (N\bar{x}(N), N\bar{w}(N)),$$

where

$$\begin{aligned} x^* &= \frac{\big(B + (1-\beta)W\big)B}{\big(B + 2(1-\alpha)\big)\big(B + W(1+\beta)\big) + 2B\alpha}, \\ w^* &= \frac{2\beta W B}{\big(B + 2(1-\alpha)\big)\big(B + W(1+\beta)\big) + 2B\alpha}, \end{aligned} \tag{19}$$

and the limiting prices in the licensed and open access bands are given by

$$\begin{aligned} p &= \frac{(1-\alpha)\big(B + W(1+\beta)\big) + B\alpha}{\big(B + 2(1-\alpha)\big)\big(B + W(1+\beta)\big) + 2B\alpha}, \\ p^w &= \frac{(1-\alpha)\big(B + W(1+\beta)\big)}{\big(B + 2(1-\alpha)\big)\big(B + W(1+\beta)\big) + 2B\alpha}. \end{aligned} \tag{20}$$

The proof is given in Appendix A.8. There the expressions for $\bar{x}(N)$, $\bar{w}(N)$ are given for arbitrary N.

Theorem 4.2 has the following implications:

1. From (19), the congestion, or load in the open access band (users/total bandwidth) is

$$\frac{w^*}{\beta W} = 2\frac{x^*}{B + (1-\beta)W}. \tag{21}$$

   Therefore, as in the $N = 2$ case, the congestion in the open access band is greater than the congestion in the licensed/proprietary band. As $N \to \infty$, these congestion differ by a factor of 1/2. More generally, for arbitrary N, the expressions in the appendix show that the congestion in the open access band is $2N/(N+1)$ times the congestion in the proprietary bands, which increases with N. Having the congestion cost in open access band shared by more competitors increases each SP's incentive to shift more traffic onto that band. Also, from (20), $p > p^w$ for all $\alpha > 0$, i.e., this shift in traffic is accompanied with a increase in price for the less congested licensed service.

2. Unlike the classical Cournot model of competition, the prices do not converge to zero as the number of competing agents becomes large. The presence of congestion deters the SPs from reducing their prices too aggressively.

3. In the special case where the shared band is always available, i.e., $\alpha = 1$, we have

$$p = \frac{1}{W(1+\beta) + B + 2}, \; p^w = 0. \tag{22}$$

   That is, the price for the open access band is zero. This is analogous to the equilibrium with

Bertrand (price) competition, derived in [18]. There it is also observed that the price of the open access band is zero, although here that occurs only when the number of SPs goes to infinity. This is similar to the classical Cournot model, in which as the number of firms grows, the prices goes to zero, though here this is only true for the price in the open access band.

Theorem 4.3 *As $N \to \infty$, consumer surplus is maximized when $\beta = 1$ (open access). However, total revenue and social welfare are maximized when $\beta = 0$ (licensed access).*

From (19), the total traffic carried is given by

$$x^* + w^* = \frac{B\big(B + W(1+\beta)\big)}{\big(B + 2(1-\alpha)\big)\big(B + W(1+\beta)\big) + 2B\alpha}.$$

Hence the total traffic along with consumer surplus is maximized when $\beta = 1$. The rest of the proof is given in Section A.9.

This theorem shows that to maximize consumer surplus, all of the new spectrum should be allocated as open access, while to maximize social welfare it should all be allocated as licensed shared access, i.e., the choice of licensing can be viewed as a trade-off between these two objectives.

Recall for $N = 2$, we showed numerically that open access can maximize social welfare. This shows that the conclusions of this theorem depend critically on there being sufficiently many SPs. Figure 4 illustrates the change in social welfare that takes place as N increases. The plots show social welfare versus N for both $\beta = 0$ (all licensed) and $\beta = 1$ (all open access), and for different values of W. Focusing on the bottom two curves for $W = 1$, the curves cross when $N \leq 3$, i.e., for $N < 3$, open access achieves higher social welfare than licensed access, and vice versa for $N > 3$. This is consistent with Theorem 4.3. Note that the corresponding crossover value of N increases as W increases.

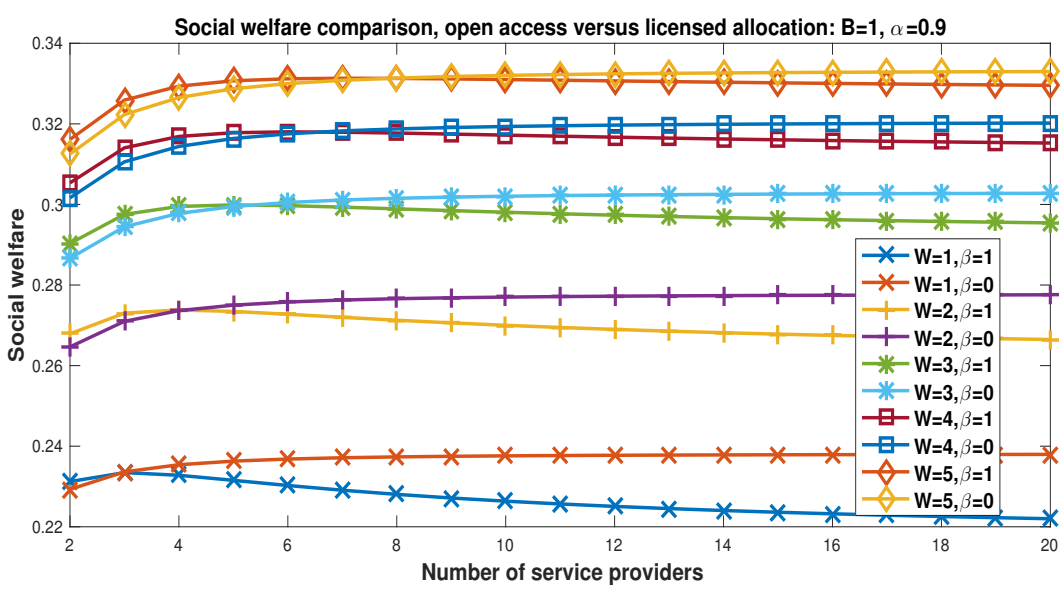


Figure 4: Social welfare achieved for $\beta = 0$ and $\beta = 1$ as a function of the number of (symmetric) SPs N for different values of W. The total proprietary bandwidth $B = 1$ and $\alpha = 0.9$

For consumer welfare a similar trade-off does not occur as N increases. Namely, as shown in Lemma 4.4 below, for any N, in a symmetric model, consumer surplus is maximized when $\beta = 1$ (all

open access).

4.1 Degraded Sharing

So far we have assumed that the latency experienced by serving traffic load x with bandwidth W is x/W regardless if the traffic load comes from a single SP or multiple SPs. In practice, because less coordination is expected among SPs in the open access band, it could be that the latency experienced for a given load in the open access band is *greater* than if the same load were served by a single SP. That would make licensed access more attractive and with enough degradation, the conclusion of Theorem 4.3 that open access maximizes consumer surplus may no longer hold. In this section we examine this possibility.

We model the degradation associated with open access by introducing a degradation factor $d \leq 1$ and assume that when traffic load x is served with open access bandwidth W, that the congestion cost is

$$\ell_W(x) = \frac{x}{dW}.$$

In other words, the "effective bandwidth" seen by the users of an open access band is $dW < W$. (Equivalently, the latency increases by $1/d$.) The next result shows that there is a sharp threshold on d must be less than for licensed shared access to maximize consumer welfare.

Lemma 4.4 *In the symmetric model with $N \geq 2$ firms, if $d > \frac{N+1}{2N}$, allocating all of the shared spectrum as open access maximizes consumer welfare, whereas if $d < \frac{N+1}{2N}$, allocating all of the shared spectrum as licensed maximizes consumer welfare.*

The proof is given in Appendix A.8.1, and is a consequence of the first property following Theorem 4.2. Note that the threshold $\frac{N+1}{2N}$ does not depend on α, W or B. This threshold is 3/4 when $N = 2$ and decreases to 1/2 as N becomes large. Hence for large N, unless the latency in the open access spectrum is more than twice that for licensed use due to lack of coordination, given the same load, open access still achieves larger consumer surplus.

4.2 Latency, price, and social welfare

In our model there is a clear trade-off between latency and price: charging a higher price, will reduce demand and thus lead to a lower latency. To gain further insight into the effects of open access bandwidth on how these factors are traded-off, Figures 5a and 5b show parametric plots of average price, consumer surplus, and total welfare versus average latency as the fraction of open access bandwidth β increases from zero to one. The average price is given by Lemma 2.1, and the average latency is similarly

$$\bar{\ell}(N) = \frac{\bar{x}(N)}{\bar{x}(N) + \bar{w}(N)} \bar{\ell}_p(N) + \frac{\bar{w}(N)}{\bar{x}(N) + \bar{w}(N)} \bar{\ell}_w(N) \tag{23}$$

where

$$\bar{\ell}_p(N) = \alpha \frac{N\bar{x}(N)}{B + (1-\beta)W} + (1-\alpha)\frac{N(\bar{x}(N) + \bar{w}(N))}{B} \tag{24}$$

and

$$\bar{\ell}_w(N) = \alpha \frac{N\bar{w}(N)}{\beta W} + (1-\alpha)\frac{N(\bar{x}(N) + \bar{w}(N))}{B} \tag{25}$$

are the latencies associated with the proprietary and shared bands, respectively. The figure shows plots for $N = 200$ and $N = 2$, and $\alpha = 1$.

No sharing ($\beta = 0$) corresponds to the lowest latency on each curve (left-most point), and as β increases from zero to one, the latency increases to the highest value (right-most point), and then subsequently decreases to the final point corresponding to full sharing ($\beta = 1$). Focusing on Fig. 5a, as β increases from zero, the average price increases slightly as latency increases. This is because when βW is small, the shift in load from the proprietary to shared band congests the shared band, increasing both average latency and price. In this region the consumer surplus and total welfare decrease. As β increases further, the SPs lower the price to continue to shift load to the shared band, and the average latency continues to increase. In this region the consumer surplus *increases* while the total welfare continues to *decrease* due to the decrease in SP revenue. Finally, as β is further increased towards one, both the price and latency fall, and consumer surplus increases more rapidly, causing total welfare to increase. Even so, the total welfare with full sharing is slightly below that with no sharing, as expected from Theorem 4.3.

Comparing Figure 5a with 5b, the additional competition with $N = 200$ results in a lower price and higher consumer surplus. Furthermore, the increase in open access bandwidth has a more pronounced effect on the quantities shown. Further examples with $\alpha < 1$ show consistent trends, but with less variation with latency due to the diminished benefit of adding the shared bandwidth.

4.3 Effects of Increasing W

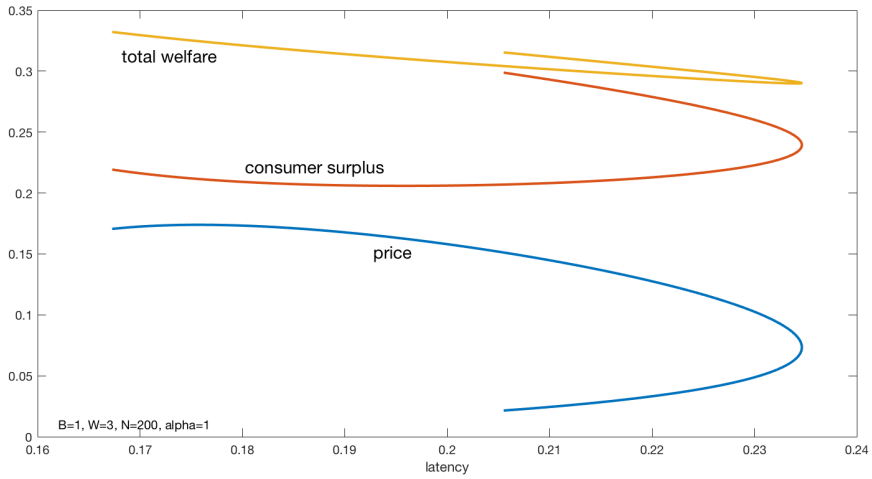
Figure 6 illustrates the effect of increasing W on social welfare, consumer surplus, and revenue for large N. Social welfare is shown for the cases where the shared bandwidth is entirely open access ($\beta = 1$) and proprietary ($\beta = 0$). Both curves are monotonically increasing, but their shape changes from convex to concave when the shared bandwidth changes from open access to proprietary. In particular, the slope at $W = 0$ is zero when the shared band is open access, but is positive when the shared band is proprietary. This behavior has also been observed within the Bertrand model of price competition [18]. There, with a small number of SPs, adding a small amount of open access bandwidth can *decrease* the social welfare (*i.e.*, the slope at $W = 0$ can be negative). Here the additional incremental shared bandwidth increases social welfare for smaller values of N (not shown), but the increase tends to zero as N becomes large. The curves for revenue and consumer surplus displayed in Figure 6 correspond to open access. Here revenue decreases, but for $\beta = 0$ the revenue initially increases slightly as W increases from zero (not shown).

We can obtain further insight by again letting W become large. In this limit, some of the expressions simplify, easing the analysis. For a given number of SPs N, taking this limit and using the equilibrium expressions in the appendices, the total mass of customers served is given by

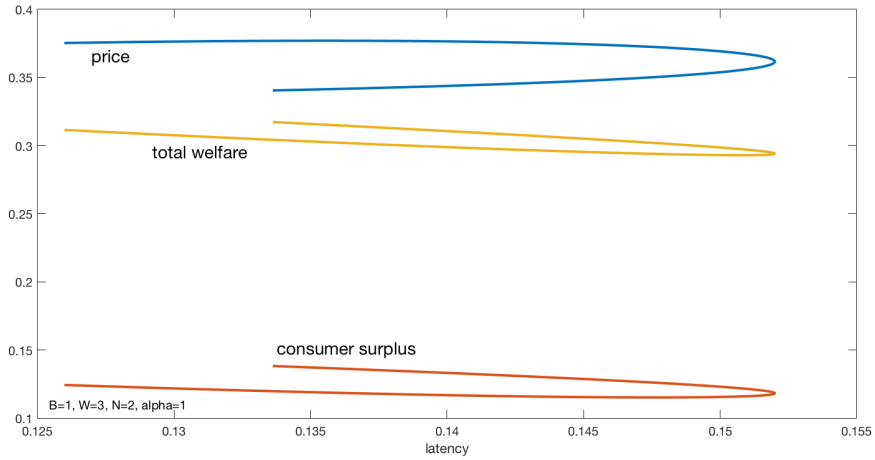
$$\rho(N) = \bar{x}(N) + \bar{w}(N) = \frac{B}{B + B/N + 2(1-\alpha)}.$$

This does not depend on β and so if there is sufficient shared bandwidth, as in the $N = 2$ case, it does not matter how it is allocated. The social welfare for licensed and open access shared bandwidth therefore become the same as illustrated in Fig. 6. Note that $\rho(N)$, and hence consumer welfare, increases with N and approaches the asymptote

$$\bar{\rho} = \frac{B}{B + 2(1-\alpha)}.$$



(a) $N = 200$



(b) $N = 2$

Figure 5: Parametric plots of average price, consumer surplus, and total welfare versus average latency as the fraction of shared bandwidth β increases from zero to one.

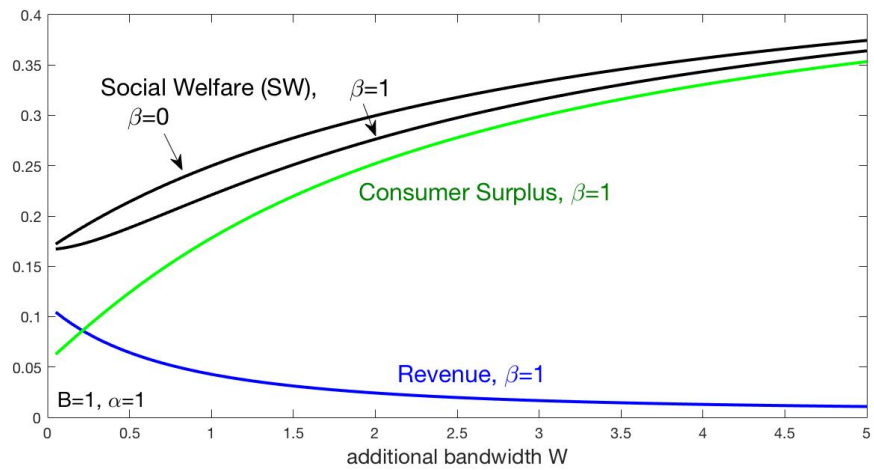


Figure 6: Total welfare versus additional bandwidth W with a large number of SPs when the shared band is open access ($\beta = 1$) and proprietary ($\beta = 0$). Revenue and consumer surplus are also shown for $\beta = 1$.

In the limit of large W, the social welfare as a function of N is given by

$$SW(N) = \rho(N) - \frac{\rho^2(N)}{2}\left(1 + \frac{2(1-\alpha)}{B}\right).$$

This expression is an increasing function of $\rho(N)$ for $\rho(N) \leq \bar{\rho}$. Hence, $SW(N)$ is also increasing with N and approaches the limiting value

$$\frac{B}{2(B + 2(1-\alpha))} = \frac{\bar{\rho}}{2}.$$

Note that $\bar{\rho}$ is a strictly increasing function of $\alpha \in [0, 1]$ and for $\alpha = 1$, we have $\bar{\rho} = 1$, meaning the entire market is served, resulting in a social welfare of $\frac{1}{2}$, which is the maximum possible for the assumed inverse demand. For $\alpha < 1$, we have $\bar{\rho} < 1$, meaning that even with an unbounded amount of shared spectrum, some users are not served due to the intermittent nature of that spectrum, so that some potential welfare is not obtained.

As we have noted for arbitrary W, the previous results show that when $\alpha < 1$ and $N \to \infty$, the aggregate profit of the SPs is strictly positive. In this case, the limiting aggregate firm profit is given by

$$\bar{\rho} - \bar{\rho}^2\left(1 + \frac{(1-\alpha)}{B}\right) = \frac{B(1-\alpha)}{(B + 2(1-\alpha))^2}.$$

Differentiating this with respect to α, it can be seen that for $B < 2$, the aggregate firm profit first increases with α and then decreases, with the maximum firm profits occurring when $\alpha = 1 - B/2$. For $B \geq 2$, aggregate firm profits decrease with α, and so the maximum occurs when $\alpha = 0$. In other words, given any value of B, the SPs would prefer that the shared spectrum is intermittent, and if B is large enough, they would prefer that the shared spectrum is never available. Adding new spectrum to the market reduces congestion, but also intensifies competition. The latter effect becomes more pronounced the less intermittent the spectrum becomes and apparently dominates the impact on the providers' profits.

5 Asymmetric Providers

To provide insight into the effects of asymmetric (large and small) SPs with different amounts of bandwidth, we now consider the following two scenarios:

- There is a single SP with proprietary bandwidth B_1. A second band B_2 is split evenly among N small SPs, where N is assumed to be large.
- Bands B_1 and B_2 are each split among N SPs. We will assume that $B_1 \geq B_2$.

Varying B_1 relative to B_2 then captures varying degrees of asymmetry. The two scenarios differ in the amount of competition experienced by the SP(s) with the larger bandwidth allocation.

As before, the shared band is split between an open access part (bandwidth βW) and proprietary part (bandwidth $(1-\beta)W$). The open access part is shared among all SPs, large and small. The proprietary part is further split into two sub-bands with bandwidths W_1 and W_2 allocated to the large and small SPs, respectively. The shared band is intermittently available, so that a small SP has proprietary bandwidth B_2/N, which is always available, plus W_2/N, which is available with probability α.

For the first scenario, let $x_1(N)$ and $w_1(N)$ denote the quantities served by the large SP in its proprietary and open access spectrum, respectively. The corresponding quantities for the ith small SP are $x_{2,i}(N)$ and $w_{2,i}(N)$, respectively. By symmetry $x_{2,i}(N)$ and $w_{2,i}(N)$ are independent of i. As $N \to \infty$, the equilibrium quantities are defined as

$$(x_1^*, w_1^*, x_2^*, w_2^*) = \lim_{N\to\infty} [x_1(N), w_1(N), Nx_{2,i}(N), Nw_{2,i}(N)] \tag{26}$$

It is shown in Appendix A.10 that those quantities are the solution to a set of four linear equations. Similarly, in the second scenario, $x_1(N)$ and $w_1(N)$ are replaced by $x_{1,j}(N)$ and $w_{1,j}(N)$, where j denotes an SP in the larger group. From symmetry those quantities are independent of j, and as $N \to \infty$, the corresponding equilibrium quantities are

$$(x_1^*, w_1^*, x_2^*, w_2^*) = \lim_{N\to\infty} [Nx_{1,j}(N), Nw_{1,j}(N), Nx_{2,i}(N), Nw_{2,i}(N)] \tag{27}$$

As for the scenario with two asymmetric SPs, here also a larger SP does not always use the open access spectrum. In addition, for the second asymmetric scenario considered here with many larger SPs, a *smaller* SP may not use the open access spectrum. The associated conditions are stated next.

Theorem 5.1 *For the first scenario with asymmetric SPs, if*

$$B_1 + W_1 + 2(1-\alpha)\frac{W_1}{B_1} > 2(1-\alpha)\frac{2\beta W + W_2}{B_2} \tag{28}$$

then there exists an N^* such that for all $N \geq N^*$, SP 1 does not use the open access spectrum, i.e.*, $w_1^*(N) = 0$.*

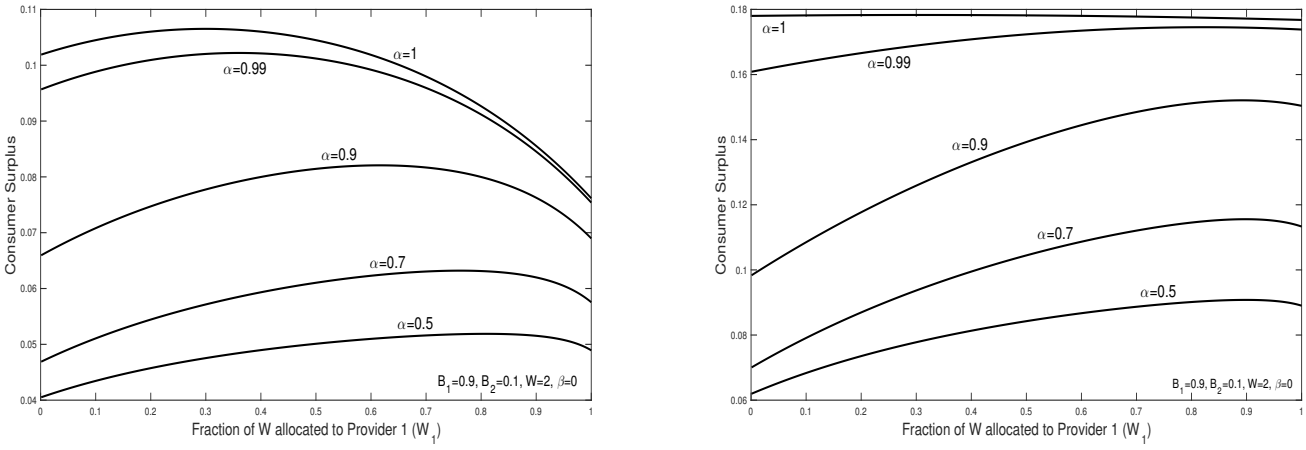
For the second scenario, a larger (smaller) SP i does not use the open access spectrum for all $N \geq N^{**}$ (for some N^{**}) when

$$\frac{W_i}{B_i} > \frac{2\beta W + W_j}{B_j}, \quad i \neq j. \tag{29}$$

where j corresponds to a smaller (larger) SP.

The proof can be found in Appendix A.11.4.

The first condition (28) resembles, but is not identical to the condition in Theorem 3.10. This is because here a portion of the licensed spectrum is also intermittent. Note that for $\alpha = 1$ the larger



(a) $N = 2$ (b) $N = 60$

Figure 7: Consumer surplus versus the fraction of shared bandwidth assigned to SP 1 (W_1/W) with different values of α.

SP *always* vacates the open access spectrum. Also, for $\alpha = 1$, the price in the shared spectrum is zero, which is also true for the analogous model with Bertrand competition [18]. The first condition in Theorem 5.1 is satisfied when β and W_2 are small. In that case, the smaller SPs congest the open access band, lowering the price, and thereby make it less desirable for the large SP(s). For small β the second condition (29) becomes $W_i/B_i > W_j/B_j$. If $W_i > W_j$, then the condition can be satisfied with $B_i < B_j$, i.e., the smaller SPs vacate the open access band. This is due to competition among the larger SPs, which causes them to shift traffic to the open access band, increasing congestion in that band and lowering the price so that the smaller SPs have no incentive to use it.

The condition (29) does not depend on α, in contrast to (28), because for large N, the additional congestion caused by intermittency is bounded, and is shared among the N large SPs. Hence that additional congestion does not significantly affect an individual SP. As α decreases, the threshold N^{**} must increase in order for the condition to apply.

Fig. 7 illustrates how the split of the shared band W into W_1 and W_2 affects consumer surplus. Here $B_1 = 0.9$, $B_2 = 0.1$, $W = 2$, $\beta = 0$ (all of W is split between the SPs), and plots are shown for different values of α. The figures show consumer welfare as a function of W_1/W for $N = 2$ (Fig. 7a) and $N = 60$ (Fig. 7b). As α increases, Fig. 7a shows that the fraction of bandwidth that maximizes consumer surplus shifts to the left. This is due to the tradeoff between the larger SP's ability to handle intermittent traffic, and competition. That is, when α is small, most of the shared band should be allocated to the larger SP, since the larger SP is better able to handle the intermittent availability of the shared band. As α increases, so that the band becomes more reliable, the consumer surplus increases by shifting bandwidth to the smaller SP to increase competition. In contrast, Fig. 7b shows that with many competing SPs it is always best to give most of the shared bandwidth to the larger SPs, independent of α.

As for the symmetric case, numerical examples show that social welfare decreases with β when $N \to \infty$, which is the same as for the symmetric case. In contrast, for $N = 2$ the total welfare *increases* with β, as discussed in Section 3. Hence at these extreme values designating the entire band W as open access ($N = 2$) or proprietary ($N \to \infty$) maximizes total welfare. Additional numerical examples indicate that this is also true for arbitrary N.

6 Extensions to General Latency and Demand

In this section we establish existence of a unique equilibrium for the Cournot game with more general demand and latency functions. The model allows the shared band to be split between licensed and open access. When the shared band is available, the total licensed bandwidth of a SP changes and hence so does the latency cost experienced by customers served on the licensed band. In addition, with a linear inverse demand function, linear latencies in the open access bands and general convex increasing latencies in the licensed bands, we also prove that the game is a potential game.[8] This fact is used in deriving some of our earlier results.

Assume that when the intermittent band is available, the latency function is given by $\ell_{i,w}(\cdot)$, and $\ell_i(\cdot)$ when not available.

Theorem 6.1 *For the Cournot game with $N \geq 2$ providers, each with proprietary spectrum and additional intermittently available shared spectrum, if the inverse demand $P(\cdot)$ is concave decreasing, and the latencies $\ell_i(\cdot)$, $\ell_{i,w}(\cdot)$ and $\ell_w(\cdot)$ are convex increasing, then an equilibrium always exists. The equilibrium is unique if either $P'(0) < 0$ and $\ell'_w(0) > 0$ or $\ell'_i(0) > 0$ and $\ell'_{i,w}(0) > 0$ for all $i = 1, 2, \ldots, N$.*

In the absence of open access spectrum, Theorem 6.1 holds without the condition on $\ell'_w(\cdot)$ and $\ell'_{i,w}(\cdot)$ for all $i = 1, 2, \ldots, N$. When bandwidths $W_1, W_2, \ldots, W_N$ with intermittent availability α are added, this is equivalent to the set of *non*-intermittent bands $T_1, \cdots, T_N$, where T_i is give in (13). With linear decreasing inverse demand, the existence and uniqueness of an equilibrium follows from Proposition 2 in [22].

We now assume two SPs with propriety spectrum *only*, *i.e.*, any shared spectrum is licensed and always available, so that we can assume $W = 0$. The proofs of the following propositions are in Appendix A.12.

Proposition 6.2 *Given an equilibrium (interior point) with two providers, concave decreasing inverse demand, and convex increasing latency, if a marginal amount of bandwidth is given to provider k, then a sequence of best responses converges to a new equilibrium in which the quantity x_k and revenue R_k each increase and x_{-k} and R_{-k} each decrease.*

According to Theorem 6.1, the sequence of best responses must converge to a unique equilibrium. This extends Theorem 3.2, and states that in this more general setting an increase in one provider's bandwidth again causes a decrease in the competitor's quantity and revenue.

Proposition 6.3 *For the scenario in Prop. 6.2, giving a marginal amount amount of bandwidth to SP k increases both consumer surplus and total welfare.*

This states that although from Proposition 6.2, adding this marginal bandwidth increases x_k and decreases x_{-k}, the *total* quantity of customers served increases. Similarly, although the revenue R_{-k} decreases, the total welfare increases.

Suppose now that we wish to give the bandwidth to the SP which will increase consumer surplus the most. That means allocating the bandwidth to maximize the total incremental quantity customers served. In general, this depends on the derivative $P'(\cdot)$ and second derivative $\ell''(\cdot)$, and is somewhat complicated (see Appendix A.12); however, for linear latencies $\ell_k(x) = c_k x$, it reduces to finding

$$\arg\max_k -\frac{c_k x_k}{B_k^2}\left(P'(x_1 + x_2) - \frac{2c_{-k}}{B_{-k}}\right) \tag{30}$$

[8] While the result in [22] proves the existence of a unique equilibrium for linear inverse demand and latencies (with a more general shared model), it does not provide the potential game characterization.

Further constraining $P(x) = 1-ax$, and using the best response conditions for x_k, x_{-k}, the bandwidth should be given to agent k if

$$B_{-k} > \sqrt{\frac{c_{-k}}{c_k}} B_k + \frac{2}{a}(\sqrt{c_k c_{-k}} - c_{-k}) \tag{31}$$

Otherwise, it should be given to agent $-k$. Recall that when allocating additional *intermittent* spectrum ($\alpha < 1$), the consumer surplus is given by (15)-(16). Here we effectively have $\alpha = 0$ so that $T_i = B_i + W_i$, which replaces B_i in the preceding condition. When $c_k = c_{-k}$ the condition reduces to $B_{-k} > B_k$, so that any marginal bandwidth should attempt to equalize the bandwidth allocation. If $c_k < c_{-k}$, however, the allocation is biased towards SP k, which provides lower latency.

7 Conclusions

We have presented a model for sharing intermittently available spectrum that captures licensed and open access sharing modes, congestion as a function of offered load, and competitive pricing for spectrum access. Our analysis suggests that allocating shared bandwidth as open access is better for consumer surplus than licensing the bandwidth for exclusive use. While latencies will be high, that is offset by lower prices, which has the effect of expanding the demand for services. Allocating additional bandwidth as licensed is good for revenue, because SPs generally choose to lower congestion by raising prices. The trade-off among revenue, consumer surplus, and congestion depends greatly on the market structure. With many SPs, competition may be enough so that total welfare (revenue plus consumer surplus) is maximized by licensing the intermittent bandwidth. With asymmetric SPs having different amounts of bandwidth, it is also possible that only a subset of the SPs use the open access band to maintain higher prices, thereby containing congestion.

The model might be enhanced in several different ways. We have not directly accounted for investment, which may be used to mitigate congestion, although we have shown that our main conclusions are robust with respect to a congestion penalty for open access. We have also generally assumed that access to the shared band is free, and have not considered pricing mechanisms, which could be used to allocate the shared spectrum as a combination of licensed and open access. Those features might also be combined with an extended model that allows SPs without proprietary spectrum to bid for open access spectrum, potentially combining both price and quantity competition.

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A Appendix

A.1 Proof of Lemma 2.1

Given the traffic allocations of SP i as stated in the lemma, the resulting price in the proprietary spectrum will be

$$p_i = p_d - (1-\alpha)\frac{x_i + w_{i,L} + w_{i,0}}{B_i} - \alpha\frac{x_i}{B_i}$$

and the price in the shared spectrum will be

$$p_i^{w,L} = p_d - (1-\alpha)\frac{x_i + w_{i,L} + w_{i,0}}{B_i} - \alpha\frac{w_{i,L}}{W_i}.$$

Note that if the SP changes x_i and $w_{i,L}$ while keeping $\tilde{x}_i = x_i + w_{i,L}$ fixed, this affects p_i and $p_i^{w,L}$ but leaves all other prices for all other SPs and bands fixed at the same values. Hence, at any equilibrium with given $\tilde{x}_i$, the values of x_i and $w_{i,L}$ must solve:

$$\begin{aligned} \max \;\; & x_i p_i + w_{i,L} p_i^{w,L} \\ \text{s.t.} \;\; & x_i + w_{i,L} = \tilde{x}_i. \end{aligned}$$

We can replace the objective function of this optimization problem by

$$-x_i\frac{x_i}{B_i} - w_{i,L}\frac{w_{i,L}}{W_i}$$

since all of the other terms only depend on the sum $x_i + w_{i,L}$. From the first order conditions for optimality, it follows that

$$\frac{x_i}{B_i} = \frac{w_{i,L}}{W_i}.$$

This implies that the price charged in each of these bands must be the same. Further, since

$$\frac{x_i}{B_i} = \frac{w_{i,L}}{W_i} = \frac{\tilde{x}_i}{B_i + W_i},$$

this price can be written as (11). ■

A.2 Proof of Lemma 3.2

Using the expression for T_i in (13), by direct calculation we have

$$m_i = \frac{\alpha B_i^2}{(B_i + (1-\alpha)W_i)^2}.$$

The first four properties, then follow directly from this expression. To see the fifth property, note that again using (13), we can re-write the previous expression for m_i as

$$m_i = \frac{\alpha T_i^2}{(B_i + W_i)^2}.$$

The conclusion then follows from noting that if $B_i > B_j$ and $T_i \leq T_j$, then for $\alpha < 1$, from property 4, it must be that $W_i < W_j$. Using this in (13), it follows that $\frac{T_i}{B_i+W_i} > \frac{T_j}{B_j+W_j}$. ■

A.3 Proof of Theorem 3.3

Theorem 3.3 follows in the standard way by deriving the best response functions of each provider and determining their intersection. Hence, many of the details are omitted. The revenue of SP i is given by

$$R_i = p_i \left(1 - x_i \left(1 + \frac{1}{T_i}\right) - x_{-i}\right).$$

Using that p_i is a linear function of x_i, this can be written as a quadratic function of x_i given x_{-i}. SP i's best response can then be determined by considering the first order optimality for maximizing R_i over x_i. Doing this and solving for the intersection of the best response functions gives a unique solution. The corresponding equilibrium quantity and price for SP 1 are

$$x_1^* = \frac{T_1}{T_1 \frac{3T_2+4}{T_2+2} + \frac{4T_2+4}{T_2+2}}, \quad p_1^* = \frac{T_1+1}{T_1 \frac{3T_2+4}{T_2+2} + \frac{4T_2+4}{T_2+2}},$$

with symmetric expressions holding for SP 2. Simplifying these, yields the expressions given in the theorem. ■

A.4 Proof of Theorem 3.4

To prove Theorem 3.4, note that the equilibrium quantity and price for SP 1 from Theorem 3.3 can be written as $x_1^* = \frac{T_1}{bT_1+a}$ and $p_1^* = \frac{T_1+1}{bT_1+a}$, where

$$b = \frac{3T_2+4}{T_2+2} = 3 - \frac{2}{T_2+2} \in [2,3), \quad a = \frac{4T_2+4}{T_2+2} = 4 - \frac{4}{T_2+2} \in [2,4). \tag{32}$$

Using these, the revenue of SP 1 is given by $R_1^* = p_1^* x_1^* = \frac{T_1(T_1+1)}{(bT_1+a)^2}$ so that

$$\frac{\partial R_1}{\partial T_1} = \frac{(2T_1+1)(bT_1+a) - 2bT_1(T_1+1)}{(bT_1+a)^3} = \frac{(2a-b)T_1 + a}{(bT_1+a)^3},$$

and

$$\frac{\partial^2 R_1}{\partial T_1^2} = \frac{(2a-b)(bT_1+a) - 3b\big((2a-b)T_1+a\big)}{(bT_1+a)^4} = \frac{-2b(2a-b)T_1 - 2a(2b-a)}{(bT_1+a)^4}.$$

Using the expressions for a and b, we get

$$2a - b = 5 - \frac{6}{T_2+2} \in [2,5), \text{ and } 2b - a = 2.$$

Therefore, the revenue of provider 1 is strictly concave and increasing in T_1 for any given value of T_2.

Both b and a are increasing in T_2 and so it follows that R_1^* is decreasing in T_2. ■

A.5 Proof of Theorem 3.5

Consider allocating an arbitrarily small amount of shared spectrum W. Then in a second price auction the bid of SP i, b_i will be proportional to

$$b_i = \frac{\partial R_i}{\partial T_i} m_i - \frac{\partial R_i}{\partial T_j} m_j$$

for $j \neq i$. Here, as in Lemma 3.2, m_i is the marginal change in T_i per unit change in W. Further, using Lemma 3.2, since initially $W_i = 0$ for both SPs, we have $m_1 = m_2 = \alpha$. Hence, determining the winner of the second price auction involves seeing which SP has the larger value of

$$b_i/\alpha = \frac{\partial R_i}{\partial T_i} - \frac{\partial R_i}{\partial T_j}.$$

Likewise, since initially $W_i = 0$, we have $T_i = B_i$ for each SP. Using the expressions for $R_i(T_1, T_2)$ from the Proof of Theorem 3.3, it can then be shown that

$$\lim_{B_1 \to \infty} b_1/\alpha = \frac{4}{27B_2^2}.$$

Likewise, we have,

$$\lim_{B_1 \to \infty} b_2/\alpha = \frac{(2a_2 - b_2)B_2 + a_2}{(b_2 B_2 + a_2)^3}$$

where a_2 and b_2 are the corresponding parameters from the proof of Theorem 3.3. Finally note that as B_2 then becomes small, the limiting expression for b_1/α grows without bound, while the limiting expression for b_2/α approaches 1/4. The claim in the theorem then follows. ■

A.6 Proof of Theorem 3.7

Let $S(T_1, T_2) = x_1^* + x_2^*$ denote the total number of customers served in equilibrium, as a function of T_1 and T_2, which from (16) satisfies

$$S(T_1, T_2) = \frac{2T_1T_2 + 2(T_1 + T_2)}{3T_1T_2 + 4 + 4(T_1 + T_2)}.$$

By direct calculation, it can be seen that $S(T_1, T_2)$ is jointly concave. The consumer welfare optimal allocation is then the solution to the optimization

$$\begin{aligned} &\max S(T_1, T_2) \\ \text{subject to: } &W_1 + W_2 \leq W, \\ &W_i \geq 0, \forall i. \end{aligned} \tag{33}$$

Recall, the relation between T_i and W_i is given by Lemma 3.1. We use this to prove each of the properties in this theorem:

1.) Note that $S(T_1, T_2)$ is increasing in each T_i and each T_i is increasing in W_i. Hence, to maximize $S(T_1, T_2)$ it must be that $W_1 + W_2 = W$.

2.) When $\alpha = 0$, $T_i = B_i + W_i$, and so the constraints in (33) can be re-written in terms of the T_i's as

$$\begin{aligned} T_1 + T_2 &= B_1 + B_2 + W, \\ T_i &\geq B_i, \forall i \end{aligned} \tag{34}$$

where we have also used the first property to write the first constraint as an equality. From the symmetry of S in T_1 and T_2 and the first order optimality conditions, the result then follows.

3.) Suppose that at optimality both users have $W_i > 0$. Then the first order optimality conditions of (33) can be written for $i = 1, 2$ as

$$\frac{\partial S}{\partial T_i}\frac{\partial T_i}{\partial W_i} = \lambda.$$

Using the symmetry and concavity of S and the properties of $\frac{\partial T_i}{\partial W_i}$ in Lemma 3.2, it can be seen that the only way this can occur if $B_1 = B_2$ is for $W_1 = W_2$. Likewise, using these properties it can be seen that at optimality both users should have $W_i > 0$.

4.) Suppose that $\alpha < 1$ and $B_1 > B_2$ and the unlicensed spectrum is allocated so that $T_1 \leq T_2$. Then from Property 5 in Lemma 3.2, it follows that the welfare can be improved by increasing W_1 and decreasing W_2. Hence, at optimality it must be that $T_1 > T_2$.

5.) From Property 4 in Lemma 3.2, if an infinitesimal amount,δ of spectrum is allocated, then $\frac{\partial T_i}{\partial W_i} \approx \alpha\delta$ for each SP i. Hence the optimal allocation is to given this to the SP for which $\frac{\partial S}{\partial T_i}$ is larger, which from the concavity of S will be the smaller SP.

A.7 Proof of Theorem 3.8

If an interior equilibrium exists, then:

$$\begin{aligned}
p_i &= 1 - x_i(1 + \tfrac{1}{B_i}) - x_{-i} - w_i(1 + \tfrac{(1-\alpha)\beta}{B_i}) - w_{-i} \\
p_i^w &= 1 - x_i(1 + \tfrac{1-\alpha}{B_i}) - x_{-i} - w_i(1 + \tfrac{\alpha}{W} + \tfrac{(1-\alpha)\beta}{B_i}) - w_{-i}(1 + \tfrac{\alpha}{W})
\end{aligned}$$

Using this, the revenue of provider i is:

$$\begin{aligned}
R_i &= p_i x_i + p_i^w w_i \\
&= (1 - x_{-i} - w_{-i})x_i - x_i w_i(2 + \tfrac{(1-\alpha)(1+\beta)}{B_i}) - x_i^2(1 + \tfrac{1}{B_i}) + (1 - x_{-i} - w_{-i}(1 + \tfrac{\alpha}{W}))w_i \\
&\quad - w_i^2(1 + \tfrac{\alpha}{W} + \tfrac{(1-\alpha)\beta}{B_i})
\end{aligned}$$

Assuming the revenue is jointly concave in (x_i, w_i) (which is true for $\beta = 1$ the case of interest), the best response functions are obtained by setting the following (partial) derivatives to 0, namely,

$$\begin{aligned}
\frac{\partial R_i}{\partial x_i} &= 1 - x_{-i} - w_{-i} - w_i(2 + \tfrac{(1-\alpha)(1+\beta)}{B_i}) - 2x_i(1 + \tfrac{1}{B_i}) \\
\frac{\partial R_i}{\partial w_i} &= 1 - x_{-i} - w_{-i}(1 + \tfrac{\alpha}{W}) - x_i(2 + \tfrac{(1-\alpha)(1+\beta)}{B_i}) - 2w_i(1 + \tfrac{\alpha}{W} + \tfrac{(1-\alpha)\beta}{B_i}).
\end{aligned}$$

In the symmetric case of $B_1 = B_2 = B/2$, we search for a symmetric equilibrium using the above to get the following linear equations

$$\begin{aligned}
x(3 + \tfrac{4}{B}) + w(3 + \tfrac{2(1-\alpha)(1+\beta)}{B}) &= 1 \\
x(3 + \tfrac{2(1-\alpha)(1+\beta)}{B}) + w(3 + \tfrac{3\alpha}{W} + \tfrac{4(1-\alpha)\beta}{B}) &= 1
\end{aligned}$$

Solving this yields the quantities in the theorem. The resulting prices are

$$\begin{aligned}
p_1 = p_2 &= 1 - x(2 + \tfrac{2}{B}) - w(2 + \tfrac{2(1-\alpha)}{B}) \\
p_1^w = p_2^w &= 1 - x(2 + \tfrac{2(1-\alpha)}{B}) - w(2 + \tfrac{2\alpha}{W} + \tfrac{2(1-\alpha)}{B})
\end{aligned}$$

From the equations of the equilibrium, it can be gleaned that the prices are positive. Thus, an interior equilibrium exists.

A.8 Proof of Lemma 4.1 and Theorem 4.2.

For the analysis we assume that $\min\big(B+(1-\beta)W, \beta W\big) > 0$. The results for the equilibrium quantities when this condition does not hold follow by continuity with the additional assumption that $B=0$ is necessarily accompanied with $\alpha = 1$.

The revenue of SP i is

$$\begin{aligned} R_i =& p_i x_i + p_i^w w_i \\ =& \left(1 - \sum_{j=1}^N x_j - \sum_{j=1}^N w_j - \alpha N \frac{x_i}{B+(1-\beta)W} - (1-\alpha)N\frac{w_i+x_i}{B}\right) x_i \\ &+ \left(1 - \sum_{j=1}^N x_j - \sum_{j=1}^N w_j - \alpha \frac{\sum_{j=1}^N w_j}{\beta W} - (1-\alpha)N\frac{w_i+x_i}{B}\right) w_i \end{aligned}$$

Taking the derivative of R_i with respect to x_i

$$\begin{aligned} \frac{\partial R_i}{\partial x_i} =& \left(1 - \sum_{j=1}^N x_j - \sum_{j=1}^N w_j - \alpha N \frac{x_i}{B+(1-\beta)W} - (1-\alpha)N\frac{w_i+x_i}{B}\right) \\ &- \left(1 + \alpha N \frac{1}{B+(1-\beta)W} + (1-\alpha)N\frac{1}{B}\right) x_i - (1 + \frac{(1-\alpha)N}{B}) w_i \end{aligned}$$

We will derive a symmetric equilibrium. It will follow from a subsequent theorem, Theorem 6.1, that the unique equilibrium is symmetric. If we set $x_i = x$ and $w_i = w$ for all i, we get:

$$\begin{aligned} 1 =& \left(Nx + Nw + \alpha N \frac{x}{B+(1-\beta)W} + (1-\alpha)N\frac{x+w}{B}\right) \\ &+ \left(1 + \alpha N \frac{1}{B+(1-\beta)W} + (1-\alpha)N\frac{1}{B}\right) x + (1 + \frac{(1-\alpha)N}{B}) w \end{aligned}$$

This implies

$$1 = x\left(1 + N + 2N\frac{\alpha}{B+(1-\beta)W} + 2N\frac{(1-\alpha)}{B}\right) + w\left(1 + N + 2N\frac{1-\alpha}{B}\right) \tag{35}$$

Taking derivative with respect to w_i

$$\begin{aligned} \frac{\partial R_i}{\partial w_i} =& -\left(1 + \frac{(1-\alpha)N}{B}\right) x_i \\ &+ \left(1 - \sum_{j=1}^N x_j - \sum_{j=1}^N w_j - \alpha \frac{\sum_{j=1}^N w_j}{\beta W} - (1-\alpha)N\frac{w_i+x_i}{B}\right) - \left(1 + \frac{\alpha}{\beta W} + \frac{(1-\alpha)N}{B}\right) w_i \end{aligned}$$

Using an argument similar to that above we get

$$1 = x\left(1 + N + 2N\frac{1-\alpha}{B}\right) + w\left(1 + \frac{\alpha}{\beta W} + N + N\frac{\alpha}{\beta W} + 2N\frac{1-\alpha}{B}\right) \tag{36}$$

Solving for the equilibrium using (35) and (36), we obtain

$$\begin{aligned} \bar{x}(N) &= \frac{\frac{\alpha(N+1)}{\beta W}}{\left(1 + N + \frac{2(1-\alpha)N}{B}\right)\left(\frac{2\alpha N}{B+(1-\beta)W} + \frac{\alpha(N+1)}{\beta W}\right) + \frac{2\alpha N}{B+(1-\beta)W}\frac{\alpha(N+1)}{\beta W}}, \\ \bar{w}(N) &= \frac{\frac{2\alpha N}{B+(1-\beta)W}}{\left(1 + N + \frac{2(1-\alpha)N}{B}\right)\left(\frac{2\alpha N}{B+(1-\beta)W} + \frac{\alpha(N+1)}{\beta W}\right) + \frac{2\alpha N}{B+(1-\beta)W}\frac{\alpha(N+1)}{\beta W}}. \end{aligned}$$

Rearranging we get

$$N\bar{x}(N) = \frac{B^2 + (1-\beta)BW}{\left(B\frac{1+N}{N} + 2(1-\alpha)\right)\left(2\beta W\frac{N}{N+1} + B + (1-\beta)W\right) + 2\alpha B}, \tag{37}$$

$$N\bar{w}(N) = \frac{2\beta BW\frac{N}{N+1}}{\left(B\frac{1+N}{N} + 2(1-\alpha)\right)\left(2\beta W\frac{N}{N+1} + B + (1-\beta)W\right) + 2\alpha B}. \tag{38}$$

The results then follow in a straight-forward manner by taking the limit as N increases to infinity. We also note that the congestion in the shared band is exactly $\frac{2N}{N+1}$ times the congestion in the proprietary bands. ■

A.8.1 Consumer Surplus

The total traffic carried $\rho(N)$ is then

$$\begin{aligned}\rho(N) &= N\bar{x}(N) + N\bar{w}(N)\\ &= \frac{B^2 + \left(1+\beta\frac{N-1}{N+1}\right)BW}{\left(B\frac{1+N}{N} + 2(1-\alpha)\right)\left(2\beta W\frac{N}{N+1} + B + (1-\beta)W\right) + 2\alpha B}\\ &= \frac{B\left(B + \left(1+\beta\frac{N-1}{N+1}\right)W\right)}{\left(B + \frac{B}{N} + 2(1-\alpha)\right)\left(B + \left(1+\beta\frac{N-1}{N+1}\right)W\right) + 2\alpha B},\end{aligned}$$

which is of the form $\frac{af(\beta)}{(a+b)f(\beta)+c}$ where $f(\beta)$ is an increasing function of β, and $a, b, c > 0$. Therefore, it is immediate that $\rho(N)$ is also an increasing function of β, and so is maximized at $\beta = 1$.

Next we present the proof of Lemma 4.4, which studies consumer surplus with degraded shared spectrum. When we allow the shared spectrum to get degraded, i.e., reduce by a factor d, then we can use the formulae in (37) and (38) to analyze the equilibrium by setting the shared bandwidth βW to $d\beta W$. Define $d^* = \frac{2dN}{N+1}$. Then the congestion in the shared band is still $\frac{2N}{N+1}$ times the congestion in the proprietary band, i.e., denoting $(\bar{x}_\beta(N), \bar{w}_\beta(N))$ as the equilibrium quantities have the following expressions:

$$\begin{aligned}N\bar{x}_\beta(N) &= \frac{B^2 + (1-\beta)BW}{\left(B\frac{1+N}{N} + 2(1-\alpha)\right)\left(2d\beta W\frac{N}{N+1} + B + (1-\beta)W\right) + 2\alpha B},\\ N\bar{w}_\beta(N) &= \frac{2d\beta BW\frac{N}{N+1}}{\left(B\frac{1+N}{N} + 2(1-\alpha)\right)\left(2d\beta W\frac{N}{N+1} + B + (1-\beta)W\right) + 2\alpha B},\\ \Leftrightarrow \quad \frac{N\bar{w}_\beta(N)}{d\beta W} &= \frac{2N}{N+1}\frac{N\bar{x}_\beta(N)}{B + (1-\beta)W},\end{aligned}$$

where we used the increased latency in the shared band for the comparison. Using this we have the total traffic served $\rho_\beta(N)$ is then

$$\begin{aligned}\rho_\beta(N) &= N\bar{x}_\beta(N) + N\bar{w}_\beta(N)\\ &= \frac{B\left(B + (1+\beta(d^*-1))W\right)}{\left(B + \frac{B}{N} + 2(1-\alpha)\right)\left(B + (1+\beta(d^*-1))W\right) + 2\alpha B},\end{aligned}$$

which is an increasing function of β if $d^* > 1$ and a decreasing function of β if $d^* < 1$. This conclusion then directly implies that the consumer surplus is maximized at $\beta = 1$ if $d^* > 1$ and at $\beta = 0$ if $d^* < 1$.

A.8.2 Total Surplus

Assuming no degradation of the shared band the revenue of SP i is

$$\begin{aligned}
R_i &= \left(1 - N\bar{x}(N) - N\bar{w}(N) - \alpha N \frac{\bar{x}(N)}{B + (1-\beta)W} - (1-\alpha)N\frac{\bar{w}(N) + \bar{x}(N)}{B}\right)\bar{x}(N) \\
&+ \left(1 - N\bar{x}(N) - N\bar{w}(N) - \alpha \frac{N\bar{w}(N)}{\beta W} - (1-\alpha)N\frac{\bar{w}(N) + \bar{x}(N)}{B}\right)\bar{w}(N) \\
&= \left(1 - \rho(N) - \alpha N \frac{\bar{x}(N)}{B + (1-\beta)W} - (1-\alpha)\frac{\rho(N)}{B}\right)\bar{x}(N) \\
&+ \left(1 - \rho(N) - \alpha \frac{N\bar{w}(N)}{\beta W} - (1-\alpha)\frac{\rho(N)}{B}\right)\bar{w}(N)
\end{aligned}$$

Hence, total revenue is

$$(1-\rho(N))\rho(N) - \frac{(1-\alpha)}{B}\rho^2(N) - \frac{\alpha N^2 \bar{x}(N)^2}{B + (1-\beta)W} - \frac{\alpha N^2 \bar{w}(N)^2}{\beta W}.$$

As consumer surplus is $\frac{\rho(N)^2}{2}$ it follows that total surplus is

$$\rho(N) - \frac{\rho(N)^2}{2} - \frac{(1-\alpha)}{B}\rho^2(N) - \frac{\alpha N^2 \bar{x}(N)^2}{B + (1-\beta)W} - \frac{\alpha N^2 \bar{w}(N)^2}{\beta W}.$$

Plots for various parameter values suggest that social welfare is a convex function of β for each N. If true, maximization over β is achieved at one of the endpoints, i.e., either 0 or 1. The maximizer $\beta^*(N)$ is initially 0, and it jumps to 1 and remains there for large enough N. Most examples show this is between 2 and 3 (see Figure 4).

A.9 Proof of Theorem 4.3

Recall that the total traffic carried is given by

$$\rho := x^* + w^* = \frac{B\big(B + W(1+\beta)\big)}{\big(B + 2(1-\alpha)\big)\big(B + W(1+\beta)\big) + 2B\alpha}.$$

It is straightforward to verify that the total quantity carried, and hence, the consumer surplus are both maximized at $\beta = 1$.

Now social welfare as a function of β denoted $SW(\beta)$ is given by:

$$SW(\beta) = \frac{(B + W + W\beta)^2(\frac{B^2}{2} + B(1-\alpha)) + B^2\alpha(B + W - W\beta).}{\left[(B + W + W\beta)(B + 2(1-\alpha)) + 2B\alpha\right]^2}.$$

The derivative of $SW(\beta)$ with respect to β when set to zero has a unique solution, β^*. For $0 \le \beta < \beta^*$ the derivative is negative, for $\beta^* < \beta \le 1$, it is positive.[9] Thus to find the maximum value it is sufficient to compare the values at the two extremes. Now,

$$SW(0) = \frac{(B + W)^2(\frac{B^2}{2} + B(1-\alpha)) + B^2\alpha(B + W)}{\left[(B + W)(B + 2(1-\alpha)) + 2B\alpha\right]^2}$$

and

$$SW(1) = \frac{(B + 2W)^2(\frac{B^2}{2} + B(1-\alpha)) + B^3\alpha)}{\left[(B + 2W)(B + 2(1-\alpha)) + 2B\alpha\right]^2}.$$

[9] As $SW(\beta)$ is the ratio of two affine functions with a positive denominator, it is quasi-convex.

Now $SW(0) > SW(1)$ implies

$$\frac{(B+W)^2(B+2(1-\alpha))+2B\alpha(B+W)}{\left[(B+W)(B+2(1-\alpha))+2B\alpha\right]^2} > \frac{(B+2W)^2(B+2(1-\alpha))+2B^2\alpha)}{\left[(B+2W)(B+2(1-\alpha))+2B\alpha\right]^2}.$$

$$\Rightarrow \frac{B+2(1-\alpha)+\frac{2B\alpha}{B+W}}{(B+2(1-\alpha)+\frac{2B\alpha}{B+W})^2} > \frac{B+2(1-\alpha)+\frac{2B^2\alpha}{(B+2W)^2}}{(B+2(1-\alpha)+\frac{2B\alpha}{B+2W})^2}$$

$$\Rightarrow (B+2(1-\alpha)+\frac{2B\alpha}{B+2W})^2 > (B+2(1-\alpha)+\frac{2B\alpha}{B+W})(B+2(1-\alpha)+\frac{2B^2\alpha}{(B+2W)^2})$$

If we let $C = \frac{B+2(1-\alpha)}{2B\alpha}$ the expression above simplifies to

$$(C+\frac{1}{B+2W})^2 > (C+\frac{1}{B+W})(C+\frac{B}{(B+2W)^2})$$

which is clearly true.

A.10 Proof of Theorem 5.1

We will assume that there are $B > 0$ units of always available spectrum and $W > 0$ units of intermittent spectrum with availability $\alpha \in (0,1]$. We will also assume that there is one "big" SP (labeled as 1) and N "small" ones (for $N \geq 1$) with the ith small provider labeled as $(2,i)$. The allocation of resources is as specified below:

1. The big SP owns the license to B_1 units of always available spectrum and W_1 units of intermittent spectrum;
2. Each of the small SPs owns the license to B_2/N units of always available spectrum and W_2/N units of intermittent spectrum;
3. βW units of intermittent spectrum is available to use by all the involved providers (big and small) as shared spectrum for $\beta \in [0,1]$.

We will insist that $B_1, B_2 \geq 0$ with $B_1+B_2 = B$, and also that $W_1, W_2 \geq 0$ with $W_1+W_2 = (1-\beta)W$. However, for ease of analysis we will assume that $B_1, B_2, W_1, W_2 > 0$ and $\beta \in (0,1)$.

We will denote the amounts served by provider 1 as x_1 in licensed spectrum and w_1 in shared spectrum. The corresponding quantities for the ith small provider are $x_{2,i}$ and $w_{2,i}$, respectively. Then we have the following expressions for the revenue of the providers:

$$\begin{aligned}
R_1 = &\left(1 - x_1 - \sum_{j=1}^{N} x_{2,j} - \frac{\alpha x_1}{B_1+W_1} - \frac{(1-\alpha)(x_1+w_1)}{B_1} - w_1 - \sum_{j=1}^{N} w_{2,j}\right) x_1 \\
&+ \left(1 - x_1 - \sum_{j=1}^{N} x_{2,j} - \frac{\alpha\left(w_1 + \sum_{j=1}^{N} w_{2,j}\right)}{\beta W} - \frac{(1-\alpha)(x_1+w_1)}{B_1} - w_1 - \sum_{j=1}^{n} w_{2,j}\right) w_1 \\
R_{2,i} = &\left(1 - x_1 - \sum_{j=1}^{N} x_{2,j} - \frac{\alpha N x_{2,i}}{B_2+W_2} - \frac{(1-\alpha)n(x_{2,i}+w_{2,i})}{B_2} - w_1 - \sum_{j=1}^{N} w_{2,j}\right) x_{2,i} \\
&+ \left(1 - x_1 - \sum_{j=1}^{N} x_{2,j} - \frac{\alpha\left(w_1 + \sum_{j=1}^{N} w_{2,j}\right)}{\beta W} - \frac{(1-\alpha)N(x_{2,i}+w_{2,i})}{B_2} - w_1 - \sum_{j=1}^{N} w_{2,j}\right) w_{2,i}
\end{aligned}$$

Taking partial derivatives, and then setting $x_{2,i} \equiv x_2$ and $w_{2,i} \equiv w_2$ (the response of all the small SPs will be the same at equilibrium as can be argued from the symmetry of the potential function)

we get the partial derivatives in the quantities as

$$\frac{\partial R_1}{\partial x_1} = 1 - 2x_1\left(1 + \frac{\alpha}{B_1 + W_1} + \frac{1-\alpha}{B_1}\right) - 2w_1\left(1 + \frac{1-\alpha}{B_1}\right) - Nx_2 - Nw_2$$
$$\frac{\partial R_1}{\partial w_1} = 1 - 2x_1\left(1 + \frac{1-\alpha}{B_1}\right) - 2w_1\left(1 + \frac{\alpha}{\beta W} + \frac{1-\alpha}{B_1}\right) - Nx_2 - Nw_2\left(1 + \frac{\alpha}{\beta W}\right)$$
$$\frac{\partial R_{2,i}}{\partial x_2} = 1 - x_1 - w_1 - Nx_2\left(1 + \frac{1}{N} + \frac{2\alpha}{B_2 + W_2} + \frac{2(1-\alpha)}{B_2}\right) - Nw_2\left(1 + \frac{1}{N} + \frac{2(1-\alpha)}{B_2}\right)$$
$$\frac{\partial R_{2,i}}{\partial w_2} = 1 - x_1 - w_1\left(1 + \frac{\alpha}{\beta W}\right) - Nx_2\left(1 + \frac{1}{N} + \frac{2(1-\alpha)}{B_2}\right) - Nw_2\left(1 + \frac{1}{N} + \frac{\alpha}{\beta W} + \frac{2(1-\alpha)}{B_2}\right)$$

The equilibrium quantities are the unique set of non-negative numbers (x_1, w_2, x_2, w_2) such at

$$\frac{\partial R_1}{\partial x_1} \leq 0, \frac{\partial R_1}{\partial w_1} \leq 0, \qquad \frac{\partial R_{2,i}}{\partial x_2} \leq 0, \frac{\partial R_{2,i}}{\partial w_2} \leq 0 \quad \forall i = 1, \ldots, N$$
$$\frac{\partial R_1}{\partial x_1} x_1 = \frac{\partial R_1}{\partial w_1} w_1 = 0, \qquad \frac{\partial R_{2,i}}{\partial x_2} x_2 = \frac{\partial R_{2,i}}{\partial w_2} w_2 = 0 \quad \forall i = 1, \ldots, N$$

Note that the inequalities give the set of the first-order conditions for maximizing the potential function, and the equations the set of complementary slackness conditions for the non-negativity constraints.

Next we will take the limit of $N \to \infty$ where we will identify the equilibrium quantities as x_1^*, w_1^*, x_2^* and w_2^* with the understanding[10] that $\lim_{n\to\infty}(x_1, w_1, Nx_2, Nw_2) = (x_1^*, w_1^*, x_2^*, w_2^*)$. We will denote the limiting values of the derivatives by $\Delta_x R_1$, $\Delta_w R_1$, $\Delta_x R_2$ and $\Delta_w R_2$, respectively. Then we have

$$\Delta_x R_1 = 1 - 2x_1^*\left(1 + \frac{\alpha}{B_1 + W_1} + \frac{1-\alpha}{B_1}\right) - 2w_1^*\left(1 + \frac{1-\alpha}{B_1}\right) - x_2^* - w_2^*$$
$$\Delta_w R_1 = 1 - 2x_1^*\left(1 + \frac{1-\alpha}{B_1}\right) - 2w_1^*\left(1 + \frac{\alpha}{\beta W} + \frac{1-\alpha}{B_1}\right) - x_2^* - w_2^*\left(1 + \frac{\alpha}{\beta W}\right)$$
$$\Delta_x R_2 = 1 - x_1^* - w_1^* - x_2^*\left(1 + \frac{2\alpha}{B_2 + W_2} + \frac{2(1-\alpha)}{B_2}\right) - w_2^*\left(1 + \frac{2(1-\alpha)}{B_2}\right)$$
$$\Delta_w R_2 = 1 - x_1^* - w_1^*\left(1 + \frac{\alpha}{\beta W}\right) - x_2^*\left(1 + \frac{2(1-\alpha)}{B_2}\right) - w_2^*\left(1 + \frac{\alpha}{\beta W} + \frac{2(1-\alpha)}{B_2}\right)$$

We will also have

$$\Delta_x R_1 \leq 0, \Delta_w R_1 \leq 0, \qquad \Delta_x R_2 \leq 0, \Delta_w R_2 \leq 0 \quad \forall i = 1, \ldots, n$$
$$\Delta_x R_1 x_1^* = \Delta_w R_1 w_1^* = 0, \qquad \Delta_x R_2 x_2^* = \Delta_w R_2 w_2^* = 0 \quad \forall i = 1, \ldots, n$$

Given the asymmetry between the SP 1 and the small ones, we will have to consider the possibility of SP 1 not using the shared spectrum. Using the asymptotic equilibrium quantities we will next provide[11] an inequality for the parameters which when satisfied will imply the existence of an N^* such that for all $N \geq N^*$, in equilibrium SP 1 will abandon the shared spectrum. If the parameters are such that the inequality does not hold, then we will always have an interior point equilibrium for any n but with the possibility that the limiting w_1^* is zero. It is easily argued that x_1^*, x_2^*, w_2^* have to be positive.

The results can summarized as follows:

[10] From the uniqueness of the solutions that any limit point has to satisfy, it is easily verified that the limits exist: existence of limit points holds from compactness and uniqueness of solutions using a potential function proves the remainder.

[11] A full proof is omitted as the logic is exactly the same as in the proof of Theorem 3.10.

1. If the parameters are such that

$$B_1 + W_1 + 2(1-\alpha)\frac{W_1}{B_1} > 2(1-\alpha)\frac{2\beta W + W_2}{B_2}, \tag{39}$$

then there exists an N^* such that for all $N \geq N^*$, SP 1 abandons the shared spectrum so that the asymptotic equilibrium quantities are $(x_1^*, w_1^* = 0, x_2^*, w_2^*)$ where (x_1^*, x_2^*, w_2^*) are obtained as the solution to

$$\begin{aligned}
1 &= 2x_1^*\left(1 + \frac{\alpha}{B_1+W_1} + \frac{1-\alpha}{B_1}\right) + x_2^* + w_2^* \\
1 &= x_1^* + x_2^*\left(1 + \frac{2\alpha}{B_2+W_2} + \frac{2(1-\alpha)}{B_2}\right) + w_2^*\left(1 + \frac{2(1-\alpha)}{B_2}\right) \\
1 &= x_1^* + x_2^*\left(1 + \frac{2(1-\alpha)}{B_2}\right) + w_2^*\left(1 + \frac{\alpha}{\beta W} + \frac{2(1-\alpha)}{B_2}\right)
\end{aligned} \tag{40}$$

2. If, instead, we have

$$B_1 + W_1 + 2(1-\alpha)\frac{W_1}{B_1} \leq 2(1-\alpha)\frac{2\beta W + W_2}{B_2}, \tag{41}$$

then for all N we have an interior point equilibrium so that the asymptotic equilibrium quantities $(x_1^*, w_1^*, x_2^*, w_2^*)$ solve

$$\begin{aligned}
1 &= 2x_1^*\left(1 + \frac{\alpha}{B_1+W_1} + \frac{1-\alpha}{B_1}\right) + 2w_1^*\left(1 + \frac{1-\alpha}{B_1}\right) + x_2^* + w_2^* \\
1 &= 2x_1^*\left(1 + \frac{1-\alpha}{B_1}\right) + 2w_1^*\left(1 + \frac{\alpha}{\beta W} + \frac{1-\alpha}{B_1}\right) + x_2^* + w_2^*\left(1 + \frac{\alpha}{\beta W}\right) \\
1 &= x_1^* + w_1^* + x_2^*\left(1 + \frac{2\alpha}{B_2+W_2} + \frac{2(1-\alpha)}{B_2}\right) + w_2^*\left(1 + \frac{2(1-\alpha)}{B_2}\right) \\
1 &= x_1^* + w_1^*\left(1 + \frac{\alpha}{\beta W}\right) + x_2^*\left(1 + \frac{2(1-\alpha)}{B_2}\right) + w_2^*\left(1 + \frac{\alpha}{\beta W} + \frac{2(1-\alpha)}{B_2}\right)
\end{aligned} \tag{42}$$

Note that equality in (41) implies that $w_1^* = 0$ so that asymptotically SP 1 reduces the quantity served in shared spectrum to 0.

Note that inequality in (41) resembles the condition from Theorem 3.10, but with a few terms on the RHS omitted owing to many small providers assumption. It is easily verified at $\alpha = 1$ that the big SP always vacates the shared spectrum. It is also easily verified that at $\alpha = 1$, the price in the shared spectrum is 0 in the limit (LHS-RHS of the third equation in (40) is the price) so that we get the same results as Bertrand competition.

Now consider the second scenario with asymmetric providers, and let x_{ij} and w_{ij} denote the quantities in the proprietary and shared bands, respectively, for provider i in subset j. The announced prices for provider i in subset j are

$$p_{ij} = 1 - \sum_{j=1}^{2}\sum_{i}(x_{ij} + w_{ij}) - (1-\alpha)\frac{x_{ij} + w_{ij}}{B_j/n_j} - \alpha\frac{x_{ij}}{(B_j + W_j)/n_j} \tag{43}$$

$$p_{ij}^w = 1 - \sum_{j=1}^{2}\sum_{i}(x_{ij} + w_{ij}) - \alpha\frac{\sum_{j=1}^{2}\sum_i w_{ij}}{\beta W} - (1-\alpha)\frac{x_{ij} + w_{ij}}{B_j/n_j}. \tag{44}$$

and the corresponding revenue is $R_{ij} = p_{ij}x_{ij} + p_{ij}^w w_{ij}$. In what follows we will drop the i subscript since the equilibrium values will be the same within each subset of providers.

Evaluating the first-order conditions for best response and letting $N \to \infty$ gives

$$\left(1 + \frac{2(1-\alpha)}{B_j} + \frac{2\alpha}{B_j + W_j}\right)\bar{x}_j + \left(1 + \frac{2(1-\alpha)}{B_j}\right)\bar{w}_j + \bar{x}_{\bar{j}} + \bar{w}_{\bar{j}} = 1 \tag{45}$$

$$\left(1 + \frac{2(1-\alpha)}{B_j}\right)\bar{x}_j + \left(1 + \frac{2(1-\alpha)}{B_j} + \frac{\alpha}{\beta W}\right)\bar{w}_1 + (1 + \frac{\alpha}{\beta W})\bar{w}_{\bar{j}} + \bar{x}_{\bar{j}} = 1 \tag{46}$$

where $\bar{x}_j = n_j x_j$, $\bar{w}_j = n_j w_j$, $j, \bar{j} \in \{1, 2\}$ and $j \neq \bar{j}$. Note that x_j and w_j each tend to zero as $N \to \infty$, but $\bar{x}_j$ and $\bar{w}_j$ converge to nonnegative constants.

The preceding conditions apply provided that $\bar{x}_j$ and $\bar{w}_j$ are nonnegative. Otherwise, the providers in one of the subsets do not make use of the shared band, i.e., $\bar{w}_i = 0$ for some i. In that scenario, we have $\partial R_i / \partial w_i < 0$ at $w_i = 0$, which gives

$$\bar{x}_i \left(1 + \frac{2(1-\alpha)}{B_i}\right) + \bar{x}_{\bar{i}} + \bar{w}_{\bar{i}} \left(1 + \frac{\alpha}{\beta W}\right) > 1 \tag{47}$$

where $\bar{x}_j$, $j = 1, 2$, and $\bar{w}_{\bar{i}}$ are determined from the three conditions (45) with $j = 1, 2$ and (46) with $j = \bar{i}$. Combining (47) with the latter conditions gives the condition in Proposition 5.1. Note that the condition resembles (41) from above, but now with a few terms on the LHS omitted owing to the many providers setting (the $2(1-\alpha)$ term is then cancelled on both sides). In contrast to the first scenario with one large SP, here the condition (47) can be satisfied for either a large ($i = 1$) or small ($i = 2$) SP.

A.11 Proof of Theorem 6.1

We show that under fairly general conditions the game with N providers has a unique Nash equilibrium, and with some restrictions we also obtain a potential game. Assume there are N firms and assume that prices can be negative; in equilibrium the prices will be non-negative. We divide the proof up into several cases.

A.11.1 Linear inverse demand and latency

With linear inverse demand and linear latency, W_0 units of intermitted secondary band set aside for unlicensed access, and assuming that firm $i \in \{1, 2, \ldots, N\}$ has B_i units of always available spectrum, W_i units of the intermittent secondary band, the utility of firm $i \in \{1, 2, \ldots, N\}$ is given by

$$\begin{aligned} u_i(y_i, w_i, \mathbf{y}_{-i}, \mathbf{w}_{-i}) &= \left(1 - \sum_{j=1}^{N} y_j\right) y_i - (1-\alpha)\frac{y_i}{B_i} y_i - \alpha \frac{y_i - w_i}{B_i + W_i}(y_i - w_i) - \alpha \frac{\sum_{j=1}^{N} w_j}{W_0} w_i \\ &= y_i - y_i^2 - \sum_{j \in -i} y_j y_i - (1-\alpha)\frac{y_i^2}{B_i} - \alpha \frac{(y_i - w_i)^2}{B_i + W_i} - \alpha \frac{w_i^2}{W_0} - \alpha \frac{\sum_{j \in -i} w_j w_i}{W_0} \end{aligned}$$

Define the following function $\Phi(\mathbf{y}, \mathbf{w})$ given by

$$\begin{aligned} \Phi(\mathbf{y}, \mathbf{w}) = &\sum_{i=1}^{N} y_i - \sum_{i=1}^{N} y_i^2 \left(1 + \frac{1-\alpha}{B_i}\right) - \sum_{i=2}^{N} \sum_{j=1}^{i-1} y_i y_j - \alpha \sum_{i=1}^{N} \frac{(y_i - w_i)^2}{B_i + W_i} \\ &- \frac{\alpha}{W_0} \sum_{i=1}^{N} w_i^2 - \frac{\alpha}{W_0} \sum_{i=2}^{N} \sum_{j=1}^{i-1} w_i w_j \end{aligned}$$

Then it is easily verified that

$$\Phi(y_i^1, w_i^1, \mathbf{y}_{-i}, \mathbf{w}_{-i}) - \Phi(y_i^2, w_i^2, \mathbf{y}_{-i}, \mathbf{w}_{-i}) = u_i(y_i^1, w_i^1, \mathbf{y}_{-i}, \mathbf{w}_{-i}) - u_i(y_i^2, w_i^2, \mathbf{y}_{-i}, \mathbf{w}_{-i})$$

Therefore, we have a potential game. Furthermore, it is easily verified that $\Phi(\mathbf{y}, \mathbf{w})$ is jointly concave in $\mathbf{y}$, $\mathbf{w}$ with the Hessian positive definite if $\alpha > 0$. If the unique maximum (under our convex and

compact constraint set) also leads to non-negative prices, then it is the equilibrium. In fact, one can impose non-negative prices as constraints on the actions, and then the resulting unique maximum is a generalized equilibrium [24].

A.11.2 Linear inverse demand and convex latency

We can generalize the potential game characterization to the case where all providers have proprietary latency functions that are convex and (strictly) increasing. However, we still have to assume that the inverse demand function and the latency function in whitespace are both linear. A fact that we will use is the following: $l(x)$ convex and non-decreasing for $x \geq 0$ implies that $xl(x)$ is also convex, and $l(x)$ being monotonically increasing implies that $xl(x)$ is strictly convex. Again assume that W_0 units of the intermittent secondary band is set aside for unlicensed access.

The profit of firm $i \in \{1, 2, \ldots, N\}$ is now given by

$$\begin{aligned}
&u_i(y_i, w_i, \mathbf{y}_{-i}, \mathbf{w}_{-i}) \\
&= \left(1 - \sum_{j=1}^{N} y_j\right) y_i - (1-\alpha) l_i(y_i) y_i - \alpha l_{i,w}(y_i - w_i)(y_i - w_i) - \alpha \frac{\sum_{j=1}^{N} w_j}{W_0} w_i \\
&= y_i - y_i^2 - \sum_{j \in -i} y_j y_i - (1-\alpha) l_i(y_i) y_i - \alpha l_{i,w}(y_i - w_i)(y_i - w_i) - \alpha \frac{w_i^2}{W_0} - \alpha \frac{\sum_{j \in -i} w_j w_i}{W_0}
\end{aligned}$$

Then, the potential function $\Phi(\mathbf{y}, \mathbf{w})$ is given by

$$\begin{aligned}
\Phi(\mathbf{y}, \mathbf{w}) = &\sum_{i=1}^{N} y_i - \sum_{i=1}^{N} y_i^2 - \sum_{i=2}^{N} \sum_{j=1}^{i-1} y_i y_j - (1-\alpha) \sum_{i=1}^{N} l_i(y_i) y_i - \alpha \sum_{i=1}^{N} l_{i,w}(y_i - w_i)(y_i - w_i) \\
&- \frac{\alpha}{W_0} \sum_{i=1}^{N} w_i^2 - \frac{\alpha}{W_0} \sum_{i=2}^{N} \sum_{j=1}^{i-1} w_i w_j
\end{aligned}$$

If the inverse demand function is $P(y) = 1 - \gamma y$ for some $\gamma > 0$, then, the potential function $\Phi(\mathbf{y}, \mathbf{w})$ is given by

$$\begin{aligned}
\Phi(\mathbf{y}, \mathbf{w}) = &\sum_{i=1}^{N} y_i - \gamma \sum_{i=1}^{N} y_i^2 - \gamma \sum_{i=2}^{N} \sum_{j=1}^{i-1} y_i y_j - (1-\alpha) \sum_{i=1}^{N} l_i(y_i) y_i - \alpha \sum_{i=1}^{N} l_{i,w}(y_i - w_i)(y_i - w_i) \\
&- \frac{\alpha}{W_0} \sum_{i=1}^{N} w_i^2 - \frac{\alpha}{W_0} \sum_{i=2}^{N} \sum_{j=1}^{i-1} w_i w_j
\end{aligned}$$

A.11.3 Concave inverse demand and convex latency

Finally, we consider the existence of pure equilibria in the general case where, the inverse demand is a general concave decreasing function $P(\cdot)$ and the latency function in whitespace is a general convex increasing function $l_w(\cdot)$. Assuming the latency cost to be a function of the normalized load[12] incorporating the capacity provisioned is a special case of our general setting. In this case the utility

[12] These correspond to the latency cost for proprietary spectrum of provider i being $\ell_i(x) = f_i(x/B_i)$ for some $B_i > 0$ and $\ell_{i,w}(x) = f_i(x/(B_i + W_i))$ for some $W_i \geq 0$ with $f_i(\cdot)$ convex and increasing, and $l_w(w) = f_w(w/W)$ for some $W > 0$ and $f_w(\cdot)$ convex and increasing.

of firm $i \in \{1, 2, \ldots, N\}$ is now given by

$$u_i(y_i, w_i, \mathbf{y}_{-i}, \mathbf{w}_{-i}) = y_i P\left(y_i + \sum_{j \in -i} y_j\right) - (1-\alpha) l_i(y_i) y_i$$
$$- \alpha l_{i,w}(y_i - w_i)(y_i - w_i) - \alpha w_i l_w \left(w_i + \sum_{j \in -i} w_j\right),$$

where $-i := \{1, 2, \ldots, N\} \setminus \{i\}$. It is easily verified that given the strategy of the opponents, namely $(\mathbf{y}_{-i}, \mathbf{w}_{-i})$, the utility of firm i is jointly concave in (y_i, w_i), and where $(\mathbf{y}, \mathbf{w})$ are to be chosen from a compact and convex set[13].Therefore, we have a concave game and existence of pure equilibria follows from the results of [24].

Following up regarding the uniqueness of equilibria, using [24] (taking $r_i \equiv 1$ for all $i = 1, 2, \ldots, N$) and working with variables $x_i = y_i - w_i$ and w_i, we need to determine the Jacobian G of the gradient vector g and show that $H = G + G^T$ is negative definite, where for $i = 1, 2, \ldots, N$,

$$\begin{aligned}
g_{x_i} &= \frac{\partial u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i} \\
&= P(\sum_{j=1}^{N} x_j + w_j) + (x_i + w_i) P'(\sum_{j=1}^{N} x_j + w_j) - (1-\alpha)(l_i(x_i + w_i) + (x_i + w_i) l_i'(x_i + w_i)) \\
&\quad - \alpha(l_{i,w}(x_i) + x_i l_{i,w}'(x_i)) \\
g_{w_i} &= \frac{\partial u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i} \\
&= P(\sum_{j=1}^{N} x_j + w_j) + (x_i + w_i) P'(\sum_{j=1}^{N} x_j + w_j) - (1-\alpha)(l_i(x_i + w_i) + (x_i + w_i) l_i'(x_i + w_i)) \\
&\quad - \alpha(l_w(\sum_{j=1}^{N} w_j) + w_i l_w'(\sum_{j=1}^{N} w_j))
\end{aligned}$$

where we use the original strategy space (x_i, w_i) for each of the providers and label each component by the corresponding variable. Note that for $i, j \in \{1, 2, \ldots, N\}$,

$$G_{x_i,x_j} = \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial x_j}, \qquad G_{x_i,w_j} = \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial w_j},$$
$$G_{w_i,x_j} = \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial x_j}, \qquad G_{w_i,w_j} = \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial w_j}$$

and

$$\begin{aligned}
H_{x_i,x_j} = H_{x_j,x_i} &= \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial x_j} + \frac{\partial^2 u_j(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial x_j}, \\
H_{x_i,w_j} = H_{w_j,x_i} &= \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial w_j} + \frac{\partial^2 u_j(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial w_j} \\
H_{w_i,x_j} = H_{x_j,w_i} &= \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial x_j} + \frac{\partial^2 u_j(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial x_j} \\
H_{w_i,w_j} = H_{w_j,w_i} &= \frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial w_j} + \frac{\partial^2 u_j(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial w_j}
\end{aligned}$$

[13]The constraints are $\sum_i y_i \leq 1$, $y_i \geq w_i \geq 0$ for all $i \in \{1, 2, \ldots, N\}$ and the prices being non-negative.

We have the following for $i = 1, 2, \ldots, N$,

$$\begin{aligned}
\frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial x_i} &= 2P'(\sum_{k=1}^{N} x_k + w_k) + (x_i + w_i)P''(\sum_{k=1}^{N} x_k + w_k) \\
&\quad - (1-\alpha)(2l_i'(x_i + w_i) + (x_i + w_i)l_i''(x_i + w_i)) \\
&\quad - \alpha(2l_{i,w}'(x_i) + x_i l_{i,w}''(x_i)) \\
\frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial w_i} &= 2P'(\sum_{k=1}^{N} x_k + w_k) + (x_i + w_i)P''(\sum_{k=1}^{N} x_k + w_k) \\
&\quad - (1-\alpha)(2l_i'(x_i + w_i) + (x_i + w_i)l_i''(x_i + w_i)) \\
&\quad - \alpha(2l_w'(\sum_{k=1}^{N} w_k) + w_i l_w''(\sum_{k=1}^{N} w_k)) \\
\frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial w_i} &= 2P'(\sum_{k=1}^{N} x_k + w_k) + (x_i + w_i)P''(\sum_{k=1}^{N} x_k + w_k) \\
&\quad - (1-\alpha)(2l_i'(x_i + w_i) + (x_i + w_i)l_i''(x_i + w_i))
\end{aligned}$$

For $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$ we have

$$\begin{aligned}
\frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial x_j} &= P'(\sum_{k=1}^{N} x_k + w_k) + (x_i + w_i)P''(\sum_{k=1}^{N} x_k + w_k) \\
\frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial x_i \partial w_j} &= P'(\sum_{k=1}^{N} x_k + w_k) + (x_i + w_i)P''(\sum_{k=1}^{N} x_k + w_k) \\
\frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial x_j} &= P'(\sum_{k=1}^{N} x_k + w_k) + (x_i + w_i)P''(\sum_{k=1}^{N} x_k + w_k) \\
\frac{\partial^2 u_i(x_i, w_i, \mathbf{x}_{-i}, \mathbf{w}_{-i})}{\partial w_i \partial w_j} &= P'(\sum_{k=1}^{N} x_k + w_k) + (x_i + w_i)P''(\sum_{k=1}^{N} x_k + w_k) \\
&\quad - \alpha(l_w'(\sum_{k=1}^{N} w_k) + w_i l_w''(\sum_{k=1}^{N} w_k))
\end{aligned}$$

Therefore, $z^T H z$ is given by

$$\begin{aligned}
\mathbf{z}^T H \mathbf{z} = {} & 2P'\left((\mathbf{x}+\mathbf{w})^T \mathbf{1}\right)\left[(\mathbf{z}^T \mathbf{1})^2 + (\mathbf{z_x} + \mathbf{z_w})^T (\mathbf{z_x} + \mathbf{z_w})\right] \\
& + 2P''\left((\mathbf{x}+\mathbf{w})^T \mathbf{1}\right)(\mathbf{z_x} + \mathbf{z_w})^T \mathbf{1}(\mathbf{x}+\mathbf{w})^T (\mathbf{z_x} + \mathbf{z_w}) \\
& - \alpha \sum_{i=1}^{N} \left(2l_{i,w}'(x_i) + x_i l_{i,w}''(x_i)\right) z_{x_i}^2 \\
& - (1-\alpha) \sum_{i=1}^{N} \left(2l_i'(x_i + w_i) + (x_i + w_i)l_i''(x_i + w_i)\right)(z_{x_i} + z_{w_i})^2 \\
& - 2\alpha l_w' \left(\mathbf{w}^T \mathbf{1}\right)\left[(\mathbf{z_w}^T \mathbf{1})^2 + \mathbf{z_w}^T \mathbf{z_w}\right] - 2\alpha l_w'' \left(\mathbf{w}^T \mathbf{1}\right) \mathbf{z_w}^T \mathbf{1} \mathbf{w}^T \mathbf{z_w}
\end{aligned}$$

If either $P'(0) < 0$ and $l_w'(0) > 0$ or $l_i'(0) > 0$ and $l_{i,w}'(0) > 0$ for all $i = 1, 2, \ldots, N$, then it follows that H is negative definite, where we've also used the fact that $\mathbf{1}(\mathbf{x}+\mathbf{w})^T$ and $\mathbf{1}\mathbf{w}^T$ are positive semidefinite since $\mathbf{x}$ and $\mathbf{w}$ are non-negative vectors. Under these conditions we have a unique equilibrium. Note that $P'(0) < 0$ and $l_w'(0) > 0$ is a sufficient condition for $xP(x)$ being strictly concave and $xl_w(x)$ being strictly convex, and similarly, $l_i'(0) > 0$ and $l_{i,w}'(0) > 0$ for all $i = 1, 2, \ldots, N$ is a sufficient condition for $xl_i(x)$ and $xl_{i,w}(x)$ being strictly convex for all $i = 1, 2, \ldots, N$.

The case of no shared spectrum is equivalent to $\alpha = 0$ above with the understanding that provider

i only chooses $y_i = x_i + w_i$. In that case (with dimension of H being $N \times N$) we have

$$\mathbf{z}^T H \mathbf{z} = 2P'\left(\mathbf{y}^T\mathbf{1}\right)\left[(\mathbf{z}^T\mathbf{1})^2 + \mathbf{z}^T\mathbf{z}\right] + 2P''\left(\mathbf{y}^T\mathbf{1}\right)\mathbf{z}^T\mathbf{1}\mathbf{y}^T\mathbf{z} - \sum_{i=1}^{N}\left(2l_i'(y_i) + y_i l_i''(y_i)\right) z_{y_i}^2$$

If either $P'(0) < 0$ or $l_i'(0) > 0$ for all $i = 1, 2, \ldots, N$, then it follows that H is negative definite, and we have a unique equilibrium.

Generalizations: The existence and uniqueness of the Nash equilibria also extends to more general availability scenarios, with a similar proofs. For example, we can allow the proprietary spectrum of different providers to also have a general distribution of availability with possibly multiple bands with the only restriction being that every service provider always has a minimum non-zero amount of spectrum available for proprietary use. Similarly, we can also allow multiple shared bands and also the proprietary spectrum with a general distribution, with the restriction that whenever the total amount of shared bandwidth is zero, there is non-zero proprietary bandwidth available at every provider and every service provider always has a minimum non-zero amount of spectrum available for proprietary use.

A.11.4 Structure

Theorem 6.1 gives us a unique equilibrium. It is easy to see that prices being 0 at equilibrium can only occur if the quantity is also zero: if not, then reducing the quantity by ϵ leads to non-zero price and an increase in profit. The unique equilibrium also maximizes a concave potential function when the demand function is linear and all the latencies are linear too. We use this and the KKT theorem to characterize the structure of the equilibrium. We establish the following results:

1. Considering the case of $N = 2$ in Result 1 we provide necessary and sufficient conditions for a provider to not use the shared spectrum in equilibrium.
2. Again specializing to $N = 2$, in Result 2 we show that the proprietary spectrum bands are always used in equilibrium. We also show that the logic extends to $N > 2$ also.
3. In Result 3 we provide the counterpoint to Result 1 to determine necessary and sufficient conditions for both providers to use all available spectrum bands.
4. In Result 4 we generalize Result 1 to the case of $N > 2$ and provide necessary and sufficient conditions for all but one provider to not use the shared spectrum in equilibrium.
5. In Result 5 we further generalize Results 1 through 4 to the case when some of the licensed bands are also intermittent.

Result 1: The equilibrium is such that $w_i^* = 0$, $y_i^* = x_i^* > 0$, and $y_{-i}^* > w_{-i}^* > 0$ if and only if $B_i \geq 2W + 4(1-\alpha)\frac{W}{B_{-i}} + 2B_{-i} + 2$, so that provider i does not use the whitespace spectrum while provider $-i$ gets proprietary access.

Proof: Note that $N = 2$. Let $i = 1$ wlog so that $-i = 2$. Then we have

$$\begin{aligned}
\frac{\partial\Phi}{\partial y_1} &= 1 - 2y_1\left(1 + \frac{1-\alpha}{B_1}\right) - y_2 - \frac{2\alpha}{B_1}(y_1 - w_1) \\
\frac{\partial\Phi}{\partial y_2} &= 1 - 2y_2\left(1 + \frac{1-\alpha}{B_2}\right) - y_1 - \frac{2\alpha}{B_2}(y_2 - w_2) \\
\frac{\partial\Phi}{\partial w_1} &= -\frac{2\alpha}{B_1}(w_1 - y_1) - \frac{2\alpha}{W}w_1 - \frac{\alpha}{W}w_2 \\
\frac{\partial\Phi}{\partial w_2} &= -\frac{2\alpha}{B_2}(w_2 - y_2) - \frac{2\alpha}{W}w_2 - \frac{\alpha}{W}w_1
\end{aligned}$$

If the optimum (and also equilibrium) is such that $w_1^* = 0$, $y_1^* = x_1^* > 0$, and $y_2^* > w_2^* > 0$, then $\frac{\partial\Phi}{\partial y_1} = 0$, $\frac{\partial\Phi}{\partial y_2} = 0$, $\frac{\partial\Phi}{\partial w_2} = 0$ and $\frac{\partial\Phi}{\partial w_1} \leq 0$ at the optimum. Substituting the variables and using the

above constraints, we get

$$
\begin{aligned}
0 &= 1 - 2y_1^*\left(1 + \frac{1-\alpha}{B_1}\right) - y_2^* - \frac{2\alpha}{B_1}y_1^* \\
0 &= 1 - 2y_2^*\left(1 + \frac{1-\alpha}{B_2}\right) - y_1^* - \frac{2\alpha}{B_2}(y_2^* - w_2^*) \\
0 &= -\frac{2\alpha}{B_2}(w_2^* - y_2^*) - \frac{2\alpha}{W}w_2^* \\
0 &\geq \frac{2\alpha}{B_1}y_1^* - \frac{\alpha}{W}w_2^*
\end{aligned}
$$

The third equation implies that

$$
y_2^* = \left(1 + \frac{B_2}{W}\right)w_2^* > w_2^* \quad (\text{If } w_2^* > 0)
$$

Using this we get two equations in two unknowns, y_1^* and w_2^*. Solving these yields

$$
\begin{aligned}
w_2^* &= \frac{1 + \frac{2}{B_1}}{(4(1+\frac{1}{B_1})(1+\frac{1}{B_2}) - 1)(1+\frac{B_2}{W}) - \frac{4\alpha}{B_2}(1+\frac{1}{B_1})} \\
y_1^* &= \frac{1 + \frac{2(1-\alpha)}{B_2} + \frac{B_2}{W} + \frac{2}{W}}{(4(1+\frac{1}{B_1})(1+\frac{1}{B_2}) - 1)(1+\frac{B_2}{W}) - \frac{4\alpha}{B_2}(1+\frac{1}{B_1})}
\end{aligned}
$$

Note that both are positive. With some algebra it is also verified that all the prices are non-negative. Thus, the last condition that must hold is $\frac{\partial\Phi}{\partial w_1} \leq 0$, which implies that $\frac{2}{B_1}y_1^* \leq \frac{1}{W}w_2^*$, i.e.,

$$
B_1 \geq 2W + 4(1-\alpha)\frac{W}{B_2} + 2B_2 + 2
$$

Additionally, it can also be verified for all B_1 satisfying the above inequality

$$
B_2 < 2W + 4(1-\alpha)\frac{W}{B_1} + 2B_1 + 2
$$

This proves the result. □

Remarks:

1. The proof above holds by checking for conditions when the chosen equilibrium maximizes the potential function; the condition is equivalent to the partial derivative of the potential function in w_i being non-positive. For fixed B_{-i}, W and α for all B_i sufficiently large, the condition above will hold. Keeping B_i, W and α fixed such that $B_i \geq 2W+2+4\sqrt{2(1-\alpha)W}$ (RHS is minimum value of lower bound as B_{-i} is varied), there are two values B_{-i}^{lb} and B_{-i}^{ub} (corresponding to solution to quadratic with equality in constraint) such that if $B_{-i} \in [B_{-i}^{lb}, B_{-i}^{ub}]$, then provider i vacates the share spectrum, and otherwise she uses it; if $B_i < W + 2 + 4\sqrt{2(1-\alpha)W}$, then provider i always uses the shared spectrum.
2. From constrained optimization theory we know that at the equilibrium we will have $\frac{\partial\Phi}{\partial w_1} \leq 0$ and $\frac{\partial\Phi}{\partial w_2} \leq 0$. Rewriting these in terms of the equilibrium variables (and assuming $\alpha > 0$) we get

$$
\frac{x_i^*}{B_i} \leq \frac{w_i^* + \frac{w_{-i}^*}{2}}{W} \leq \frac{w_1^* + w_2^*}{W},
$$

which is equivalent to stating that the congestion level in the proprietary bands is always less than the congestion level in the shared band; note that if the provider i uses the shared band, then the first inequality is tight. Additionally, if in equilibrium both providers carry non-zero traffic in the shared band, then the congestion level in the shared band is strictly greater, by

exactly $\frac{w^*_{-i}}{2W}$ for provider $i \in \{1,2\}$. Furthermore, if in equilibrium provider $i \in \{1,2\}$ does not use the shared band, then the proprietary band for provider $-i$ and the shared band have the same congestion level that is at least two times the level of the congestion in provider i's proprietary band. Interestingly, this is the only reason for provider i to not use the shared band, as opposed to others such as the (corresponding) price becoming negative if the traffic carried is positive, etc.

Result 2: At the equilibrium $x_i^* > 0$ for all $i = 1, 2$, irrespective of the parameters.
Proof: As before let $i = 1$ wlog so that $-i = 2$. If $x_1^* = 0$, then $y_1^* = w_1^* > 0$ (easy to see that at least one of x_1^* or w_1^* should be positive). This then implies that

$$\frac{\partial \Phi}{\partial y_1} \leq 0, \text{ and } \frac{\partial \Phi}{\partial w_1} \geq 0.$$

The second inequality can be rewritten as

$$-\frac{2\alpha}{W} w_1^* - \frac{\alpha}{W} w_2^* \geq 0,$$

which then implies that $w_1^* = w_2^* = 0$. This is a contradiction. □
Using the same logic, this result holds for $N > 2$ too. For the $N = 2$ case, we next show that Result 2 implies Result 3.

Result 3: For $N = 2$, if the conditions of Result 1 don't hold, then the equilibrium is always an interior point equilibrium.
Proof: From Result 2 we know that $x_i^* > 0$ for all $i = 1, 2$. Since the conditions of Result 1 don't apply, we can either have $w_1^* = w_2^* = 0$ or $w_1^*, w_2^* > 0$. The former cannot hold as this would imply

$$\frac{\partial \Phi}{\partial w_i} \leq 0 \Leftrightarrow \frac{2\alpha}{B_i} x_i^* \leq 0 \Leftrightarrow x_i^* = 0,$$

which is a contradiction. □

Result 4: For general N, the equilibrium is $y_i^* = x_i^* > 0$, $w_i^* = 0$ for all $i = 1, 2, \ldots, N-1$ and $y_N^* > w_N^* > 0$ if and only if $B_i \geq 2W + 2B_N + 2 + 4(1-\alpha)\frac{W}{B_N}$ for all $i = 1, 2, \ldots, N-1$. Note that all providers except for provider N vacate the whitespace spectrum
Proof: Using the potential function and the KKT conditions for convex optimization, the equilibrium is $y_i^* = x_i^* > 0$, $w_i^* = 0$ for all $i = 1, 2, \ldots, N-1$ and $y_N^* > w_N^* > 0$ if and only if the following hold at the equilibrium

$$\frac{\partial \Phi}{\partial y_i} = 0, \quad i = 1, 2, \ldots, N \tag{48}$$

$$\frac{\partial \Phi}{\partial w_i} \leq 0, \quad i = 1, 2, \ldots, N-1 \tag{49}$$

$$\frac{\partial \Phi}{\partial w_N} = 0 \tag{50}$$

Equation (50) yields

$$\frac{\partial \Phi}{\partial w_N} = -\frac{2\alpha}{B_N}(w_N - y_N) - \frac{2\alpha}{W} w_N = 0 \Leftrightarrow y_N^* = \left(1 + \frac{B_N}{W}\right) w_N^* \tag{51}$$

For $i = 1, 2, \ldots, N-1$, inequalities (49) yield

$$\frac{\partial \Phi}{\partial w_i} = \frac{2y_i}{B_i} - \frac{w_N}{W} \leq 0 \Leftrightarrow \frac{2y_i^*}{B_i} \leq \frac{w_N^*}{W} \tag{52}$$

For $i = N$, equation (48) in combination with (51) yields

$$\begin{aligned}\frac{\partial \Phi}{\partial y_i} &= 1 - 2y_N\left(1+\frac{1}{B_N}\right) - \sum_{j=1}^{N-1} y_j + \frac{2\alpha w_N}{B_N} = 0 \\ \Leftrightarrow \sum_{j=1}^{N-1} y_j^* &= 1 - 2w_N^*\left[\left(1+\frac{B_N}{W}\right)\left(1+\frac{1}{B_N}\right) - \frac{\alpha}{B_N}\right]\end{aligned} \tag{53}$$

For $i = 1, 2, \ldots, N-1$, equations (48) in combination with (51) and (53) yield

$$\begin{aligned}\frac{\partial \Phi}{\partial y_i} &= 1 - \sum_{j=1, j\neq i}^{N} y_j - 2y_i\left(1+\frac{1}{B_i}\right) = 0 \\ \Leftrightarrow y_i^*\left(1+\frac{2}{B_i}\right) &= w_N^*\left[\left(1+\frac{B_N}{W}\right)\left(1+\frac{2}{B_N}\right) - \frac{2\alpha}{B_N}\right]\end{aligned} \tag{54}$$

Substituting the result of (54) into (53) yields

$$w_N^* = \frac{1}{2\left(1+\frac{B_N}{W}\right)\left(1+\frac{1}{B_N}\right) - \frac{2\alpha}{B_N} + \left[\left(1+\frac{B_N}{W}\right)\left(1+\frac{2}{B_N}\right) - \frac{2\alpha}{B_N}\right]\sum_{j=1}^{N-1}\frac{1}{1+\frac{2}{B_j}}} > 0 \tag{55}$$

With some algebra it can be shown that all the prices are all positive as well. Therefore, the only conditions that need to be satisfied are given by (52), which simplify to

$$B_i \geq 2W + 2B_N + 2 + 4(1-\alpha)\frac{W}{B_N}, \quad \forall i = 1, 2, \ldots, N-1.$$

This proves the result. □

Result 5: Let the W units of secondary band be split into $(W_0, W_1, \ldots, W_N)$ such that firm i gets W_i units and W_0 units are assigned to shared access, then the following are true:

1. For all $N \geq 2$ the equilibrium is such that $x_i^* > 0$ for all $i = 1, 2, \ldots, N$.
2. The equilibrium is $y_i^* = x_i^* > 0, w_i^* = 0$ for all $i = 1, 2, \ldots, N-1$ and $y_N^* > w_N^* > 0$ if and only if for all $i = 1, 2, \ldots, N-1$

$$B_i + W_i + 2(1-\alpha)\frac{W_i}{B_i} \geq 2(W_0 + W_N) + 2B_N + 2 + 4(1-\alpha)\frac{W_0 + W_N}{B_N}. \tag{56}$$

3. For $N = 2$, if the condition in (56) is not satisfied for either firm 1 or 2, then we necessarily have an interior point equilibrium, i.e., $y_1^* > w_1^* > 0$ and $y_2^* > w_2^* > 0$.

Proof: The proof follows by repeating all the steps of the proofs of Results 1 to 4, and is omitted. □

Remark: Similar to Remark 2 at the bottom of Result 1, at equilibrium we need $\frac{\partial \phi}{\partial w_i} \leq 0$ and this implies that for all $i = 1, \ldots, N$

$$\frac{x_i^*}{B_i + W_i} \leq \frac{w_i^* + \frac{\sum_{j\in -i} w_j^*}{2}}{W_0} \leq \frac{\sum_{j=1}^{N} w_j^*}{W_0}.$$

This again implies that conditioned on the intermittent spectrum being available, the congestion in the proprietary band is less than the congestion in the shared band. Since $w_i^* > 0$ implies that the first inequality is tight, the congestion level in the proprietary band of provider i is exactly lower by $\frac{\sum_{j\in -i} w_j^*}{2W_0}$ and strictly greater than *half* the congestion level in the shared band. If instead, $w_i^* = 0$, then the congestion level in the propriety band of provider i is less than *half* the congestion level in the shared band. Both these statements hold when conditioned on the intermittent spectrum being available.

A.12 Proof of Proposition 6.2

Given inverse demand $P(\cdot)$ and latency $\ell(\cdot)$, the revenue for provider k is

$$R_k = x_k[P(x_1 + x_2) - \ell(x_k/B_k)]. \tag{57}$$

where B_k is provider k's bandwidth. Setting $\partial R_k/\partial x_k = 0$ gives

$$P(x_1 + x_2) - \ell(x_k/B_k) + x_k[P'(x_1 + x_2) - \ell'(x_k/B_k)B_k^{-1}] = 0 \tag{58}$$

Taking $\partial/\partial B_k$ gives

$$\frac{\partial x_k}{\partial B_k} G(x_k, x_{-k}) = -\frac{2x_k}{B_k^2}\ell'(x_k/B_k) - \frac{x_k^2}{B_k^3}\ell''(x_k/B_k) \tag{59}$$

where

$$G(x_k, x_{-k}) = 2P'(x_1 + x_2) + x_k P''(x_1 + x_2) - \frac{2}{B_k}\ell'(x_k/B_k) - \frac{1}{B_k^2}\ell''(x_k/B_k), \tag{60}$$

and is negative. The right-hand side is also negative, hence $\partial x_k/\partial B_k > 0$.

The best response condition (58) gives x_k implicitly as a function of x_{-k}. Swapping k and $-k$, we can compute

$$\frac{\partial x_{-k}}{\partial x_k} G(x_{-k}, x_k) = -P'(x_1 + x_2) - x_{-k}P''(x_1 + x_2) \tag{61}$$

Since $G < 0$ and the right-hand side is positive, $\partial x_{-k}/\partial x_k < 0$. Hence a marginal increase in B_k leads to a marginal increase in x_k and a marginal decrease in x_{-k}.

Now let $x_{k,-k} = \partial x_k/\partial x_{-k}$ and consider the sequence of best responses for $n = 1, 2, \cdots$, which result from adding δB_k starting from the equilibrium $(x_k^{(0)}, x_{-k}^{(0)})$:

1. Update $x_k^{(1)} = x_k^{(0)} + \delta x_k^{(1)}$, where $\delta x_k^{(1)} = (\partial x_k/\partial B_k) \cdot \delta B_k > 0$.
2. Update $x_{-k}^{(n)} = x_{-k}^{(n-1)} + \delta x_{-k}^{(n)}$ where $\delta x_{-k}^{(n)} = x_{-k,k} \cdot \delta x_k^{(n)} < 0$.
3. Update $x_k^{(n+1)} = x_k^{(n)} + \delta x_k^{(n+1)}$ where $\delta x_{-k}^{(n+1)} = x_{k,-k} \cdot \delta x_{-k}^{(n)} > 0$
4. Iterate steps 2 and 3.

The total change in x_k is then the geometric series $\delta x_k(1 + \rho + \rho^2 + \cdots)$ where $\rho = x_{k,-k}x_{-k,k}$. Since $0 < \rho < 1$, this sequence of best responses converges to a new equilibrium with quantities $(x_k + \delta x_{k,eq}, x_{-k} + \delta x_{-k,eq})$ where

$$\delta x_{k,eq} = \frac{1}{1-\rho} \cdot \delta x_k^{(1)} > 0 \tag{62}$$

$$\delta x_{-k,eq} = x_{-k,k} \cdot \delta x_{k,eq} < 0. \tag{63}$$

Furthermore, the change in total quantity

$$\delta x_{k,eq} + \delta x_{-k,eq} = (1 + x_{-k,k})\frac{1}{1-\rho}\delta x_k^{(1)} > 0 \tag{64}$$

We must have $\partial R_k/\partial B_k > 0$, since the revenue increases with B_k when x_k is fixed, hence optimizing x_k can only increase the revenue further. The incremental revenue for agent $-k$ is given by

$$\begin{aligned}\delta R_{-k} = {} & x_{-k}P'(x_1 + x_2)(\delta x_{k,eq} + \delta x_{-k,eq}) + [P(x_1 + x_2) - \ell_{-k}(x_{-k}/B_{-k})]\delta x_{-k,eq} \\ & - \frac{x_{-k}}{B_{-k}}\ell'_{-k}(x_{-k}/B_{-k})\delta x_{-k,eq}\end{aligned} \tag{65}$$

where the first term (< 0) is due to the reduction in total price, the second term (< 0) is due to the reduced quantity at the announced price, and the last term (> 0) is due to reduction in latency.

Combining with the best response condition (58), this simplifies to

$$\delta R_{-k} = x_{-k} P'(x_1 + x_2) \cdot \delta x_{k,eq} < 0. \tag{66}$$

A.13 Proof of Proposition 6.3

To prove Proposition 6.3, we note that the increase in consumer surplus follows directly from (64). The increase in total welfare can be shown by adding the incremental areas when x_k and x_{-k} are incremented by $\delta x_{k,eq}$ and $\delta x_{-k,eq}$. (can add this later...)

To determine which agent should receive the bandwidth to generate the most consumer surplus, we maximize the incremental quantity in (64). Equivalently, we wish to determine

$$\arg\max_k (1 + x_{-k,k}) \frac{\partial x_k}{\partial B_k} \tag{67}$$

Substituting for the derivatives, this becomes

$$\arg\max \left(P'(x_1 + p_2) - \frac{2}{B}_{-k} \ell'_{-k}(\bar{x}_{-k}) - \frac{1}{B_{-k}^2} \ell''_{-k}(\bar{x}_{-k}) \right) \left(\frac{2x_k}{B_k^2} \ell_2(\bar{x}_k) + \frac{x_k^2}{B_k^3} \ell''_k(\bar{x}_k) \right) \tag{68}$$

where $\bar{x}_k = x_k / B_k$. For linear latencies $\ell_k(x) = c_k x$ this reduces to finding

$$\arg\max -\frac{c_k x_k}{B_k^2} \left(P'(x_1 + x_2), -\frac{2c_{-k}}{B_{-k}} \right) \tag{69}$$

which this reduces to (30), and further constraining $P(x) = 1 - ax$ gives the condition (31).