Variable damping in seismic tomography based on ray coverage

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Summary

In the seismic tomography problem, the subsurface slowness distribution is estimated from the travel-times computed along generally curved rays. For a dense and uniform set of rays, the slowness distribution will be correctly reconstructed. However, for an uneven distribution of rays, the estimated slowness distribution may be influenced by the ray configuration.

For the linear case, the seismic tomography problem can be written as $A(x-x_0)=(d^{obs}-d^{pred})$ where d^{obs} and d^{pred} are the observed and predicted travel-times, x and x_0 are the true and prior slowness distributions, and A is the $(m\times n)$ matrix of partial derivatives $\partial T_i / \partial u_j$. Prior estimates of model uncertainty can be used to stabilize the inverse problem (Tarantola, 1987). The stochastic linear inverse can then be written

$$x \approx x_0 + (A^* C_d^{-1} A + C_{x_0}^{-1})^{-1} A^* C_d^{-1} (d^{obs} - d^{pred})$$

where C_d is the data covariance matrix and C_{x_0} is the prior model covariance associated with x_0. Operationally, the diagonal elements of $C_{x_0}^{-1}$ provide damping of the solution and the off-diagonal elements provide smoothing. The final covariance estimate is then $C_x = (A^* C_d^{-1} A + C_{x_0}^{-1})^{-1}$. One might consider using the final covariance matrix C_x instead of C_{x_0} in the inverse formula above, however this will lead to an incorrect estimate (Claerbout, 1992).

In addition to the prior uncertainties in the model, C_{x_0} can also include a discretization correction for variable block sizes (Nolet, 1987). For the case of constant damping and equal ray coverage, the larger blocks will be emphasized at the expense of the smaller blocks. In a very simple example, Nolet (1987) used the underdetermined formulation to invert for two variable length blocks from one travel-time. Using a constant damping this leads to unequal estimates of the slowness parameters in which $s_1/s_2 = l_1/l_2$, where s_i is the slowness and l_i is the length for each of the two blocks. Incorporating a variable model covariance with diagonal values of σ_1^2 and σ_2^2 results in the ratio of the slowness estimates of $s_1/s_2 = \sigma_1^2 l_1 / \sigma_2^2 l_2$. In order to equalize the slowness estimates from the bias resulting from the unequal block sizes, the diagonal elements of the model covariance matrix can be chosen to be proportional to the inverse of the block lengths. In a 3-D model, the diagonal elements would be proportional to the inverse of the block volumes (Nolet, 1987). For spline parameterizations, the block volumes can be estimated by integration of the individual basis functions.

In addition to the effects of unequal block sizes, unequal ray coverage will affect the inverse solution. One approach to correct for this is to reconfigure the experimental design in such a way as to approximately equalize the ray coverage through the model. Although this is possible in laboratory experiments, in real earth cases this may be not be feasible to do. As an alternative, one can specifically construct a model parameterization with unequal cell sizes in such a way as to equalize the ray coverage within each of the cells (Vesnaver, 1996, Curtis and Snieder, 1997).

The effects of unequal ray coverage can also be corrected for by using variable damping. Wang (1993) suggested a variable damping based on the total ray length within a cell. He noted that the diagonal elements of the matrix A^*A can be written as $\sum_{i=1}^{m}(\partial T_i/\partial u_j)^2$. For non-overlapping block parameterization the diagonal values are $\sum_{i=1}^{m}(d_{ij})^2$, where d_{ij} is the ray length in the jth block from the ith ray. Taking the square-root then results in the Euclidean "length" of vector of ray lengths within a given block. Incorporating the prior covariance, Wang (1993) chose a variable model damping proportional to $G=diag\sqrt{A^*C_{x_0}^{-1}A}$ Incorporating this, as well as operational damping, he obtained the following inverse formula

$$A_g^{-1}=(A^*C_d^{-1}A+\varepsilon G+\theta^2 D^*D)^{-1}A^*C_d^{-1}$$

where $\varepsilon>0$ and D is a general damping or smoothing operator. Wang (1993) applied this formulation to several synthetic examples using curved-ray refraction tomography, as well as to data from the KRISP90 refraction experiment from Kenya, and found that the addition of the G term reduced artifacts of the inversion results due to ray concentrations.

These results can be extended by using the general square-root operator and the SVD decomposition. We first look at the inverse operator $(A^*A+\varepsilon\, sqrt(A^*A))^{-1}A^*$. The SVD decomposition of A can be written as $A=(U_p\,U_0)\begin{pmatrix}\Lambda_p & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_p^*\\ V_0^*\end{pmatrix}$ where U_p and U_0 span the range-space and orthogonal range-space, V_0 and V_p span the null-space and orthogonal null-space, and Λ_p are the p nonzero singular values λ_i (Lanczos, 1961). Assuming that the rank p is equal to n, the size of the model space, then the inverse operator above can be written as $A_{g1}^{-1}=V_p\left(\frac{\lambda_i}{\lambda_i^2+\varepsilon\lambda_i}\right)U_p^*=V_p\left(\frac{1}{\lambda_i+\varepsilon}\right)U_p^*$. In contrast, the simple, damped least squares solution can be written $A_{dls}^{-1}=(A^*A+\varepsilon^2 I)^{-1}A^*=V_p\left(\frac{\lambda_i}{\lambda_i^2+\varepsilon^2}\right)U_p^*$. A comparison of these formulas shows that the small singular values of A_{dls}^{-1} go to zero for $\lambda_i<\varepsilon$, whereas the small singular values of A_{g1}^{-1} go to $1/\varepsilon$. Thus for A_{g1}^{-1}, the small singular values are weighted more

heavily than for the damped least squares solution. In the tomography problem, the small singular values correspond to block with less ray coverage.

Although the inverse solution A_{g1}^{-1} is nonsingular when constructed using the SVD decomposition, by using a least squares formulation the inverse will be singular when there are zero singular values, or when blocks with no rays occur. Using least squares, this can be avoided by using an additional small damping as was done by Wang (1993). Using a simple damping this can be written $A_{g2}^{-1} = (A^* A + \varepsilon sqrt(A^* A) + \vartheta^2 I)^{-1} A^* = V_p \left(\frac{\lambda_i}{\lambda_i^2 + \varepsilon \lambda_i + \vartheta^2} \right) U_p^*$ where $\varepsilon > \vartheta$. The additional small damping eliminates the very small and zero singular values in the least squares formulation.

Finally, the zero singular value components can be incorporated into the solution by damping the SVD formulation such that $A_{g3}^{-1} = (V_p \; V_\varepsilon) \begin{pmatrix} 1/(\lambda_i + \varepsilon) & 0 \\ 0 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} U_p^* \\ U_\varepsilon^* \end{pmatrix}$ where U_ε and V_ε are the augmented data and model vectors associated with zero singular values modified to $1/\varepsilon$. However, it's as yet unclear whether this is physically reasonable. By truncating the $1/\varepsilon$ singular values, the A_{g3}^{-1} inverse reduces to the A_{g1}^{-1} inverse above.

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