I If $X_1$ and $X_2$ are standard normal random variables, then $X_1^2 + X_2^2$ is an exponential random variable with parameter $1/2$. Use this fact and the structure of the contours of the bivariate normal distribution to devise a technique for generating standard normal random variates. (Your method will result in generating standard normal random variates in pairs.)

II Suppose the random variable $X$ has the density function $f(x), x \in D$, and that the uniform $(0, 1)$ random variable $U$ is independent of $X$. Show that the double $(U, X)$ is uniformly distributed “under the surface $f(x)$,” that is $(U, X) \sim \mathcal{U}\{(x, u) : 0 < u < f(x)\}$. This result should provide much of the intuition for acceptance-rejection methods.

III Devise a method to generate random variates from the truncated normal distribution $f_{\tilde{\mu}}(x; \mu, \sigma^2) = \exp\left\{-\frac{1}{2\sigma^2}(x - \mu)^2\right\}\mathbb{I}\{x \geq \tilde{\mu}\}$ using the acceptance rejection method with the shifted truncated exponential $g_\alpha(x) = \alpha \exp\left\{-(z - \tilde{\mu})\right\}\mathbb{I}\{z \geq \tilde{\mu}\}$. What is the optimal choice of $\alpha$?

IV A commonly held (incorrect) notion is that, amongst all distributions supported on $[0, 1]$, the uniform distribution is the most “dispersed” and hence has the largest variance. Give an example of a random variable that takes on some or all values between 0 and 1, and that has larger variance than $U$.

The entropy for a continuous random variable $X$ with density function $f_X(\cdot)$ is defined as $e(f_X) = -\int_A f_X(y) \log f_X(y) \, dy$, where $A$ is the support. It so happens that entropy is a notion that quantifies the amount of dispersion of a random variable over its support. (So, random variables that have any “accumulation” over specific regions have lower entropy.) Prove that amongst all distributions supported on the interval $[0, 1]$ the uniform distribution has the highest entropy.

V The density $f(x)$ is called log-concave if the logarithm of the function $f(x)$ is concave. The log-concave class turns out to be a large and useful class of density functions.

Let $S_n$ be a set of points $x_i, i = 0, 1, \ldots, n + 1$ in the support of $f$ such that $h(x_i) = \log f(x_i)$ is known up to the same constant.

(i) Use the points $(x_i, h(x_i)), i = 1, 2, \ldots, n + 1$ and the fact that $h(x)$ is concave to construct piece-wise linear lower and upper envelopes to the function $h(x)$.

(ii) Construct an acceptance-rejection-squeeze algorithm that generates random variates from $f(x)$ by using the constructed envelopes.
(iii) Notice that the algorithm you constructed becomes more and more accurate as the number of generated points tends to infinity, i.e., the probability of rejecting a generated point tends to zero as $n \to \infty$. Prove this.

VI The scalar $\rho \in [-1, 1]$ is called feasible with respective to distribution functions $F_1, F_2$ if there exist random variables $X_1, X_2$ having distributions $F_1, F_2$ such that $\text{Corr}(X_1, X_2) = \rho$. Calculate the range of feasible correlations when $X_1$ and $X_2$ are each exponential with parameter $\lambda$.

VII Recall the setup for the Alias Method used to generate random variates from a probability mass function $F$. Assume that $F$ is supported on $\{x_1, x_2, \ldots, x_{n+1}\}$ and that $F(x_i) = p_i, i = 1, 2, \ldots, n+1$. Recall that the important step in the Alias Method is expressing the function $F$ as an equi-mixture of $n$ two-point distributions $G_i$, that is,

$$F(x) = \sum_{i=1}^{n} \frac{1}{n} G_i(x), \quad x \in \{x_1, x_2, \ldots, x_{n+1}\},$$

where $G_i(x)$ is strictly positive for at most two points $x = x_{i1}, x = x_{i2}$ and zero everywhere else. Devise an algorithm for constructing $G_i, i = 1, 2, \ldots, n$.

(To get you started, choose $x_{11}, x_{12}$, and $G_1$ so that $F(x_{11}) \leq \frac{1}{n}$ and

$$F(x) = \frac{1}{n} G_1(x) + \frac{n-1}{n} H_1(x),$$

where $H_1(x)$ is now supported on $\{x_1, x_2, \ldots, x_{n+1}\} \setminus \{x_{11}\}$. Now repeat the procedure for $H_1$.)