Coupled-Mode Tutorial

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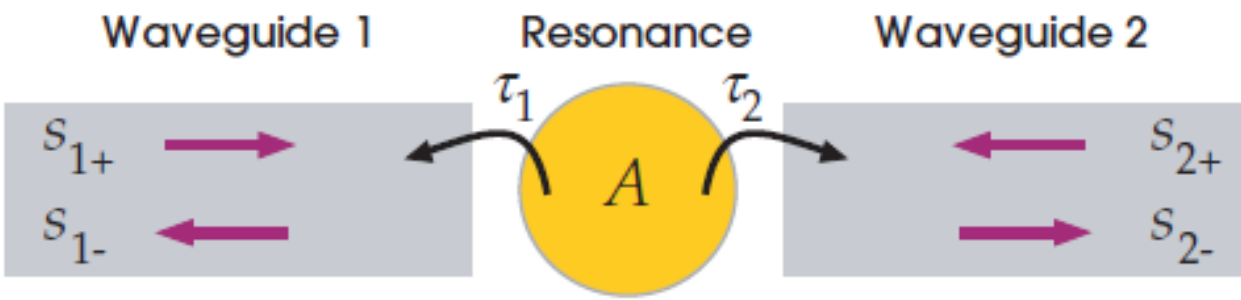
Coupled-Mode Theory: Basic Concepts



H. Haus, *Waves & Fields in Optoelectronics*, Chap. 7 (1984)
W. Suh et al., *IEEE J. Quantum Electron.* **40**, 1511 (2004)
J.D. Joannopoulos et al., *Photonic Crystals*, Chap. 10 (2008).

- Energy exists in 2 forms:
  - » Localized resonant modes: $\{A_i\}$
  - » Traveling waveguide modes: $\{S_{i+},\ S_{i-}\}$
- Key assumptions:
  - » Weak coupling between modes
  - » Linearity (i.e., the validity of superposition)
  - » Time-reversal symmetry and conservation of energy
  - » Time-invariance

Derivation of Coupled Mode Equations

- Assume that:
  - » Energy of resonant modes is given by $U_i = |A_i|^2$
  - » Incident power of waveguide modes is given by $|S_{i+}|^2$
- Resonator *i* oscillates in phase at frequency ω_i, hence:

$$\frac{dA_i}{dt} = -j\omega_i A_i$$

- Resonator energy decays at rate proportional to energy present:

$$\frac{dU_i}{dt} = -\frac{2U_i}{\tau_i}$$

$$\frac{dA_i}{dt} = -j\omega_i A_i - \frac{A_i}{\tau_i}$$

Derivation of Coupled Mode Equations

- By linearity, coupling of waveguides into modes given by:

$$\frac{dA_i}{dt} = \cdots + \sum_j \alpha_{ij} S_{j+}$$

- For similar reasons, outgoing waveguide modes given by:

$$S_{i-} = \beta_i S_{i+} + \sum_j \gamma_{ij} A_j$$

- By conservation of energy, inputs must be stored or lost:

$$\sum_i \left[|S_{i+}|^2 - |S_{i-}|^2 - \frac{dU_i}{dt} \right] = 0$$

- Special cases can be used to obtain coefficients: $\{\alpha_{ij},\ \beta_i,\ \gamma_{ij}\}$

Derivation of Coupled Mode Equations

- In absence of coupling to resonant modes, conservation of energy requires $|\beta_i| = 1$. Phase depends on convention.
- In absence of input waveguide, must have:

$$0 = |S_{i-}|^2 + \frac{dU_i}{dt}$$

$$0 = |S_{i-}|^2 - \frac{2U_i}{\tau_i}$$

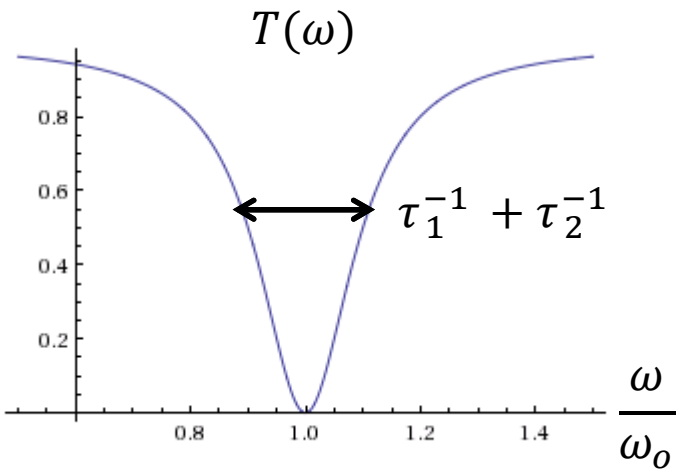
$$0 = |\gamma_i|^2 U_i - \frac{2U_i}{\tau_i}$$

- Thus, $\gamma_i = \sqrt{2/\tau_i}$
- Finally, time reversal implies $\alpha_i = \gamma_i$

Application to Add-Drop Filters

- For simplest case: 2 waveguides + 1 resonator with 1 input:

$$S_{1-} = S_{1+} - \sqrt{2/\tau_1}A$$
$$S_{2-} = \sqrt{2/\tau_2}A$$
$$\frac{dA}{dt} = -j\omega_o A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}}S_{1+}$$



- Transmission can be calculated as quotient:

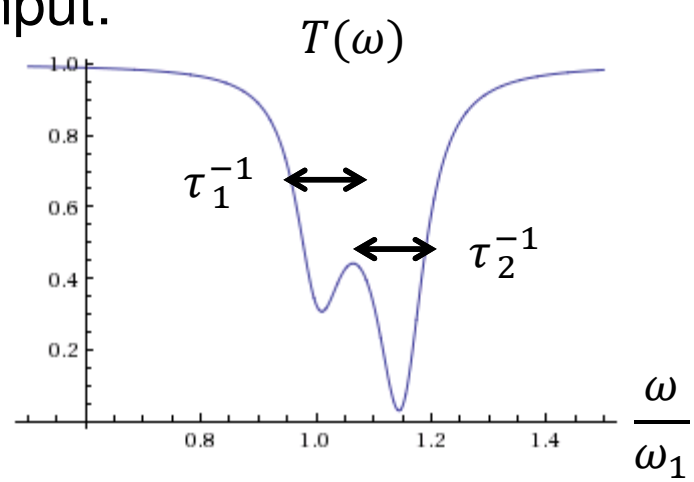
$$T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \frac{(\omega - \omega_o)^2 + (\tau_1^{-1} - \tau_2^{-1})^2}{(\omega - \omega_o)^2 + (\tau_1^{-1} + \tau_2^{-1})^2}$$

- Result: a Lorentzian dip in transmission, centered at resonant frequency ω_o

Application to Add-Drop Filters

- Consider 4 channels + 2 resonators with 1 input:

$$S_{1-} = S_{1+} - \sqrt{2/\tau_1}A_1 - \sqrt{2/\tau_2}A_2$$
$$S_{2-} = \sqrt{2/\tau_1}A_1 + \sqrt{2/\tau_2}A_2$$
$$S_{34-} = \sqrt{2/\tau_3}A_1 + \sqrt{2/\tau_4}A_2$$
$$\frac{dA_1}{dt} = -j\omega_1 A_1 - \sum_i \frac{A_1}{\tau_i} + \sqrt{\frac{2}{\tau_1}}S_{1+}$$



- Transmission can be calculated as quotient:

$$T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \left|1 - \frac{2/\tau_1}{j(\omega_1 - \omega) + \Gamma} - \frac{2/\tau_2}{j(\omega_2 - \omega) + \Gamma}\right|^2$$

- Result: a Fano lineshape encompassing both resonances ω_1 and ω_2

Conclusions

- In general, CMT works for a broad range of systems with well-defined and relatively weakly coupled resonances
- Can be readily extended to cases with weak losses, by treating them as additional ‘waveguides’
- Furthermore, in the linear case, most problems can be solved analytically
- Can extend CMT to nonlinear systems (e.g., Kerr media) or time-varying systems, but generally must use ODE solvers to find numerical solutions

Any Questions?

Contact Peter Bermel <pbermel@purdue.edu>