Coupled-Mode Tutorial

Peter Bermel
Assistant Professor
Purdue University
pbermel@purdue.edu
http://web.ics.purdue.edu/~pbermel/

March 3, 2013
Coupled-Mode Theory: Basic Concepts

- Energy exists in 2 forms:
  - Localized resonant modes: \( \{ A_i \} \)
  - Traveling waveguide modes: \( \{ s_{i+}, s_{i-} \} \)

- Key assumptions:
  - Weak coupling between modes
  - Linearity (i.e., the validity of superposition)
  - Time-reversal symmetry and conservation of energy
  - Time-invariance

Derivation of Coupled Mode Equations

• Assume that:
  » Energy of resonant modes is given by \( U_i = |A_i|^2 \)
  » Incident power of waveguide modes is given by \( |S_{i+}|^2 \)

• Resonator \( i \) oscillates in phase at frequency \( \omega_i \), hence:
  \[
  \frac{dA_i}{dt} = -j\omega_i A_i
  \]

• Resonator energy decays at rate proportional to energy present:
  \[
  \frac{dU_i}{dt} = -\frac{2U_i}{\tau_i}
  \]
  \[
  \frac{dA_i}{dt} = -j\omega_i A_i - \frac{A_i}{\tau_i}
  \]
Derivation of Coupled Mode Equations

• By linearity, coupling of waveguides into modes given by:
\[ \frac{dA_i}{dt} = \ldots + \sum_j \alpha_{ij}S_{j+} \]

• For similar reasons, outgoing waveguide modes given by:
\[ S_{i-} = \beta_i S_{i+} + \sum_j \gamma_{ij}A_j \]

• By conservation of energy, inputs must be stored or lost:
\[ \sum_i \left[ |S_{i+}|^2 - |S_{i-}|^2 - \frac{dU_i}{dt} \right] = 0 \]

• Special cases can be used to obtain coefficients: \( \{\alpha_{ij}, \beta_i, \gamma_{ij}\} \)
Derivation of Coupled Mode Equations

• In absence of coupling to resonant modes, conservation of energy requires $|\beta_i| = 1$. Phase depends on convention.

• In absence of input waveguide, must have:

\[
0 = |S_{i-}|^2 + \frac{dU_i}{dt}
\]

\[
0 = |S_{i-}|^2 - \frac{2U_i}{\tau_i}
\]

\[
0 = |\gamma_i|^2U_i - \frac{2U_i}{\tau_i}
\]

• Thus, $\gamma_i = \sqrt{2/\tau_i}$

• Finally, time reversal implies $\alpha_i = \gamma_i$
Application to Add-Drop Filters

- For simplest case: 2 waveguides + 1 resonator with 1 input:

\[ S_{1-} = S_{1+} - \frac{\sqrt{2}}{\tau_1} A \]
\[ S_{2-} = \sqrt{2} / \tau_2 A \]
\[ \frac{dA}{dt} = -j\omega_o A - \frac{A}{\tau_1} - \frac{A}{\tau_2} + \sqrt{\frac{2}{\tau_1}} S_{1+} \]

- Transmission can be calculated as quotient:

\[ T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \frac{(\omega - \omega_o)^2 + (\tau_1^{-1} - \tau_2^{-1})^2}{(\omega - \omega_o)^2 + (\tau_1^{-1} + \tau_2^{-1})^2} \]

- Result: a Lorentzian dip in transmission, centered at resonant frequency \( \omega_o \)
Application to Add-Drop Filters

• Consider 4 channels + 2 resonators with 1 input:

\[
S_{1-} = S_{1+} - \sqrt{2/\tau_1 A_1} - \sqrt{2/\tau_2 A_2}
\]
\[
S_{2-} = \sqrt{2/\tau_1 A_1} + \sqrt{2/\tau_2 A_2}
\]
\[
S_{34-} = \sqrt{2/\tau_3 A_1} + \sqrt{2/\tau_4 A_2}
\]
\[
\frac{dA_1}{dt} = -j\omega_1 A_1 - \sum_i \frac{A_1}{\tau_i} + \sqrt{\frac{2}{\tau_1}} S_{1+}
\]

• Transmission can be calculated as quotient:

\[
T(\omega) = \frac{|S_{1-}|^2}{|S_{1+}|^2} = \left| 1 - \frac{2/\tau_1}{j(\omega_1 - \omega) + \Gamma} - \frac{2/\tau_2}{j(\omega_2 - \omega) + \Gamma} \right|^2
\]

• Result: a Fano lineshape encompassing both resonances \(\omega_1\) and \(\omega_2\)
Conclusions

• In general, CMT works for a broad range of systems with well-defined and relatively weakly coupled resonances
• Can be readily extended to cases with weak losses, by treating them as additional ‘waveguides’
• Furthermore, in the linear case, most problems can be solved analytically
• Can extend CMT to nonlinear systems (e.g., Kerr media) or time-varying systems, but generally must use ODE solvers to find numerical solutions
Any Questions?
Contact Peter Bermel <pbermel@purdue.edu>