Solving Ordinary Differential Equations

• For RC and RL circuits, we obtain:
  
  \[ X = X_\infty + (X_0 - X_\infty)e^{-(t-t_0)/\tau} \]

• Solution steps:
  
  – Choose \( X \)
    
    • RC circuits: \( X=Q \) or \( X=V \)
    
    • LR circuits: \( X=I \)
  
  – Find \( X_0 \) (simplifying diagram)
  
  – Find \( X_\infty \) (simplifying diagram)
  
  – Find \( R_{th} \) (for inductor/capacitor)
  
  – Find time constant \( \tau \):
    
    • RL circuits: \( \tau = L/R_{th} \)
    
    • RC circuits: \( \tau = R_{th}C \)
  
  – If circuit changes at \( t_1 \), use \( X(t_1) \) from prior solution for initial values
Example 1

• For this RC circuit, find the current flow $I(t)$ at all times, including the discontinuity at $t=0$.

\[ -36 + 108 \cdot \Theta(t) \]

\[ \begin{array}{c}
6 \text{ k}\Omega \\
100 \text{ } \mu\text{F} \\
30 \text{ k}\Omega
\end{array} \]
Solution

For t<0: no capacitor current,
\[ I = \frac{-36 \text{ V}}{30 + 6 \text{ kΩ}} = -1 \text{ mA} \]
\[ V_C = \frac{30}{36} \times (-36 \text{ V}) = -30 \text{ V} \]

At t=0^+, \ V_C is continuous, so \[ I = \frac{72 + 30}{6} = 17 \text{ mA} \]

As t \to \infty, \ I = 2 \text{ mA} \]

Time constant \[ \tau = R_{th} C = (5 \text{ kΩ}) \times (100 \text{ µF}) = 0.5 \text{ s} \]
Solution

Matching our solution with the boundary conditions yields:

\[ I(t) = \begin{cases} 
-1 \text{ mA,} & t < 0 \\
2 + 15e^{-2t}, & t > 0 
\end{cases} \]
Solution

Current (mA)
Linearity

• Much like with networks of resistors and sources, networks with capacitors and resistors obey these principles:
  – Linearity
  – Superposition
  – Proportionality

• Initial conditions can be viewed as another superimposed source that shuts off at the beginning
Example 2

- Using superposition, calculate $V_C$ for zero input ($V_o=25 \text{ V}$), $I_1$ only (50 mA), $I_2$ only (25 mA), all combined, and when the sources are cut in half.
Solution

\[ V_C \text{ for zero input: capacitor sees two resistors of 1 k}\Omega; R_{th}=500 \\Omega:\]

\[ V_C = 25 e^{-0.4t} \]
$V_C$ from $I_1$ only: source transformation yields $V_{oc} = 30$ V; combine series resistors; transform for $I_{sc} = 30$ mA; $R_{th} = 500$ Ω; transform again for $V_{oc} = 15$ V. With $\tau = (5 \text{ mF})(500 \text{ Ω}) = 2.5$ s, we obtain:

$$V = 15(1 - e^{-0.4t})$$
V_C from I_2 only: source transformation yields V_{oc} = -10 V; combine series resistors; transform for I_{sc} = -10 mA; R_{th} = 500 Ω; transform again for V_{oc} = -5 V. With \( \tau = (5 \text{ mF})(500 \text{ Ω}) = 2.5 \text{ s} \), we obtain:

\[
V = -5(1 - e^{-0.4t})
\]
Putting everything together, we obtain:

\[ V = 10 + 15e^{-0.4t} \]

If the input sources are both cut in half, we get:

\[ V = 5 + 20e^{-0.4t} \]
Example 3

• What waveform is produced by this circuit with a mechanical switch S that flips on for 1 s, then off for 1 s, and so on?
Solution

• Consider 2 cases separately:
  - Switch open: \( RC = (20 \text{ k}\Omega)(100 \text{ \mu F}) = 2 \text{ s}, \ V_\infty = 10 \text{ V} \)
  - Switch closed: \( RC = (1 \text{ k}\Omega)(100 \text{ \mu F}) = 0.1 \text{ s}, \ V_\infty = 0.5 \text{ V} \)

• After closing switch, will quickly reach final value
Solution

Thus, to an excellent approximation:

\[ V_c(t) = \begin{cases} 
10 - 9.5e^{-t/2}, & 0 \leq t < 1 \\
0.5 + 3.74e^{-10(t-1)}, & 1 \leq t < 2 
\end{cases} \]
Solution

Voltage (V)
Homework

- HW #18 due today by 4:30 pm in EE 326B
- HW #19 due Wed.: DeCarlo & Lin, Chapter 8:
  - Problem 18(a),(b)
  - Problem 21(a),(b)
  - Problem 22